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Multi-sublinear maximal operators on product total Morrey-Guliyev spaces

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Abstract. *Let $\mathcal{M}$ be the multi-sublinear maximal operator. We study the boundedness of the multi-sublinear maximal operator $\mathcal{M}$ on product total Morrey-Guliyev spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$.*

Keywords. total Morrey-Guliyev spaces, multi-sublinear maximal function, commutator, BMO.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

The classical Morrey spaces were introduced by Morrey [21] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [4,25]. The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [7], extend the Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. The total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L^p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [2,3,6,8–12,19,23, 24].

Let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. We denote by $\vec{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\vec{y} = (y_1, \ldots, y_m)$ and $d\vec{y} = dy_1 \cdots dy_m$.

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In the past twenty years, the multilinear Calderón-Zygmund theory was developed a lot and studied by many authors. Grafakos and Torres [5] introduced the multilinear Calderón-Zygmund operator and studied the boundedness of such operators. Later in [18], Lerner et al. introduced the following multi-sublinear maximal function $\mathcal{M}(\vec{f})(x)$ defined as (see, also [16,30–32])

$$\mathcal{M}(\vec{f})(x) = \sup_{B \ni x} \prod_{i=1}^{m} \frac{1}{|B|} \int_B |f_i(y_i)| dy_i.$$

When $m = 1$, we get $M \equiv M_1$ the classical Hardy-Littlewood maximal operator is given by

$$M(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,$$

where f is a locally integrable function. Obviously, it is easy to see $\mathcal{M}(\vec{f})(x) \leq \prod_{j=1}^{m} M(f_j)(x)$.

In this paper, we obtain the boundedness of the multi-sublinear maximal operator $\mathcal{M}$ on product total Morrey-Guliyev spaces $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

This paper is organized as follows: In Section 2, we give some theorems about the boundedness of multi-sublinear maximal operator $\mathcal{M}$ on the product total Morrey-Guliyev spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, see also [13,17,20,22,26–29].

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Multi-sublinear maximal operator on product total Morrey-Guliyev spaces

In this section, we investigate the boundedness of multi-sublinear maximal operator $\mathcal{M}$ on product total Morrey-Guliyev spaces.

Definition 2.1 *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space , by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space [6], and by $L^{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions f with the finite quasi-norms*

$$\|f\|_{L^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))}, \ \|f\|_{\widetilde{L}^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,t))},$$

$$\|f\|_{L^{p,\lambda,\mu}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L^p(B(x,t))},$$

respectively.

Definition 2.2 *Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ [6] and the weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions f with finite quasi-norms*

$$\|f\|_{WL^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} t^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))}, \ \|f\|_{W\widetilde{L}^{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{WL^{p,\lambda,\mu}} = \sup_{x \in \mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{WL^p(B(x,t))},$$

respectively.

Lemma 2.1 *[7, 9] If $0 < p < \infty$, $0 \le \mu \le \lambda \le n$, then*

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n),$$
$$WL^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n)$$

and

$$\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\},$$
$$\|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}}\right\},$$

respectively.

The following local estimate is valid.

Lemma 2.2 *[1, Lemma 3.3] Let $1 \le p < \infty$ and $B(x,r)$ be any ball in $\mathbb{R}^n$. Then, for $p > 1$ the inequality*

$$\|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L^p(B(x,t))} \tag{2.1}$$

holds for all $B(x,r)$ and for all $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.
Moreover if $p = 1$, then the inequality

$$\|Mf\|_{WL^1(B(x,r))} \lesssim r^n \sup_{t>2r} t^{-n} \|f\|_{L^1(B(x,t))}$$

holds for all $B(x,r)$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 2.1 *[7] 1. If $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \le C_{1,\lambda,\mu} \|f\|_{L^{1,\lambda,\mu}}, \tag{2.2}$$

where $C_{1,\lambda,\mu}$ is independent of f.
2. If $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \le \lambda < n$ and $0 \le \mu < n$, then $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and

$$\|Mf\|_{L^{p,\lambda,\mu}} \le C_{p,\lambda,\mu} \|f\|_{L^{p,\lambda,\mu}}, \tag{2.3}$$

where $C_{p,\lambda,\mu}$ depends only on p,λ,μ and n.

Lemma 2.3 *[15] Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and $\vec{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \times \ldots \times L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$*

$$\mathcal{M}\vec{f}(x) \le \prod_{j=1}^m \left[M\big(f_j^{\frac{p_j}{p}}\big)(x)\right]^{\frac{p}{p_j}}, \tag{2.4}$$

When $m \ge 2$, we find out $\mathcal{M}$ also have the same properties by providing the following multi-version of the Theorem 2.1.

Theorem 2.2 *Let p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \text{ for } 0 \le \lambda_j < n, \frac{\mu}{p} = \sum_{j=1}^m \frac{\mu_j}{p_j} \text{ for } 0 \le \mu_j < n. \tag{2.5}$$

(i) If $p > 1$, then the operator $\mathcal{M}$ is bounded from product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{L^{p,\lambda,\mu}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

(ii) If $p = 1$, then the operator $\mathcal{M}$ is bounded from product total Morrey space $L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$ to weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f} \in L^{p_1,\lambda_1,\mu_1}(\mathbb{R}^n)\times \ldots \times L^{p_m,\lambda_m,\mu_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda,\mu}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.$$

Proof. (i) If $p > 1$, by (2.4) and the Hölder inequality, we get

$$\begin{aligned}
&\Big([t]_1^{-\lambda}\,[1/t]_1^{\mu} \int_{B(x,t)} |\mathcal{M}\vec{f}(y)|^p dy\Big)^{\frac{1}{p}} \\
&\le \Big([t]_1^{-\lambda}\,[1/t]_1^{\mu} \int_{B(x,t)} \Big(\prod_{j=1}^{m} \Big[M(f_j^{\frac{p_j}{p}})(y)\Big]^{\frac{p}{p_j}}\Big)^p dy\Big)^{\frac{1}{p}} \\
&\le \prod_{j=1}^{m} \Big([t]_1^{-\lambda_j}\,[1/t]_1^{\mu_j} \int_{B(x,t)} \Big[M(f_j^{\frac{p_j}{p}})(y)\Big]^p dy\Big)^{\frac{1}{p_j}}.
\end{aligned}$$

Taking the p-th root of both sides and applying Theorem 2.1 with $p > 1$ and $\left|f_j\right|^{\frac{p_j}{p}} \in L^{p,\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\begin{aligned}
\|\mathcal{M}\vec{f}\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,\, t>0} \Big([t]_1^{-\lambda}\,[1/t]_1^{\mu} \int_{B(x,t)} |\mathcal{M}\vec{f}(y)|^p dy\Big)^{\frac{1}{p}} \\
&= \prod_{j=1}^{m} \Big\|M(f_j^{\frac{p_j}{p}})\Big\|_{L^{p,\lambda_j,\mu_j}} \le \prod_{j=1}^{m} \Big\|f_j^{\frac{p_j}{p}}\Big\|_{L^{p,\lambda_j,\mu_j}} = \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}},
\end{aligned}$$

which is the desired inequality.

(ii) If $p = 1$, for any $\tau > 0$, let $\varepsilon_0 = \tau$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (2.4), we get

$$\begin{aligned}
&\{y \in B(x,t) : \mathcal{M}\vec{f}(y) > \tau\} \\
&\subset \bigcup_{j=1}^{m} \Big\{y \in B(x,t) : \Big[M(f_j^{\frac{p_j}{p}})(y)\Big]^{\frac{p}{p_j}} > \frac{\varepsilon_{j-1}}{\varepsilon_j\,[t]_1^{\frac{\lambda-\lambda_j}{p_j}}\,[1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}\Big\}.
\end{aligned}$$

Let us now take $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}} = \frac{\Big[\prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j,\mu_j}}\Big]^{s'/p_j}}{\tau^{s'/p_j}\,\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}}, \qquad j = 1, 2, ..., m.$$

Then, applying Theorem 2.1 with $p=1$ and the fact $\left|f_j\right|^{p_j} \in L^{1,\lambda_j,\mu_j}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\Big|\Big\{y\in B(x,t): \mathcal{M}\vec{f}(y)\Big| > \tau\Big\} \\
&\lesssim \sum_{j=1}^m \Big|\Big\{y\in B(x,t): M(f_j^{p_j})(y) > \Big(\frac{\varepsilon_{j-1}}{\varepsilon_j\,[t]_1^{\frac{\lambda-\lambda_j}{p_j}}\,[1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}\Big)^{p_j}\Big\}\Big| \\
&\le \sum_{j=1}^m [t]_1^{\lambda_j}\,[1/t]_1^{-\mu_j}\Big(\frac{\varepsilon_j\,[t]_1^{\frac{\lambda-\lambda_j}{p_j}}\,[1/t]_1^{-\frac{\mu-\mu_j}{p_j}}}{\varepsilon_{j-1}}\Big)^{p_j}\|f_j^{p_j}\|_{L^{1,\lambda_j,\mu_j}} \\
&= \sum_{j=1}^m [t]_1^{\lambda}\,[1/t]_1^{-\mu}\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}}\Big)^{p_j}\|f_j\|^{p_j}_{L^{p_j,\lambda_j,\mu_j}} = \sum_{j=1}^m [t]_1^{\lambda}\,[1/t]_1^{-\mu}\Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}}\Big)\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}\Big]^{p_j} \\
&= \sum_{j=1}^m [t]_1^{\lambda}\,[1/t]_1^{-\mu}\Big(\frac{1}{\tau}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}\Big)^{s'} = [t]_1^{\lambda}\,[1/t]_1^{-\mu}\Big(\frac{1}{\tau}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}\Big)^{p}.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
&\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda,\mu}} \\
&= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big([t]_1^{-\lambda}\,[1/t]_1^{\mu}\Big|\Big\{y\in B(x,t): \mathcal{M}\vec{f}(y) > \tau\Big\}\Big|\Big)^{\frac{1}{p}} \\
&\lesssim \prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j,\mu_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 2.2.

In the case $\lambda=\mu$, $\lambda_j=\mu_j$, $j=1,\ldots,m$ from Theorem 2.2 we get the following corollary

Corollary 2.1 *[14, Theorem 2] Let* p *be the harmonic mean of* $p_1,\ldots,p_m>1$ *and*

$$\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j} \ \textit{for}\ \ 0\le\lambda_j<n. \tag{2.6}$$

(i) If $p>1$*, then the operator* $\mathcal{M}$ *is bounded from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to Morrey space* $L^{p,\lambda}(\mathbb{R}^n)$*. Moreover, there exists a positive constant* C *such that the following inequality is valid for all* $\vec{f}\in L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{L^{p,\lambda}} \le C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p=1$*, then the operator* $\mathcal{M}$ *is bounded from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to weak Morrey space* $WL^{p,\lambda}(\mathbb{R}^n)$*. Moreover, there exists a positive constant* C *such that the following inequality is valid for all* $\vec{f}\in L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{WL^{p,\lambda}} \le C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}.$$

In the case $\mu=\mu_j=0$, $j=1,\dots,m$ from Theorem 2.2 we get the following corollary

Corollary 2.2 *[14, Theorem 4] Let p be the harmonic mean of $p_1,\dots,p_m>1$ and satisfy* (2.6).

(i) If $p>1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f}\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{\widetilde{L}^{p,\lambda}}\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p=1$, then the operator $\mathcal{M}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\vec{f}\in\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}\vec{f}\|_{W\widetilde{L}^{p,\lambda}}\le C\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

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