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On basicity of eigenfunctions of a spectral problem in rearrangement-invariant Banach function space

Telman B. Gasymov · Alirza Q. Ahmadov

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Abstract. *In this paper, we investigate the basicity of the eigenfunction system associated with a second-order differential operator with a discontinuity point and a spectral parameter in the discontinuity condition, within rearrangement-invariant Banach function spaces. The analysis is carried out under certain assumptions on the Boyd indices of these spaces. As a result, we prove a series of theorems establishing the basicity of the eigenfunctions in appropriate separable subspaces. These results contribute to the extension of classical spectral theory to wider classes of function spaces.*

Keywords. Discontinuous second-order differential operator · Banach function spaces · basicity · eigenfunctions.

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1 Introduction

We consider the following spectral problem with a point of discontinuity:

$$y^{''}(x) + \lambda y(x) = 0, \;\; x \in \left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, 1\right) \tag{1.1}$$

$$\left.\begin{aligned} y'(0) = y'(1) = 0 \\ y\left(\tfrac{1}{3} - 0\right) = y\left(\tfrac{1}{3} + 0\right) \\ y'\left(\tfrac{1}{3} - 0\right) - y'\left(\tfrac{1}{3} + 0\right) = \lambda m y\left(\tfrac{1}{3}\right), \end{aligned}\right\} \tag{1.2}$$

here, λ is the spectral parameter, and $m \neq 0$ is a non-zero complex constant. Spectral problems of this type naturally arise in the study of vibrations of a loaded string with free ends, particularly when solved using the Fourier method. The theoretical and practical importance of these problems is noted in several well-known monographs (see, e.g., [1–3]). The spectral problem formed by the vibration of a loaded string with fixed ends was studied in works [4-14]. In [15–17], asymptotic formulas for the eigenvalues and eigenfunctions of

T.B.Gasymov
Baku State Uneversity, Baku, Azerbaijan
E-mail: telmankasumov@rambler.ru

A.Q.Ahmadov
Karabakh University, Karabakh, Azerbaijan
Baku State Uneversity, Baku, Azerbaijan E-mail: alirza.ahmadov@karabakh.edu.az

the spectral problem (1.1),(1.2) were derived. Furthermore, theorems concerning completeness and basis properties were proved in the spaces $L_p \bigoplus C$ and L_p, as well as in Morrey spaces. In [18, 19], asymptotic formulas for the eigenvalues and eigenfunctions of the spectral problem (1.1),(1.2) in the summable potential case were obtained, and theorems on the basis properties of the eigenfunctions in the spaces $L_p \bigoplus C$ and L_p, were established. In [20-25], direct and inverse spectral problems for differential operators with discontinuity conditions in various functional spaces, were studied. Another class of boundary conditions are degenerate boundary conditions. For such boundary conditions, the spectrum of the corresponding operator is either empty or coincides with the entire complex plane (see [26] and the bibliography there).

The basicity properties of eigenfunctions corresponding to the Sturm–Liouville problem with periodic and antiperiodic boundary conditions were studied in [27] within the framework of rearrangement-invariant Banach function spaces. However, the method used in the article [27] is applicable only to a narrow class of spectral problems, since it is based on the fact of uniform equiconvergence with the trigonometric system. The basis property of the classical exponential system in separable subspaces of invariant spaces is established, for example, in the monographs [28-30]. In [31], the basis property of the trigonometric system in the weighted Morrey space is proved.

In this paper, we investigate the basis properties of a system of eigenfunctions of the spectral problem (1.1),(1.2) in Banach function spaces. Under certain conditions imposed on the Boyd indices (see [28, 32, 33]) of rearrangement-invariant Banach function spaces, we prove basis properties of the system of eigenfunctions of the spectral problem in suitable separable subspaces of these spaces. In particular, these spaces include Lebesgue, Grand-Lebesgue, Orlicz, Marcinkiewicz, and other spaces.

2 Auxiliary Facts

Recall the definitions of the $r-$bases and $r-$close systems in Banach space X.

Definition 2.1 *The bases $\{x_n\}_{n\in N}$ of Banach space X is called a $r-$bases, if for any $x \in X$*

$$\left(\sum_{n=1}^{\infty} |\langle x,\ x_n^*\rangle|^r\right)^{\frac{1}{r}} \leq C\,\|x\|\,,$$

where $\{x_n^*\}_{n\in N}$ is a conjugate system of $\{x_n\}_{n\in N}$.

Definition 2.2 *The sequence $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ of X is called a $r-$close if*

$$\sum_{n=1}^{+\infty} \|x_n - y_n\|_X^r < +\infty.$$

The following theorem is proved in [34].

Theorem 2.1 *Let X be a Banach space with $r-$basis $\{x_n\}_{n\in N}$ and a system $\{y_n\}_{n\in N}$ is $r'-$close to $\{x_n\}_{n\in N}$, $\frac{1}{r}+\frac{1}{r'}=1$. Then the following properties are equivalent:*

1 $\{y_n\}_{n\in N}$ is a complete system in X;
2 $\{y_n\}_{n\in N}$ is a minimal system in X;
3 $\{y_n\}_{n\in N}$ is a $\omega-$linear independent system in X;
4 $\{y_n\}_{n\in N}$ is a basis in X.

If one of these conditions is satisfied, then the system $\{y_n\}_{n\in N}$ forms a basis of the space X which is isomorphic to the system $\{x_n\}_{n\in N}$.

Let us consider that X can be presented as a direct decomposition $X = X_1 \bigoplus \bigoplus X_2 \bigoplus \cdots \bigoplus X_m$, where $X_i, i = \overline{1,m}$ are Banach spaces. For convenience, the element of X are identified, with vectors: $x \in X \ \Leftrightarrow x = (x_1, x_2, \ldots, x_m)$, where $x_k \in X_k, k = \overline{1,m}$. The norm of X is defined by the formula

$$\|x\|_X = \sqrt{\sum_{i=1}^{m} \|x_i\|_{X_i}^2}.$$

It is easy shown that (see [35]) $X^* = X_1^* \bigoplus X_2^* \bigoplus \cdots \bigoplus X_m^*$ and for all $f \in X^*$ and $x \in X$ it holds

$$< x; f >= \sum_{i=1}^{m} < x_i; f_i >,$$

($< \cdot\, ; \, \cdot >$ is the value of functional), where $f = (f_1, f_2, \ldots, f_m)$ and $f_k \in X_k^*, k = \overline{1,m}$. For the $x_k \in X_k$ let us denote by $\tilde{x}_k$ the element from X, which is defined by the formula

$$\tilde{x}_k = \left(\underbrace{0, \ldots, 0, x_k}_{k}, \ldots, 0 \right).$$

Suppose that a system $\{u_{in}\}_{i=\overline{1,m}, n\in N}$ is given in each space $X_i, i = \overline{1,m}$. Consider the following system in X

$$\hat{u}_{in} = \left(a_{i1}^{(n)} u_{1n}, a_{i2}^{(n)} u_{2n}, \ldots, a_{im}^{(n)} u_{mn} \right), \ \ i = \overline{1,m}, \ n \in \mathbb{N} \tag{2.1}$$

where $a_{ij}^{(n)}$ are some numbers. Let

$$A_n = \left(a_{ij}^{(n)} \right)_{i,j=\overline{1,m}}; \quad \Delta_n = \det A_n.$$

The following theorem is proved in [12].

Theorem 2.2 *If the system $\{u_{kn}\}_{k=\overline{1,m}; n\in N}$ forms basis for X and*

$$\Delta_n \neq 0, \forall n \in N$$

then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m}; n\in N}$ defined by formula (2.1) forms basis with parentheses for X. If in addition the following conditions

$$\sup_n \left\{ \|A_n\|, \left\|A_n^{-1}\right\| \right\} < \infty, \quad \sup_n \left\{ \|u_{kn}\|, \|\vartheta_{kn}\| \right\} < \infty$$

hold, where $\{\vartheta_{kn}\}_{k=\overline{1,m}; n\in N} \subset X^*$- is biorthogonal to $\{u_{kn}\}_{k=\overline{1,m}; n\in N}$, then the system $\{\hat{u}_{kn}\}_{k=\overline{1,m}; n\in N}$ forms a usual basis for X.

Let X_0 be a Banach space with the norm $\|\cdot\|_{X_0}$. Then $X = X_0 \bigoplus \mathbb{C}^m$ is also a Banach space for $\hat{u} = (u, \alpha_1, \alpha_2, \ldots, \alpha_m) \in X$, where $u \in X_0, \ \alpha_k \in \mathbb{C}, k = \overline{1,m}$, the norm is defined by the formula

$$\|\hat{u}\|_X = \left(\|u\|_{X_0}^2 + \sum_{k=1}^{m} |\alpha_k|^2 \right)^{\frac{1}{2}}.$$

$X^* = X_0^* \bigoplus \mathbb{C}^m$ is a dual space of X and latter means that the each vector $(v, \beta_1, \ldots, \beta_m) \in X_0^* \bigoplus \mathbb{C}^m$ defines the element $\hat{v} \in X^*$ by the formula [35]

$$<\hat{u}, \hat{v}> = <u, v> + \sum_{k=1}^{m} \alpha_k \overline{\beta}_k,$$

where $v \in X_0^*,\ \beta_k \in \mathbb{C}$, $k = \overline{1, m}$. In [36] (see also [37]) has been proved.

Theorem 2.3 *Let $\{\hat{u}_n\}_{n\in\mathbb{N}}$ form a basis for X, where $\hat{u}_n = (u_n, \alpha_{n1}, \alpha_{n2}, \ldots, \alpha_{nm})$ and $\{\hat{v}_n\}_{n\in\mathbb{N}}$ where $\hat{v}_n = (v_n, \beta_{n1}, \beta_{n2}, \ldots, \beta_{nm})$ is a biorthogonal conjugate system, $J = \{n_1, n_2, \ldots, n_m\} \subset \mathbb{N}$ is the set of m different natural numbers, $\mathbb{N}_J = \mathbb{N} \backslash J$. Put*

$$\delta = \det \|\beta_{n_k j}\|_{k,j=\overline{1,m}}^{n} .$$

Then for the basicity of the system $\{u_n\}_{n\in\mathbb{N}_J}$ in the space X_0 it is necessary and sufficient the fulfilment of the condition $\delta \neq 0$. Here the biorthogonal system of $\{u_n\}_{n\in\mathbb{N}_J}$ is given by

$$v_n^* = \frac{1}{\delta} \begin{vmatrix} v_n \ v_{n_1} & \ldots \ v_{n_m} \\ \beta_{n1} \ \beta_{n_1 1} & \ldots \ \beta_{n_m 1} \\ \vdots \quad \vdots & \vdots \quad \vdots \\ \beta_{nm} \ \beta_{n_1 m} & \ldots \ \beta_{n_m m} \end{vmatrix} .$$

If $\delta = 0$, then the system $\{u_n\}_{n\in\mathbb{N}_J}$ does not form a basis for X_0, moreover the system $\{u_n\}_{n\in\mathbb{N}_J}$ is not complete and minimal in X_0.

In this part we will give some definition and facts related to Banach function spaces (see, [28,29,30,38]). Let $(R, \mathfrak{G}, \mu)$ will denote a measurable space, $\mathfrak{G}$ is the σ –algebra of measurable subset of R , $\mathcal{M}$ will stand for a set of $\mu-$measurable functions on R, $\mathcal{M}^+$ is the set of non-negative functions from $\mathcal{M}$, $\mathcal{M}_0$ is the set of $\mu - a.e.$ finite functions from $\mathcal{M}$. The characteristic function of measurable subset E of R will be denote by χ_E.

Definition 2.3 *A mapping $\rho : \mathcal{M}^+ \to [0, \infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, (n = 1, 2, \ldots)$ in $\mathcal{M}^+$, for all constants $\alpha \geq 0$ and for all $\mu-$measurable subsets E of R, the following properties hold:*

1. $\rho(f) = 0 \Leftrightarrow f = 0\ \mu - a.e.$; $\rho(\alpha f) = \alpha\rho(f)$, $\rho(f + g) \leq \rho(f) + \rho(g)$;
2. $0 \leq g \leq f\ \mu - a.e. \Rightarrow \rho(g) \leq \rho(f)$;
3. $0 \leq f_n \nearrow f\ \mu - a.e. \Rightarrow \rho(f_n) \nearrow \rho(f)$;
4. $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;
5. $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, *for some constant* C_E, $0 < C_E < +\infty$, *depending on E and ρ but independent of f.*

Definition 2.4 *Let ρ be a function norm. The collection $X = X(\rho)$ all functions f in $\mathcal{M}$ for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$ define*

$$\|f\|_X = \rho(|f|) . \tag{2.2}$$

Definition 2.5 (Associated norm). *If ρ is a function norm, its associated norm ρ' is defined on $\mathcal{M}^+$ by*

$$\rho'(g) = \left\{ \int_R f g d\mu \ ; \ f \in \mathcal{M}^+,\ \rho(f) \leq 1 \right\},\ \ g \in \mathcal{M}^+ . \tag{2.3}$$

Definition 2.6 *Let ρ be a function norm and let $X = X(\rho)$ be a Banach function space determined by ρ as in definition 4. Let ρ' be the associate norm of ρ. The Banach function space $X\left(\rho'\right)$ determined by ρ' is called the associate space of X and is denoted by X'.*

It follows from (2.2), (2.3) that the norm of a function ρ in the associate space X' is given by

$$\|g\|_X' = \sup\left\{\int_R |fg|\,d\mu \ : \ f \in X,\ \|f\|_X \le 1\right\}.$$

Definition 2.7 *A function f in a Banach function space X is said to have absolutely continuous norm in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{+\infty}$ satisfying $E_n \to \emptyset$ $\mu-$ a.e. The set of all functions in X of absolutely continuous norm is denoted by X_a. If $X_a = X$, then the space X itself is said to have absolutely continuous norm.*

Definition 2.8 *Let X be a Banach function space. The closure in X of the set of simple functions is denoted by X_b.*

Proposition 2.1 *[38] The subspace X_b is the closure in X of the set of bounded functions supported in sets of finite measure. In addition, there are continuous embeddings $X_a \subset X_b \subset X$. The subspaces X_a and X_b coincide if and only if $1 \in X_a$.*

Definition 2.9 *The distribution function μ_f of a function $f \in \mathcal{M}_0$ is given by*

$$\mu_f(\lambda) = \mu\{x \in R \ : \ |f(x)| > \lambda\},\ \lambda \ge 0.$$

Two functions $f, g \in \mathcal{M}_0$ are called to be equimeasurable if $\mu_f(\lambda) = \mu_g(\lambda)$ for all $\lambda \ge 0$.

The decreasing rearrangement function of the function $f \in \mathcal{M}_0$ is defined as a function

$$f^*(t) = inf\{\lambda \ge 0 :\ \mu_f(\lambda) \le t\},\ t \ge 0.$$

Definition 2.10 *Let $(R, \mathfrak{G}, \mu)$ is totally σ –finite measure space. If for every pair equimeasurable functions $f, g \in \mathcal{M}_0$ identity $\rho(|f|) = \rho(|g|)$ holds, then the norm ρ is called rearrangement-invariant norm. The Banach function space X, generated by rearrangement-invariant norm ρ is called rearrangement-invariant Banach function space.*

Let X be a rearrangement-invariant Banach function space over $(R, \mathfrak{G}, \mu)$, $R^+ = (0, +\infty)$ and m is the Lebesgue measure on R^+. From the Luxemburg representation theorem ([28] Theorem II.4.10) there is a rearrangement-invariant function norm $\tilde{\rho}$ over (R^+, m) defined by

$$\tilde{\rho}(g) = sup\left\{\int_0^\infty f^*(t)\, g^*(t)\,dt :\ \rho'(f) \le 1\right\},$$

such that

$$\rho(f) = \tilde{\rho}(f^*),\ \forall f \in \mathcal{M}_0^+.$$

The rearrangement-invariant Banach function space generated by $\tilde{\rho}$ is denoted by $\tilde{X}$.

Definition 2.11 *For $t > 0$, the operator E_t defined by the formula*

$$(E_t f)(s) = f(st),\ \ 0 < s < +\infty,$$

Is called a dilation operator. If $h_X(t) = \left\|E_{1/t}\right\|_{\tilde{X}\to\tilde{X}}$, then the Boyd indices a_X and β_X of the space X defined by the formula

$$\alpha_X = \lim_{t\to+0} \frac{\ln h_X(t)}{\ln t}\ ,\quad \beta_X = \lim_{t\to+\infty} \frac{\ln h_X(t)}{\ln t}\ .$$

For the Boyd indices, the conditions $0 \le \alpha_X \le \beta_X \le 1$ are satisfied, moreover $\alpha_{X'} = 1 - \beta_X$, $\beta_{X'} = 1 - \alpha_X$.

We need also the following theorems [28,32,33].

Theorem 2.4 *(Boyd) Let X be a rearrangement-invariant Banach function space with the Boyd indices $\alpha_X,\ \beta_X:\ 0<\alpha_X\le\beta_X<1$. Then for every $p,\ q:\ 1\le q<\frac{1}{\beta_X}\le\frac{1}{\alpha_X}<$ $<p\le+\infty$, the continuous embeddings $L_p\subset X\subset L_q$ hold.*

Theorem 2.5 *Let $X=X(-\pi,\pi)$ be a rearrangement-invariant Banach function space on the interval $(-\pi,\pi)$. The Fourier series converges in $X_b(-\pi,\pi)$ if and only if the Boyd indices satisfy the condition*

$$0<\alpha_X\le\beta_X<1. \tag{2.4}$$

In [15] it was proved that the eigenvalues of the problem (1.1),(1.2) are asymptotically simple and consist of $\lambda_0=0$ and two series: $\lambda_{i,n}=\rho_{i,n}^2,\ i=1,2;\ n\in Z^+$, where $Z^+=\{0\}\bigcup N$ and $\rho_{i,n}$ holds the following asymptotically formulas:

$$\begin{cases}\rho_{1,n}=3\pi n+\frac{3\pi}{2}+O\left(\frac{1}{n}\right),\\ \rho_{2,n}=\frac{3\pi n}{2}+\frac{3\pi}{4}+O\left(\frac{1}{n}\right).\end{cases} \tag{2.5}$$

Also the eigenfunctions $y_0\equiv1,\ y_{i,n}(x),\ i=1,2;\ n\in Z^+$ of the problem (1.1),(1.2) corresponding to the eigenvalues λ_0 $\lambda_{i,n}=\rho_{i,n}^2$, hold the following formulas

$$y_{i,n}(x)=\begin{cases}\cos\frac{2\rho_{i,n}}{3}\cos\rho_{i,n}x, & x\in\left[0,\frac{1}{3}\right],\\ \cos\frac{\rho_{i,n}}{3}\cos\rho_{i,n}(1-x), & x\in\left[\frac{1}{3},1\right],\end{cases}\quad i=1,2;n\in Z^+. \tag{2.6}$$

Considering (2.5) in (2.6), we will get the following system:

$$\begin{cases}y_{1,n}(x)=\begin{cases}-\cos\left(3\pi n+\frac{3\pi}{2}\right)x+O\left(\frac{1}{n}\right), & x\in\left[0,\frac{1}{3}\right],\\ 0, & x\in\left[\frac{1}{3};1\right],\end{cases}\\ y_{2,n}(x)=\begin{cases}0, & x\in\left[0,\frac{1}{3}\right],\\ \alpha_n\cos\left(\frac{3\pi n}{2}+\frac{3\pi}{4}\right)(1-x)+O\left(\frac{1}{n}\right), & x\in\left[\frac{1}{3};1\right],\end{cases}\end{cases} \tag{2.7}$$

where $\alpha_n=\cos\left(\frac{\pi n}{2}+\frac{\pi}{4}\right)$. If $n=4k$and $n=4k+3$, $\alpha_n=\frac{1}{\sqrt{2}}$ and $n=4k+1$ and $n=4k+2$, $\alpha_n=-\frac{1}{\sqrt{2}}$.

The problem (1.1),(1.2) can be transformed into a linear eigenvalue problem in the space $L_r(0;1)\bigoplus\mathbb{C}$ by introducing the operator L, which is defined as

$$D(L)=\begin{cases}\hat{y}=L_r(0,1)\bigoplus C:\hat{y}=\left(y,my\left(\frac{1}{3}\right)\right),y\in W_r^2\left(0,\frac{1}{3}\right)\bigoplus W_r^2\left(\frac{1}{3},1\right),\\ y'(0)=y'(1)=0,y\left(\frac{1}{3}-0\right)=y\left(\frac{1}{3}+0\right)\end{cases}$$

and for $\hat{y}\in D(L)$

$$L\hat{y}=\left(-y'';y'\left(\frac{1}{3}-0\right)-y'\left(\frac{1}{3}+0\right)\right).$$

The eigenvalues of the operator L and problem (1.1),(1.2) coincide and $\{\hat{y}_{i,n}\}_{i=1,2,n\in Z^+}$ are eigenvectors of the operator L, where

$$\hat{y}_0=(y_0(x);m),\hat{y}_{i,n}=\left(y_{i,n}(x);my\left(\frac{1}{3}\right)\right),i=1,2;n\in Z^+. \tag{2.8}$$

It is also proved in [17] that the system $\{y_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2,n\in\mathbb{Z}^+}$ of the eigenvectors of the operator L form a basis for the space $L_r(0;1) \bigoplus \mathbb{C}$, $1 < r < +\infty$ and for $r = 2$ this basis becomes a Riesz basis. Besides, a biorthogonal system is constructed in [17] for the system $\{y_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2,n\in\mathbb{Z}^+}$.

Let X be a Banach function space. In this part we consider basicity properties of eigenfunctions of the spectral problem (1.1),(1.2) in $X \bigoplus \mathbb{C}$, and in X. Here we examine this properties rearrangement-invariant case for X.

If $X = X(I)$ be a rearrangement-invariant Banach function space on a finite interval I of the real axis and $1 \in X_a$ then $X_a = X_b = \overline{C_0^\infty}(I)$. In addition, if $I = I_1 \cup I_2$, $I_1 \cap I_2 = \emptyset$, then each element $f \in X(I)$ can be represented as a sum $f = f\chi_{I_1} + f\chi_{I_2} = f_1 + f_2$, where χ_{I_1} and χ_{I_2} are the characteristic functions of the intervals I_1 and I_2, respectively. The projectors corresponding to this decomposition will be denoted by P_1 and P_2, i.e. $P_1 f = f_1,\ P_2 f = f_2$. Denote $X(I_1) = P_1 X(I),\ \ X(I_2) = P_2 X(I)$. Then $X(I_1)$ and $X(I_2)$ are a rearrangement-invariant Banach function space in which the norms are defined as follows

$$\|f_1\|_{X(I_1)} = \|f\chi_{I_1}\|_{X(I)},\ \ \|f_2\|_{X(I_2)} = \|f\chi_{I_2}\|_{X(I)}.$$

Thus the space $X(I)$ is represented as a direct sum $X(I) = X(I_1) \bigoplus X(I_2)$. If we define a new norm in it as $\|f_1\|_{X(I_1)} + \|f_2\|_{X(I_2)}$, then it will be equivalent to the original. In addition, there is $X_b(I) = X_b(I_1) \bigoplus X_b(I_2)$.

Let $X = X(-1,1)$ be rearrangement-invariant Banach function space on the interval $(-1,1)$ and its Boyd indices α_X, β_X satisfy the conditions (2.3). According to Theorem 5, the exponential system $\{e^{i\pi nx}\}_{n\in Z}$ forms a basis in $X_b(-1,1)$, and this, in turn, is consistent with the fact that the system of trigonometric functions $\{1\} \cup \{\cos \pi nx\ , \sin \pi nx\ \}_{n\in N}$ forms a basis in $X_b(-1,1)$. Let us consider the decomposition of the space $X_b(-1,1)$ into a direct sum $X_b(-1,1) = X_b(-1,0) \bigoplus X_b(0,1)$. The following theorem holds .

Theorem 2.6 *The system of cosines* $\{\cos \pi nx\ \}_{n\in Z^+}$ *forms a basis in each of the spaces* $X_b(-1,0)$ *and* $X_b(0,1)$.

The proof of this theorem is carried out analogously to Theorem 3.1 in paper [14], which concerns the basis property of the system $\{\sin \pi nx\ \}_{n\in Z^+}$ in the spaces $X_b(-1,0)$ and $X_b(0,1)$.

Analogously to the above-mentioned case, the space $X_b(0,1)$ can be represented as a direct sum of its subspaces:

$$X\ (0,1) = X\ \left(0, \frac{1}{3}\right) \bigoplus X\ \left(\frac{1}{3}, 1\right),$$

where $X_b\left(0, \frac{1}{3}\right)$ and $X_b\left(\frac{1}{3}, 1\right)$ are the corresponding separable subspaces. Similarly to [14], it can be proved that the following statement holds.

Theorem 2.7 *The system of* $\left\{\cos\left(n + \frac{1}{2}\right)\pi x\right\}_{n\in Z^+}$ *forms a basis in the spaces* $X_b(0,1)$.

From this theorem, by means of a change of variable, the following is obtained in particular.

Corollary 2.1 *The systems of functions* $\left\{\cos\left(3\pi n + \frac{3\pi}{2}\right)x\right\}_{n\in Z^+}$ *and* $\left\{\cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x)\right\}$ *form bases in the spaces* $X\left(0, \frac{1}{3}\right)$ *and* $X\ \left(\frac{1}{3}, 1\right)$.

3 Main results

The main result of the work is the following theorem.

Theorem 3.1 *Let, X is a rearrangement-invariant Banach function space whose Boyd indices a_X, β_X satisfy $0 < a_X \le \beta_X < 1$. Then the system $\{\hat{y}_0\} \cup \{\hat{y}_{in}\}_{i=1,2,\ n\in n\in Z^+}$ of eigenfunctions corresponding to the spectral problem* (1.1), (1.2) *forms a basis in the space $X_b(0,1) \bigoplus \mathbb{C}$.*

Proof. Consider the direct sum $X_b(0,1) = X_b\left(0,\frac{1}{3}\right) \bigoplus X_b\left(\frac{1}{3},1\right)$ and represent its elements as vectors $\tilde{f} = (f_1; f_2)$, where $f_1 \in X_b\left(0,\frac{1}{3}\right),\ \ f_2 \in X_b\left(\frac{1}{3},1\right)$

Denote

$$\begin{cases} e_{1,n}(x) = \begin{cases} \cos\left(3\pi n + \frac{3\pi}{2}\right)x, & x \in \left[0,\frac{1}{3}\right], \\ 0, & x \in \left[\frac{1}{3},1\right], \end{cases} \\ e_{2,n}(x) = \begin{cases} 0, & x \in \left[0,\frac{1}{3}\right], \\ \cos\left(\frac{3\pi n}{2} + \frac{3\pi}{4}\right)(1-x), & x \in \left[\frac{1}{3},1\right]. \end{cases} \end{cases}$$

By Theorem 2.6, the systems $\{e_{1,n}\}_{n\in Z^+}$ and $\{e_{2,n}\}_{n\in Z^+}$ form bases in the spaces $X_b\left(0,\frac{1}{3}\right)$ and $X_b\left(\frac{1}{3},1\right)$respectively.Define

$$\tilde{e}_{1,n} = (e_{1,n}; 0), \tilde{e}_{2,n} = (0; e_{2,n}).$$

It then follows immediately that the combined system $\{\tilde{e}_{i,n}\}_{i=1,2;\ n\in Z^+}$ is a basis of the space $X_b(0,1)$. Consider the system $\{\tilde{u}_{i,n}\}_{i=1,2;n\in Z^+}$ defined by

$$\tilde{u}_{i,n} = b_{i,1}\tilde{e}_{1,n} + b_{i,2}\tilde{e}_{2,n},\ i = 1,2;\ n \in Z^+ \tag{3.1}$$

where b_{i1} and $b_{i,2}$, $i = 1,2$ are the entries of the matrix

$$B_n = \begin{pmatrix} -6 & 0 \\ 0 & 6\alpha_n \end{pmatrix}.$$

Since the determinant of B_n is nonzero, it follows from Theorem 2.2 that the system $\{\tilde{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ also forms a basis in $X_b(0,1)$, which is equivalent to the system $\{\tilde{e}_{i,n}\}_{i=1,2;\ n\in Z^+}$.

By Theorem 2.4, there exist numbers p and q satisfying

$$1 < q < \frac{1}{\beta_X} \le \frac{1}{\alpha_X} < p < +\infty,$$

such that the continuous embeddings

$$L_p(0,1) \subset X_b(0,1) \subset L_q(0,1)$$

hold.

Let us show that system $\{\tilde{e}_{i,n}\}_{i=1,2;n\in Z^+}$ is an $\tilde{q}$-basis in $X_b(0,1)$, where $\tilde{q} = \max\left\{q, q'\right\}$.

Assume that $1 < q \le 2$, then $q' \ge 2$ and $\tilde{q} = q'$. For any $f \in X_b(0,1)$, represented as $f = (f_1, f_2)$, with $f_1 \in X_b\left(0,\frac{1}{3}\right)$ and $f_2 \in X_b\left(\frac{1}{3},1\right)$ it follows from the Hausdorff–Young inequality and the embeddings

$$X_b\left(0,\frac{1}{3}\right) \subset L_q\left(0,\frac{1}{3}\right),\ X_b\left(\frac{1}{3},1\right) \subset L_q\left(\frac{1}{3},1\right)$$

that

$$\left(\sum_{n=0}^{\infty}|<f_1,e_{1,n}>|^{q'}\right)^{\frac{1}{q'}}\leq C\|f_1\|_{L_q\left(0,\frac{1}{3}\right)}\leq C\|f_1\|_{X_b\left(0,\frac{1}{3}\right)},$$

$$\left(\sum_{n=0}^{\infty}|<f_2,e_{2,n}>|^{q'}\right)^{\frac{1}{q'}}\leq C\|f_2\|_{L_q\left(\frac{1}{3},1\right)}\leq C\|f_2\|_{X_b\left(\frac{1}{3},1\right)}.$$

Here and in what follows, C will denote a constant that has different values in different places. From here we get

$$\left(\sum_{i=1}^{2}\sum_{n=0}^{\infty}|<f,\tilde{e}_{i,n}>|^{q'}\right)^{\frac{1}{q'}}$$

$$\leq\left(\sum_{n=0}^{\infty}|<f_1,e_{1,n}>|^{q'}\right)^{\frac{1}{q'}}+\left(\sum_{n=0}^{\infty}|<f_2,e_{2,n}>|^{q'}\right)^{\frac{1}{q'}}\leq C\|f\|_{X_b(0,1)}.$$

This implies that the system $\{\tilde{e}_{i,n}\}_{i=1,2,n\in Z^+}$ forms a $q'-$ basis for $X_b\left(0,1\right)$.

Let $q>2$, then $1<q'<2$ and $\tilde{q}=q$. By the Hausdorff–Young inequality, for all $f\in X_b\left(0,1\right)$ and using the embeddings

$$X_b\left(0,1\right)\subset L_q\left(0,1\right)\subset L_{q'}\left(0,1\right)$$

we have

$$\left(\sum_{i=1}^{2}\sum_{n=0}^{\infty}|<f,\tilde{e}_{i,n}>|^{q}\right)^{\frac{1}{q}}$$

$$\leq\left(\sum_{n=0}^{\infty}|<f_1,e_{1,n}>|^{q}\right)^{\frac{1}{q}}+\left(\sum_{n=0}^{\infty}|<f_2,e_{2,n}>|^{q}\right)^{\frac{1}{q}}$$

$$\leq C\|f\|_{L_{q'}(0,1)}\leq C\|f\|_{X_b(0,1)}.$$

This means that the system $\{\tilde{e}_{i,n}\}_{i=1,2;n\in Z^+}$ forms $q-$basis for $X_b\left(0,1\right)$. Consequently, the system $\{\tilde{e}_{i,n}\}_{i=1,2;n\in Z^+}$ forms $\tilde{q}-$ basis for $X_b\left(0,1\right)$. It follows from formulas (3.1) that the system $\{\tilde{u}_{i,n}\}_{i=1,2,n\in Z^+}$ is also a $\tilde{q}$-basis in $X_b\left(0,1\right)$. Then the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2,n\in Z^+}$, where , $\hat{u}_0=(0;1)$, $\hat{u}_{1,n}=(\tilde{u}_{1,n};0)$, $\hat{u}_{2,n}=(\tilde{u}_{2,n};\ 0)$, also forms a $\tilde{q}$-basis in the space $X_b\left(0,1\right)\bigoplus\mathbb{C}$.

Then from the formulas (2.7),(2.8) implies that, the following relations are true:

$$\begin{cases}\hat{y}_{1,n}\left(x\right)=\hat{u}_{1,n}\left(x\right)+O\left(\frac{1}{n}\right),\\ \hat{y}_{2,n}\left(x\right)=\hat{u}_{2,n}\left(x\right)+O\left(\frac{1}{n}\right).\end{cases}\tag{3.2}$$

Let's point

$$\hat{y}_{i,n}=\left(y_{i,n}\left(x\right),my_{i,n}\left(\frac{1}{3}\right)\right),i=1,2;\ n\in Z^+,$$

According to formulas (2.6),(2.7) since $y_{i,n}\left(\frac{1}{3}\right)=O\left(\frac{1}{n}\right)$, it follows from (3.2) that for the systems $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ and $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2;\ n\in Z^+}$ the following relation holds:

$$\sum_{i=1}^{2}\sum_{n=1}^{\infty}\|\hat{y}_{i,n}-\hat{u}_{i,n}\|^r_{X_b\bigoplus\mathbb{C}}<+\infty \tag{3.3}$$

for all $r\in(1,+\infty)$.

According to the results of [16], the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2,n\in Z^+}$ forms a basis of the space $L_r(0,1)\bigoplus\mathbb{C}$ for every $r\in(1,+\infty)$. Hence, it is complete and minimal in this space. Taking into account the embeddings

$$L_p(0,1)\bigoplus\mathbb{C}\subset X_b(0,1)\bigoplus\mathbb{C}\subset L_q(0,1)\bigoplus\mathbb{C}$$

we conclude that the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2;\ n\in Z^+}$ is also complete and minimal in $X_b(0,1)\bigoplus\mathbb{C}$.

Now, choosing $r=\tilde{q}'=\min\left\{q,q'\right\}$ in (3.3), we see that all the assumptions of Theorem 2.1 are satisfied. Therefore, by Theorem 2.1, the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2,n\in Z^+}$ forms a basis of the space $X_b(0,1)\bigoplus\mathbb{C}$ which is equivalent to the system $\{\hat{u}_0\}\cup\{\hat{u}_{i,n}\}_{i=1,2,n\in Z^+}$, and thus also equivalent to the system $\{\hat{e}_0\}\cup\{\hat{e}_{i,n}\}_{i=1,2,n\in Z^+}$, where $\hat{e}_0=\hat{u}_0$.

Now let us consider the basicity of the system $\{y_0\}\cup\{y_{i,n}\}^{\infty}_{i=1,2;n\in Z^+,n\neq n_0}$ with a remote function in space $X_b(0,1)$.

Theorem 3.2 *In order for the system*

$$\{y_0\}\cup\{y_{i,n}\}^{\infty}_{i=1,2;n\in Z^+,n\neq n_0}$$

of eigenfunctions and associated functions of problem (1.1),(1.2) to form a basis in $X_b(0,1)$, after eliminating any function $y_{i,n_0}(x)$, it is necessary and sufficient that the corresponding function $z_{i,n_0}(x)$ of the biorthogonal system satisfy the condition

$$z_{i,n_0}\left(\frac{1}{3}\right)\neq 0.$$

If

$$z_{i,n_0}\left(\frac{1}{3}\right)=0,$$

then after eliminating the function $y_{i,n_0}(x)$ from the system, the resulting system does not form a basis in $X_b(0,1)$. Moreover, in this case it is neither complete nor minimal in this space.

Proof. As we know from [16], the system $\{\hat{y}_0\}\cup\{\hat{y}_{i,n}\}_{i=1,2,n\in Z^+}$ has a biorthogonal conjugate vector system $\{\hat{z}_0\}\cup\{\hat{z}_{i,n}\}_{i=1,2,n\in Z^+}$, $\hat{z}_0=(z_0(x),\overline{m})$, $\hat{z}_{i,n}(x)=\left(z_{i,n}(x),\overline{m}z_{i,n}\left(\frac{1}{3}\right)\right)$ in $L_{p'}(0,1)\bigoplus\mathbb{C}\subset X_b^*(0,1)\bigoplus\mathbb{C}$, where the functions $z_{i,n}(x)$ are the eigenfunctions of the corresponding conjugate spectral problem

$$z''(x)+\lambda z(x)=0,\ \ x\in\left(0,\frac{1}{3}\right)\bigcup\left(\frac{1}{3},1\right), \tag{3.4}$$

$$\left.\begin{array}{c} z'(0) = z'(1) = 0 \\ z\left(\frac{1}{3} - 0\right) = z\left(\frac{1}{3} + 0\right) \\ z'\left(\frac{1}{3} - 0\right) - z'\left(\frac{1}{3} + 0\right) = \lambda \overline{m} z\left(\frac{1}{3}\right) \end{array}\right\}, \tag{3.5}$$

and have the form

$$z_0(x) \equiv 1,$$

$$z_{i,n}(x) = \begin{cases} c_{i,n} cos\frac{2\rho_{i,n}}{3}\, cos\rho_{i,n}\, x, & x \in \left[0, \frac{1}{3}\right], \\ c_{i,n} cos\frac{\rho_{i,n}}{3}\, cos\rho_{i,n}\, (1-x), & x \in \left[\frac{1}{3}, 1\right], \end{cases} \quad i = 1,2; n \in Z^+ \tag{3.6}$$

with the normalized numbers c_0, $c_{i,n}$ satisfying

$$c_0 = \frac{1}{1 + |m|^2}, \ \ c_{i,n} = 6 + O\left(\frac{1}{n}\right) \ i = 1,2; n \in Z^+.$$

From the formulas (3.6) for the eigenfunctions $\{z_0\} \cup \{z_{i,n}\}_{i=1,2;n\in Z^+}$ of the conjugate problem (3.4),(3.5), it follows that $z_{i,n}\left(\frac{1}{3}\right) \neq 0$. On the other hand, the eigenvectors of the adjoint operator L^* have the form $\hat{z}_0 = (1, \overline{m})$, $\hat{z}_{i,n} = \left(z_{i,n}(x), \overline{m} z_{i,n}\left(\frac{1}{3}\right)\right)$. Applying Theorem 2.3 to the system $\{\hat{y}_0\} \cup \{\hat{y}_{i,n}\}_{i=1,2,n\in Z^+}$ we see that $\delta = \overline{m} z_{i,n}\left(\frac{1}{3}\right) \neq 0$ or $\delta = \overline{m} z_0\left(\frac{1}{3}\right) = \overline{m} \neq 0$ for any $n \in \mathbb{Z}^+$, and all statement of the theorem follow from the corresponding statements of the Theorem 2.3.

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