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On Poincare and Friedrichs-type inequalities, convolution operators and Riesz potential in one class of non classical spaces

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Abstract. *In this work we generalize the results of the works [4,16] for one class of spaces defined by additive-shift operators in [19]. These spaces are characterized by behavior of shifts generated by sufficiently small vectors, and it is proved that considered spaces are isomorphic to the so-called additive-invariant spaces. In [4] problems related to compactness, compactness of embeddings with respect to additive invariant spaces, in [16] the integral operators, their compactness, Riesz potential, some integral representation theorems are studied. In the present work we give generalization of these results.*

Keywords. Poincare and Friedrichs-type inequalities, convolution operator, Riesz potential, compactness.

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1 Introduction

In recent years, as it turned out, it is necessary to introduce new spaces to solve a number of contemporary problems arising naturally in different areas of mechanics, mathematics, physics. Thus it became necessary to introduce and study such spaces from various viewpoints. We consider them mostly in the context of partial differential equations.

The emergence of new function spaces such as Morrey, grand Lebesgue, Orlicz, variable Lebesgue spaces, etc. naturally requires to develop the appropriate theory. That's why various problems in such spaces and corresponding Sobolev spaces generated by these spaces

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began to be intensively studied (see [2,3,5–12,14,20–22]). In general, many of Banach function spaces are not separable. Therefore, have no direct analogies in these spaces. It is requiried the essential modification of classical methods and a lot of preparation concerning correctness of substitution operator, problems related to the extension operator in such spaces, etc. To this aim, based on the additive shift operator $(T_\delta f)(x) = f(x+\delta)$, corresponding separable subspaces $X_s(\Omega)$ of these spaces have been introduced, in which the set of compactly supported infinitely differentiable functions is dense ([4,5,13,15–17]). Corresponding subspaces of grand Lebesgue, Marcinkiewich, weak type L_p^w, Morrey spaces are described for example in [13]. In rearrangement-invariant additive-invariant case with certain property these subspaces coincide with the set of absolutely continuous functions.

In [15] the substitution operators, extension of functions of the space $W_{X_s}^m(\Omega)$, in [17] interior Shauder-type estimates in rearrangement-invariant spaces, in [4,16] compactness criteria and boundedness of integral operators in additive-invariant spaces, in [18] one specific elliptic equation in Hardy-Banach classes are studied.

Main aim of this work is to define a wider class of Banach function spaces for which the above mentioned results of [4,16] hold true. In [19] one class of spaces defined by additive-shift operators is introduced. These spaces are characterized by behavior of shifts generated by sufficiently small vectors. It is proved that considered spaces are isomorphic to the so-called additive-invariant spaces.

2 Necessary information

Throughout the paper we follow the symbols, terminologies and agreements occurred in the paper [19]. In particular, we will use the following standard notations: N- the set of natural numbers, Z_+ will denote the set of non-negative integers, $|x| = \sqrt{x_1^2 + ... + x_n^2}$ will represent the norm of $x = (x_1, ..., x_n)$, $m = |E|$ will stand for the Lebesgue measure of the set $E \subset R^n$. supp f will denote the support of the function f. By $[X,Y]$ $([X])$ we will denote the space of bounded operators acting from Banach function space X (if $X = Y$), $\|T\|_{[X,Y]}$ will be the norm of the operator T in $[X,Y]$.

Some monographs have been dedicated to the theory of Banach function spaces. We follow the terminologies and agreements occurred in [1].

21 Conventions

We are guided by the following agreements: $K = \{(x_1, ..., x_n) : |x_i| < d\} \subset R^n$ will be some cube or $K = R^n$, (K, m) will be Lebesgue measure space, we consider only the class of the functions which are finite $m - a.e.$, χ_E will be the characteristic function of m-measurable subset E.

It is considered only bounded subdomains Ω of K and $\Omega : \bar{\Omega} \subset K$. $\Omega + \delta$ will denote the shift of the domain Ω corresponding to the vector δ, i.e. $\Omega + \delta = \{t + \delta : t \in \Omega\}$, and it is considered only those vectors , for which $\Omega + \delta \subset K$ holds true.

$X(K)$ will be a Banach function space defined on K, with the function norm ρ. For a arbitrary domain $\Omega \subset K : \overline{\Omega} \subset K$, $X(\Omega)$ means the space of restrictions of all functions from $X(K)$ on Ω, with the corresponding norm, i.e.

$$X(\Omega) = \left\{ f \in X(K) : \|f\|_{X(\Omega)} = \|f\chi_\Omega\|_{X(K)} < \infty \right\}.$$

Depending on circumstances we assume that $f \in X(\Omega, m)$ is extended by zero on K, or on whole R^n.

We will call the following set the ***possible values*** of the shift vectors

$$a(\Omega) = \{\delta \in R^n : \Omega - \delta \subset K\},$$

by $T_\delta : X(\Omega) \to X(\Omega - \delta)$, we denote the *additive-shift operator*, defined by the following way

$$(T_\delta f)(x) = \begin{cases} f(x+\delta), & x+\delta \in \Omega, \\ 0, & x+\delta \notin \Omega. \end{cases}$$

Convention. *We assume that i)* $T_\delta f \in X(K), \forall \Omega : \bar{\Omega} \subset K, \forall \delta \in a(\Omega)$.

We called the space with the relation $\|T_\delta f\|_{X(K)} = \|f\|_{X(\Omega)}$, for arbitrary realtively compact domain $\Omega \subset K$, and $\forall \delta \in a(\Omega)$ *additive-invariant space*.

It is clear that if the space $X(K)$ is a rearrangement invariant space, then

$$T_\delta \in [X(\Omega), X(\Omega - \delta)], \forall \delta \in a(\Omega),$$

is a isometric operator.

$X_a(\Omega)$ will denote the subspace of all absolutely continuous functions, $X_b(\Omega)$ is the closure of all bounded functions from $X(\Omega)$ and $C_0^\infty(\Omega)$ is the set of all compact supported infinitely differentiable functions on Ω.

Let us introduce the following characteristic.

Property $\beta)$ **.** $\forall E_n \to \emptyset \Rightarrow \|\chi_{E_n}\|_{X(K)} \to 0$.

The following statement is true.

Statement 2.1 *([4]) Let $X(K)$ be a rearrangement–invariant Banach function space with Property $\beta)$ and $\Omega : \overline{\Omega} \subset K$ be any bounded domain. Then*

$$X_s(\Omega) = X_b(\Omega) = X_a(\Omega) = \overline{C_0^\infty(\Omega)}.$$

22 Banach-Sobolev spaces

Main aggregate of our studies are Banach-Sobolev spaces generated by Banach function spaces. They are the following spaces of functions denoted by $W_X^m(\Omega)$ and $W_{X_s}^m(\Omega)$

$$W_X^m(\Omega) = \{f \in X : \partial^p f \in X, \forall p \in Z_+, |p| \le m\},$$

$$W_{X_s}^m(\Omega) = \left\{f \in W_X^m(\Omega) : \|T_\delta f - f\|_{WX^m(\Omega)} \to 0, \delta \to 0\right\},$$

with the corresponding norm

$$\|f\|_{W_X^m(\Omega)} = \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{X(\Omega)}.$$

Since the shift operator is continuous on $W_{X_s}^m(\Omega)$, it is follows that $W_{X_s}^m(\Omega)$ is a closed subspace of $W_X^m(\Omega)$.

$\overset{0}{W}{}_{X_s}^m(\Omega) = \overline{C_0^\infty}(\Omega)$ (in the space $W_X^m(\Omega)$). It is clear that $u \in \overset{0}{W}{}_{X_s}^m(\Omega) \Rightarrow u \in \overset{0}{W}{}_{X_s}^m(\Omega_1)$ for every domain .

3 Main results

31 Compactness. Poincare and Friedrichs-type inequalities

Throughout this section without loss of a generality we assume that K is a cube and $X\left(K\right)$ has the following property:

$$\begin{aligned} &\forall \Omega : \ \bar{\Omega} \subset K \, \exists C\left(\Omega\right) > 0 :\Rightarrow \\ &\qquad \Rightarrow \ \left\|T_{\delta} f\right\|_{X(\Omega-\delta)} \le C\left(\Omega\right) \left\|f\right\|_{X(\Omega)}, \forall f \in X(\Omega), \forall \delta \in a(\Omega), \end{aligned} \tag{3.1}$$

i.e. $\forall \delta \in a\left(\Omega\right) \Rightarrow T_{\delta} \in \left[X\left(\Omega\right), X\left(K\right)\right]$ and the constant independent of Ω.

In [19] it is proved that if $X\left(K\right)$ has the property (3.1) then every space $X\left(\Omega\right)$ is isomorphic to some additive-invariant space. By these reason many of facts which valid for additive-invariant spaces must be true also for the spaces with property (3.1).

Now we give some generalization of results proved in [4,16] for the spaces with Property 3.1. In particular, in [4] criteria for compactness in $X_s\left(\Omega\right)$, embedding theorems in $W^m_{X_s}\left(\Omega\right)$, are studied, and Poincare's and Fricrichs type inequalities are established. It is clear that this criteria holds place under arbitrary isomorphism. Thus the following statements are true.

Theorem 3.1 *Let $X\left(K\right)$ be some Banach function space with relation* (3.1) *and Propertyβ), $\Omega : \overline{\Omega} \subset K$ be any domain, and $U \subset X_s\left(\Omega\right)$ be any subset. For compactness of U, it is necessary and sufficient that*

1) $\exists M > 0 : \sup\limits_{u \in U} \left\|u\right\|_{X(\Omega)} \le M$, i.e. U is bounded in $X_s\left(\Omega\right)$,

2)

$$\sup_{u\in U}\sup_{|z|<h} \left\|u\left(x+z\right) - u\left(x\right)\right\|_{X_s(K)} \to 0, \ h \to 0.$$

Indeed, properties of convergences, boundedness of sequences under isomorphism are preserved.

Theorem 3.2 *Let $X\left(K\right)$ be a Banach function space with* (3.1)*, Property β), and $\Omega : \bar{\Omega} \subset K$ be any domain, which admit the extension of functions of the space $W^1_{X_s}\left(\Omega\right)$. Then the bounded set in $W^1_{X_s}\left(\Omega\right)$ is compact in $X_s\left(\Omega\right)$.*

Naturally, if there is the extension of functions of the space $W^1_{X_s}\left(\Omega\right)$ into $\overset{0}{W}{}^1_{X_s}\left(\Omega'\right)$ in $X\left(K\right)$, after isomorphic translation this property must be preserved.

Corollary 3.1 *Let $X\left(K\right)$ be Banach function space with* (3.1) *and Property β), and the domain $\Omega \subset K$ admit the extension of functions of the space $W^{m+1}_{X_s}\left(\Omega\right)$. Then the bounded set in $W^{m+1}_{X_s}\left(\Omega\right)$ is relatively compact in $W^m_{X_s}\left(\Omega\right)$.*

From classical theory it is known that in Sobolev spaces Poincare and Friedrichs inequalities play exceptional role for definition equivalent norms. In our case the following holds true.

Corollary 3.2 (Poincare inequality) *Let $X\left(K\right)$ be Banach function space with* (3.1) *and Propertyβ), $\Omega : \overline{\Omega} \subset K$ admit the extension of functions of the space $W^1_{X_s}\left(\Omega\right)$ and $E \subset \Omega : \left|E\right| > 0$, be an arbitrary measurable subset. Then the norm defined as*

$$\left\|u\right\|^{(E)}_{W^1_{X_s}(\Omega)} = \left|\int_E u dx\right| + \sum_1^n \left\|\frac{\partial u}{\partial x_i}\right\|_{X_s(\Omega)}, \ \ u \in W^1_{X_s}\left(\Omega\right),$$

is equivalent to the original norm of $W^1_{X_s}\left(\Omega\right)$.

Corollary 3.3 (Friedrichs-type inequality) *Let* $X(K)$ *be some Banach function space with Properties* (3.1) *and* β), $\Omega:\ \overline{\Omega}\subset\Omega'$, $\overline{\Omega'}\subset K$, *and the domain* Ω' *admit the extension of the functions of the space* $W^m_{X_s}(\Omega')$. *Then in* $\overset{0}{W}{}^m_{X_s}(\Omega)$ *the original norm will be equivalent to the norm defined by the following way*

$$|||u|||_{W^m_X(\Omega)_s}=\sum_{|\alpha|=m}\|\partial^\alpha u\|_{X_s(\Omega)}\,.$$

32 Convolution operator

Without loss of the generality, in the sequel, we suppose that $\Omega\pm\Omega=\{x\pm y:\ x,y\in\Omega\}\subset K$.

By the convolution of the functions $f,\ h$ defined on $\Omega\subset K$, $f\in L_1(\Omega)\,,\ h\in X(\Omega)$, we will mean the following operator

$$(f*h)(x)=\int_{R^n}f(x-y)h(y)\,dy,\ x\in\Omega,$$

denoted as $f*g$.

Let us state some statements with respect to the spaces with property (3.1), which were proved in [16] for additive-invariant case.

Lemma 3.1 *Let* $X(K)$ *be some Banach function space on* K *with property* (3.1) *and* $\Omega:\overline{\Omega}\subset K$ *be some domain. Then for arbitrary pair* $f,g\in X(\Omega)$ *the convolution* $f*g$ *belongs to* $X(\Omega)$ *and the estimate*

$$\|f*g\|_{X(\Omega)}\le\|f\|_{X(\Omega)}\|g\|_{L_1(\Omega)}\,,$$

holds true.

Corollary 3.4 *Let* $X(K)$ *be some Banach function space on* K *with property* (3.1) *and* $\Omega:\ \overline{\Omega}\subset K$ *be some domain. Then for arbitrary functions* $f\in L_1(\Omega)\,,g\in X(\Omega)$ *the convolution* $f*g$ *belongs to* $X(\Omega)$ *and the estimate*

$$\|f*g\|_{X(\Omega)}\le\|f\|_{L_1(\Omega)}\|g\|_{X(\Omega)}\,,$$

holds.

33 On Riesz potential

Let $\Omega\subset R^n$ be any relatively compact domain, $0<\alpha<n$, $A(x,y)\in L_\infty(\Omega\times\Omega)$. The operator defined as

$$(R_{A,\alpha}f)(x)=\int_\Omega\frac{A(x,y)}{|x-y|^\alpha}f(y)\,dy,$$

is called a Riesz potential.

Let $k_\alpha(x)=\frac{1}{|x|^\alpha}$, $0<\alpha<n$. Consider the integral operator K_α with the kernel $k_\alpha(x-y)=\frac{1}{|x-y|^\alpha}$, i.e. $(K_\alpha u)(x)=(k_\alpha*u)(x)=\int_\Omega\frac{u(y)}{|x-y|^\alpha}dy$.

It is clear that the boundedness of the operator K_α implies the boundedness of the operator $R_{A,\alpha}$. For the spaces under our consideration the following statements are true.

Corollary 3.5 *Let $X\,(K)$ be some Banach function space on K with property* (3.1)*, Ω : $\overline{\Omega} \subset K$ be any relatively compact domain and $\alpha \in (0,n)$. Then the integral operator K_α is bounded in $X\,(\Omega)$ and the estimate*

$$\|K_\alpha g\|_{X(\Omega)} = \|f * g\|_X \leq \|k_\alpha\,(.)\|_{L_1(\Omega)}\,\|g\|_{X(\Omega)} \leq C\,\|g\|_{X(\Omega)}\,,\,\forall g \in X\,(\Omega)\,,$$

holds. Consequently, Riesz potential is bounded in $X\,(\Omega)$.

Lemma 3.2 *Let $X\,(K)$ be some Banach function space on K with property* (3.1) *and Property β) and Ω : $\overline{\Omega} \subset \mathrm{K}$ be any relatively compact domain in R^n. Then*

$$K_\alpha \in [X_s\,(\Omega)],\ 0 < \alpha < n.$$

Corollary 3.6 *Let $X\,(K)$ be some Banach function space on K with properties* (3.1) *and β). Let Ω : $\overline{\Omega} \subset K$ be any relatively compact domain in R^n. Then the K_α can be approximated by compact integral operators. Consequently, K_α is a compact operator acting in $X\,(\Omega)$.*

Theorem 3.3 *Let $X\,(K)$ be some Banach function space on K with properties* (3.1) *and β), Ω : $\overline{\Omega} \subset K$ be any relatively compact domain in R^n. Then, for an arbitrary $u \in \overset{0}{W}{}^1_{X_s}\,(\Omega)$ the following representation is true*

$$u\,(x) = \frac{1}{\sigma_n}\sum_{i=1}^{n}\int_{\Omega}\frac{x_i - y_i}{|x-y|^n}\frac{\partial u}{\partial y_i}dy,$$

where σ_n is an area of the unit sphere in R^n, i.e. $\sigma_n = 2\pi^{\frac{n}{2}}\left(\Gamma\left(\frac{n}{2}\right)\right)^{-1}$.

Corollary 3.7 *Let $X\,(K)$ be some Banach function space on K with properties* (3.1) *and β), Ω : $\overline{\Omega} \subset K$ be any relatively compact domain. Then, for an arbitrary $u \in \overset{0}{W}{}^m_{X_s}\,(\Omega)$, the following representation*

$$v\,(x) = \frac{1}{(m-1)!\sigma_n}\sum_{|i|=m}\int_{R^n}\frac{(x_{i_1} - y_{i_1})^{l_i}\,...\,(x_{i_n} - y_{i_n})^{i_n}}{|x-y|^n}\frac{\partial^m v\,(y)}{\partial y^i}dy,$$

holds true.

Corollary 3.8 *Let $X\,(K)$ be some Banach function space on K with properties* (3.1) *and β), Ω : $\overline{\Omega} \subset K$ be any relatively compact domain. Then*

a) the space $\overset{0}{W}{}^m_{X_s}\,(\Omega)$ can be compactly embedded into $X_s\,(\Omega)$.
b) if the domain Ω : $\overline{\Omega} \subset K$ admits extension of the functions from $W^m_{X_s}\,(\Omega)$, then $W^m_{X_s}\,(\Omega)$ can be compactly embedded into $X_s\,(\Omega)$.

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