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New integral inequalities in the framework of weighted Riemann–Liouville integral operators

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Abstract. *In this article, we introduce new weighted integral operators. We develop a lemma where we derive a generalized identity using these integral operators. Leveraging this identity, we establish new generalized Simpson-type inequalities for (h,m)-convex functions.*

Keywords. (h, m)–convex modified functions, Simpson type inequality, Riemann–Liouville fractional integrals.

Mathematics Subject Classification (2010): Primary 26A51, Secondary 26D10, 26A51, 26D15

1 Introduction

The concept of convexity is fundamental in several scientific disciplines related to mathematics, such as optimization theory, numerical analysis, and computational mathematics, because it is closely associated with estimating the mean value of a function over an interval. Currently, the literature features numerous classes of convexity that extend this concept. The definition of convexity are given in the literature as follows:

A function $\mathbf{f} : [\upsilon_1, \upsilon_2] \to \mathbb{R}$ is called convex if for any $x, y \in [\upsilon_1, \upsilon_2]$ and any $\xi \in [0, 1]$, the inequality $\mathbf{f}(\xi x + (1 - \xi)y) \leq \xi \mathbf{f}(x) + (1 - \xi)\mathbf{f}(y)$ holds. If this inequality is flipped, then the function $\mathbf{f}$ is concave on $[\upsilon_1, \upsilon_2]$.

Many classes of convexity of functions are defined in the literature. In [35], a fairly wide range of convexity classes and their relations are given.

In the literature, the well-known Simpson-type inequality is presented as follows.

If $\mathbf{f} \in C^4(\upsilon_1, \upsilon_2)$ and $\left\|\mathbf{f}^{(4)}\right\|_\infty := \sup\limits_{x \in (\upsilon_1, \upsilon_2)} \left|\mathbf{f}^{(4)}(x)\right| < \infty$, then

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$$\left|\frac{\upsilon_2-\upsilon_1}{3}\left[\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2}+2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right]-\int_{\upsilon_1}^{\upsilon_2}\mathbf{f}(x)dx\right|\le\frac{(\upsilon_2-\upsilon_1)^5}{2880}\left\|\mathbf{f}^{(4)}\right\|_\infty. \tag{1.1}$$

Several recent studies have focused on refining and generalizing Simpson-type inequalities for different classes of convex functions. For instance, Alomari and Hussain [2] derived Simpson-type inequalities for quasi-convex and differentiable functions. In [3], the author obtained Hadamard- and Simpson-type parametric integral inequalities for concave and $r-$convex functions via special mean applications. The authors in [6] developed new generalized integral inequalities of Simpson and Hadamard types for convex functions or those satisfying the Lipschitz or Lagrange conditions. Dragomir et al. [15] and Liu [30] presented Simpson-type inequalities for continuously differentiable functions and explored their applications. Erden et al. In [19] provides error estimates for Simpson's second-type formula and related inequalities. In [40] Siricharuanun et al. derives new quantum Simpson's and Newton's type inequalities extending existing results in quantum and classical analysis.

In [24], Hussain and Qaisar established new Simpson-type inequalities for functions whose third derivatives are prequasiinvex and preinvex. Park [36] introduced generalized Simpson- and Hadamard-type integral inequalities for functions whose $q-$th powers of the second derivatives are decreasing and $(\alpha, m)-$geometrically convex. Studies such as [14, 22,28] presented new Simpson-type inequalities based on $s-$convexity. Furthermore, [18, 16,31] investigated new Simpson-type inequalities for extended $(s,m)-$convex and generalized $(s,m)-$preinvex functions.

Hsu et al. [23] obtained extended Simpson-type inequalities for differentiable concave functions in connection with Hadamard's inequality. Nápolis and Rabossi [34] established several Simpson-type inequalities using generalized weighted integral operators for functions whose derivatives are bounded or $(\alpha, m)-$convex. Finally, Ujević [41] introduced Simpson-type double integral inequalities and their applications in numerical integration, while Nápoles et al. [33] defined weighted integral operators and, by using them, derived generalized Simpson-type inequalities for functions whose first derivatives are modified $(h,m)-$convex. Benaissa et al. [9] introduced new fractional integral operators and established trapezoid, and midpoint inequalities for $h-$convex functions.

In references [4,7], we introduced the definitions that follow.

Definition 1.1 *Let* $\mathbf{f}: X=[0,+\infty)\to[0,+\infty)$ *and* $h:[0,1]\to(0,1]$. *If*

$$\mathbf{f}\left(\upsilon_1\xi+m(1-\xi)\upsilon_2\right)\le h^s(\xi)\mathbf{f}(\upsilon_1)+m(1-h^s(\xi))\mathbf{f}(\upsilon_2) \tag{1.2}$$

is holds true $\forall\ \upsilon_1,\upsilon_2\in X$ *and* $\xi\in[0,1]$, *where* $0\le m\le 1$, $s\in(0,1]$. *Then the function* $\mathbf{f}$ *will be called the* $(h,m)-$*convex modification of the first type on* X.

Definition 1.2 *Let* $\mathbf{f}: X=[0,+\infty)\to[0,+\infty)$ *and* $h:[0,1]\to(0,1]$. *If*

$$\mathbf{f}\left(\upsilon_1\xi+m(1-\xi)\upsilon_2\right)\le h^s(\xi)\mathbf{f}(\upsilon_1)+m(1-h(\xi))^s\mathbf{f}(\upsilon_2) \tag{1.3}$$

is holds true $\forall\upsilon_1,\upsilon_2\in X$ *and* $\xi\in[0,1]$, *where* $s\in[-1,1], 0\le m\le 1$. *Then the function* $\mathbf{f}$ *will be called the* $(h,m)-$*convex modification of the second type on* X.

Remark 1.1 Based on the definitions given above, the sets of $(h,m)-$convex modified functions of the first and second type, identified by the triple $(h(\xi),m,s)$, are represented by $N_{h,m}^{s,1}[\upsilon_1,\upsilon_2]$ and $N_{h,m}^{s,2}[\upsilon_1,\upsilon_2]$, respectively. References [7] provide examples of convex classes derived from specific instances of this triple.

Remark 1.2 If the direction of the inequality (1.3) changes, it we have concave function.

The emergence of fractional calculus, involving integrals and derivatives of arbitrary order, was inevitable ([32,38]). Owing to its applicability in many fields of science and engineering ([17,10,42,11]), this area has gained significant prominence. A key feature of this subject is that researchers seeking more efficient solutions to physical phenomena have gradually adapted to new operators with dominant kernels.

Numerous studies in recent years have investigated Simpson-type integral inequalities for various fractional integral operators (see [1,4,5,7,12,14,21,31,43] and references therein). Several recent works have also focused on integral inequalities involving weighted integrals of different types (see [5,8,26,28,29,33,34]).

To facilitate a better understanding of the research topic, we first present the definition of the Riemann–Liouville fractional integral (where $0 \leq \upsilon 1 < \xi < \upsilon 2 \leq \infty$).

Definition 1.3 *Let* $\mathbf{f} \in L_1[\upsilon_1, \upsilon_2]$. *Then the Riemann–Liouville fractional integrals of order* $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ *are defined by (right and left respectively):*

$$I^{\alpha}_{\upsilon_1+}\mathbf{f}(x) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon_1}^{x} (x-\xi)^{\alpha-1}\mathbf{f}(\xi)\, d\xi, \quad x > \upsilon_1,$$
$$I^{\alpha}_{\upsilon_2-}\mathbf{f}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\upsilon_2} (\xi-x)^{\alpha-1}\mathbf{f}(\xi)\, d\xi, \quad x < \upsilon_2.$$

Here Γ is the Euler gamma function (see [37]).

Now we present the integral operators what we will use in our work.

Definition 1.4 *Let* $\alpha \geq 0$, *and* $\mathbf{f} \in L_1[\upsilon_1, \upsilon_2]$, *and* $w : [0, +\infty) \to [0, +\infty)$. *The right and left–hand sided Weighted Riemann–Liouville integral are defined as follows:*

$$J^{\alpha,w}_{\upsilon_1+}\mathbf{f}(x) = \frac{1}{\Gamma(\alpha)} \int_{\upsilon_1}^{x} (x-\xi)^{\alpha-1} w(x-\xi)\mathbf{f}(\xi) d\xi, x > \upsilon_1,$$
$$J^{\alpha,w}_{\upsilon_2-}\mathbf{f}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\upsilon_2} (\xi-x)^{\alpha-1} w(\xi-x)\mathbf{f}(\xi) d\xi, x < \upsilon_2.$$

Remark 1.3 Readers can check, without much difficulty, that weighted operators defined above contain, as particular cases, many of the well–known integral operators, for example, if $w \equiv 1$ and $\alpha > 0$ we obtain the Riemann–Liouville fractional integrals, if additionally $\alpha \equiv 1$ we obtain the Riemann classic integral. With $\alpha = 0$ and under certain conditions for the function $w(\xi)$, we obtain the generalized fractional integral introduced by Sarıkaya and Ertuğral in [39].

The aim of this article is to develop and investigate various forms of Simpson-type inequalities for modified $(h, m)-$convex functions using the generalized weighted integral operators defined in Definition 1.4, thereby extending and generalizing the classical Simpson's inequality and unifying existing results reported in the literature. These findings are derived through the convexity property, the classical Hölder inequality, and its variant, the power mean inequality. Our results are broadly applicable due to two main factors: the type of integral operator and the concept of convexity. The integral operator's "weight" enables us to include many known operators, such as the classic Riemann and Riemann-Liouville. Furthermore, with suitable parameter choices, our concept of convexity incorporates several established convexity notions. This allows us to demonstrate that many existing results in the literature are special cases of our findings.

2 Main results

First we define the following auxiliary functions $M(\xi) = \int_0^\xi H_{\alpha,z}(s)ds$ with $H_{\alpha,z}(\xi) = \int_0^\xi u^{\alpha-1}w(zu)du$ and for brevity of writing some mathematical expressions, we will accept the following notation:

$$\mathbf{g}(\alpha,\xi) := M(1) - M(\xi) - \frac{2}{3}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)(1-\xi)$$

and

$$\mathbf{G}(\mathbf{f},\alpha,w) := \frac{\mathbf{f}(\upsilon_1) + 4\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \mathbf{f}(\upsilon_2)}{6} - \frac{2^{\alpha-1}\Gamma(\alpha)}{(\upsilon_2-\upsilon_1)^\alpha H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}\left[J^{\alpha,w}_{\upsilon_2-}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + J^{\alpha,w}_{\upsilon_1+}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right].$$

So we have:

Lemma 2.1 *Let* $\mathbf{f}$ *be a real function defined on some interval* $[\upsilon_1,\upsilon_2] \subset \mathbb{R}$ *and* $\mathbf{f} \in \mathbf{C}^2(\upsilon_1,\upsilon_2)$. *If* $\mathbf{f}'' \in L_1(\upsilon_1,\upsilon_2)$ *and* $w(\xi)$ *is a integrable function on* $[\upsilon_1,\upsilon_2]$, *then we have the following equality:*

$$\mathbf{G}(\mathbf{f},\alpha,w) = \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)} \times \int_0^1 \mathbf{g}(\alpha,\xi)\left[\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2\right) + \mathbf{f}''\left(\frac{1+\xi}{2}\upsilon_1 + \frac{1-\xi}{2}\upsilon_2\right)\right]d\xi. \tag{2.1}$$

Proof. From property of integrals we have

$$\begin{aligned}\mathbf{I} &= \int_0^1 \mathbf{g}(\alpha,\xi)\left[\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2\right) + \mathbf{f}''\left(\frac{1+\xi}{2}\upsilon_1 + \frac{1-\xi}{2}\upsilon_2\right)\right]d\xi \\ &= \int_0^1 \mathbf{g}(\alpha,\xi)\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2\right)d\xi + \int_0^1 \mathbf{g}(\alpha,\xi)\mathbf{f}''\left(\frac{1+\xi}{2}\upsilon_1 + \frac{1-\xi}{2}\upsilon_2\right)d\xi \\ &= \mathbf{I}_1 + \mathbf{I}_2.\end{aligned}$$

For $\mathbf{I}_1$ we have after integrating by parts:

$$\begin{aligned}\mathbf{I}_1 &= -\frac{2\mathbf{g}(\alpha,0)}{\upsilon_2-\upsilon_1}\mathbf{f}'\left(\frac{\upsilon_1+\upsilon_2}{2}\right) \\ &\quad - \frac{2}{\upsilon_2-\upsilon_1}\int_0^1\left(-H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(\xi) + \frac{2}{3}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\right)\mathbf{f}'\left(\frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2\right)d\xi \\ &= -\frac{2}{\upsilon_2-\upsilon_1}\mathbf{g}(\alpha,0)\mathbf{f}'\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \frac{4}{3(\upsilon_2-\upsilon_1)^2}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\left(2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \mathbf{f}(\upsilon_2)\right) \\ &\quad - \frac{4}{(\upsilon_2-\upsilon_1)^2}\int_0^1 \xi^{\alpha-1}w\left(\frac{\upsilon_2-\upsilon_1}{2}\xi\right)\mathbf{f}\left(\frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2\right)d\xi.\end{aligned}$$

By changing the variables $u = \frac{1-\xi}{2}\upsilon_1 + \frac{1+\xi}{2}\upsilon_2$ in the last integral we have

$$\mathbf{I}_1 = -\frac{2\mathbf{g}(\alpha,0)}{\upsilon_2-\upsilon_1}\mathbf{f}'\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \frac{4}{3(\upsilon_2-\upsilon_1)^2}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\left(2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+\mathbf{f}(\upsilon_2)\right)$$
$$-\frac{8}{(\upsilon_2-\upsilon_1)^3}\int_{\frac{\upsilon_1+\upsilon_2}{2}}^{\upsilon_2}\left(\frac{u-\frac{\upsilon_1+\upsilon_2}{2}}{\frac{\upsilon_2-\upsilon_1}{2}}\right)^{\alpha-1} w\left(\frac{\upsilon_2-\upsilon_1}{2}\cdot\frac{u-\frac{\upsilon_2+\upsilon_1}{2}}{\frac{\upsilon_2-\upsilon_1}{2}}\right)\mathbf{f}(u)du.$$

Result that can be written, after a simple algebraic work, in this way:

$$\mathbf{I}_1 = -\frac{2\mathbf{g}(\alpha,0)}{\upsilon_2-\upsilon_1}\mathbf{f}'\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \frac{4}{3(\upsilon_2-\upsilon_1)^2}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\left(2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+\mathbf{f}(\upsilon_2)\right)$$
$$-\left(\frac{2}{\upsilon_2-\upsilon_1}\right)^{2+\alpha}\Gamma(\alpha)J^{\alpha,w}_{\upsilon_2-}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right).$$

Similarly for $\mathbf{I}_2$ we have:

$$\mathbf{I}_2 = \frac{2\mathbf{g}(\alpha,0)}{\upsilon_2-\upsilon_1}\mathbf{f}'\left(\frac{\upsilon_1+\upsilon_2}{2}\right) + \frac{4}{3(\upsilon_2-\upsilon_1)^2}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\left(2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+\mathbf{f}(\upsilon_1)\right)$$
$$-\left(\frac{2}{\upsilon_2-\upsilon_1}\right)^{2+\alpha}\Gamma(\alpha)J^{\alpha,w}_{\upsilon_1+}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right).$$

So

$$\mathbf{I} = \mathbf{I}_1+\mathbf{I}_2 = \frac{4}{3(\upsilon_2-\upsilon_1)^2}H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)\left[\mathbf{f}(\upsilon_2)+4\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+\mathbf{f}(\upsilon_1)\right]$$
$$-\left(\frac{2}{\upsilon_2-\upsilon_1}\right)^{2+\alpha}\Gamma(\alpha)\left[J^{\alpha,w}_{\upsilon_2-}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)+J^{\alpha,w}_{\upsilon_1+}\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right].$$

Multiplying this last result by $\frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}$ we obtain the required equality. This ends the proof.

Remark 2.1 For $h(\xi)=\xi$ and $s=m=1$ we have
1) If $w(z\xi)=e^{-\lambda z\xi}$ and $\alpha>0$, then this Lemma becomes Lemma 2.1 of [12];
2) If $\alpha=0$ and $0<H_{0,z}(\xi)=\int_0^\xi \frac{w(zu)}{u}du<\infty$, then we get Lemma 2 from [43];
3) If we take $w\equiv 1$, then we obtain Lemma 1 of [21].

Theorem 2.1 *Let* $0<m\le 1$, $0\le \upsilon_1<\upsilon_2$ *and* $\mathbf{f}$ *real function defined on the interval* $[\upsilon_1,\upsilon_2]$, *and* $\mathbf{f}\in C^2(\upsilon_1,\upsilon_2)$. *If* $\mathbf{f}\in L_1[\upsilon_1,\upsilon_2]$ *and* $|\mathbf{f}''|\in N^{s,2}_{h,m}[\upsilon_1,\upsilon_2]$ *for some fixed* $s\in(0,1]$, *then the inequality*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \le \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)} \tag{2.2}$$
$$\times\left\{\left(\left|\mathbf{f}''(\upsilon_1)\right|+\left|\mathbf{f}''(\upsilon_2)\right|\right)\int_0^1|\mathbf{g}(\alpha,\xi)|\,h^s\left(\frac{1-\xi}{2}\right)d\xi\right.$$
$$\left.+m\left(\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|+\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\right)\int_0^1|\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right\}$$

holds $\forall\alpha>0$. *Here* $H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)=\int_0^1 u^{\alpha-1}w\left(\frac{\upsilon_2-\upsilon_1}{2}u\right)du.$

Proof. By using the properties of the module from (2.1), we can write

$$\begin{aligned}|\mathbf{G}(\mathbf{f},\alpha,w)| &\leq \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)} \\ &\quad\times\left|\int_0^1 \mathbf{g}(\alpha,\xi)\left[\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1+\frac{1+\xi}{2}\upsilon_2\right)+\mathbf{f}''\left(\frac{1+\xi}{2}\upsilon_1+\frac{1-\xi}{2}\upsilon_2\right)\right]d\xi\right| \\ &\leq \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}\left(|\mathbf{I}_1|+|\mathbf{I}_2|\right)\end{aligned} \tag{2.3}$$

and the fact that $|\mathbf{f}''|$ belongs to the class $N_{h,m}^{s,2}[\upsilon_1,\upsilon_2]$ for the $|\mathbf{I}_1|$,we obtain

$$\begin{aligned}|\mathbf{I}_1| &\leq \int_0^1 |\mathbf{g}(\alpha,\xi)|\left|\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1+\frac{1+\xi}{2}\upsilon_2\right)\right|d\xi \\ &= \int_0^1 |\mathbf{g}(\alpha,\xi)|\left|\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1+m\left(1-\frac{1-\xi}{2}\right)\frac{\upsilon_2}{m}\right)\right|d\xi \\ &\leq \int_0^1 |\mathbf{g}(\alpha,\xi)|\left[\left|\mathbf{f}''(\upsilon_1)\right|h^s\left(\frac{1-\xi}{2}\right)+m\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\right]d\xi \\ &= \left|\mathbf{f}''(\upsilon_1)\right|\int_0^1 |\mathbf{g}(\alpha,\xi)|\,h^s\left(\frac{1-\xi}{2}\right)d\xi \\ &\quad + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\int_0^1|\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi.\end{aligned}$$

Proceeding similarly, for the $|\mathbf{I}_2|$, we can obtain

$$\begin{aligned}|\mathbf{I}_2| &\leq \left|\mathbf{f}''(\upsilon_2)\right|\int_0^1 |\mathbf{g}(\alpha,\xi)|\,h^s\left(\frac{1-\xi}{2}\right)d\xi \\ &\quad + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|\int_0^1|\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi.\end{aligned}$$

Thus, for the $|\mathbf{I}_1|+|\mathbf{I}_2|$, we get

$$\begin{aligned}|\mathbf{I}_1|+|\mathbf{I}_2| &\leq \left(\left|\mathbf{f}''(\upsilon_1)\right|+\left|\mathbf{f}''(\upsilon_2)\right|\right)\int_0^1 |\mathbf{g}(\alpha,\xi)|\,h^s\left(\frac{1-\xi}{2}\right)d\xi \\ &\quad + m\left(\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|+\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\right)\int_0^1|\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi.\end{aligned}$$

Finally, multiplying both sides of the last inequality by $\frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}$ we obtain the required inequality (2.2).

Remark 2.2 For $h(\xi)=\xi$ and $s=m=1$, from (2.2) we have
1) If $w(z\xi)=e^{-\lambda z\xi}$, $\alpha>0$, then we obtain Theorem 3.1 of [12];
2) If we take $w\equiv 1$, then for $\alpha>0$ we obtain Theorem 3 of [21];
3) For $\alpha=0$ and $0<H_{0,z}(\xi)=\int_0^\xi \frac{w(zu)}{u}du<\infty$, we get Theorem 4 from [43].

Corollary 2.1 *If we take* $w \equiv 1$ *and* $h(\xi) = |1-2\xi|$ *with* $\alpha > 0$*, then from Theorem 2.1, for the* $(s,m)-$*convex function, we get*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \leq \frac{(\upsilon_2-\upsilon_1)^2}{24(\alpha+1)} \left\{ \left(\left|\mathbf{f}''(\upsilon_1)\right| + \left|\mathbf{f}''(\upsilon_2)\right|\right) \int_0^1 |\mathbf{g}^*(\alpha,\xi)| \xi^s d\xi \right. \tag{2.4}$$
$$\left. + m\left(\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right| + \left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\right) \int_0^1 |\mathbf{g}^*(\alpha,\xi)| (1-\xi)^s d\xi \right\},$$

where

$$\mathbf{g}^*(\alpha,\xi) = -3\xi^{\alpha+1} + 2(\alpha+1)\xi + (1-2\alpha).$$

Proof. When $w \equiv 1$ for auxiliary functions, we get:

$$H_{\alpha,z}(\xi) = \frac{\xi^\alpha}{\alpha} \text{ and } M(\xi) = \int_0^\xi \frac{s^\alpha}{\alpha} ds = \frac{\xi^{\alpha+1}}{\alpha(\alpha+1)}$$

and

$$\mathbf{g}(\alpha,\xi) = M(1) - M(\xi) - \frac{2}{3}H_{\alpha,z}(1)(1-\xi) = \frac{1}{\alpha(\alpha+1)} - \frac{\xi^{\alpha+1}}{\alpha(\alpha+1)} - \frac{2}{3\alpha}(1-\xi)$$
$$= \frac{-3\xi^{\alpha+1} + 2(\alpha+1)\xi + (1-2\alpha)}{3\alpha(\alpha+1)}.$$

If $h(\xi) = |1-2\xi|$, then $h\left(\frac{1-\xi}{2}\right) = \xi$ and taken account for auxiliary functions, inequality (2.4) obviously follows from (2.2). End of proof**.**

Corollary 2.2 *For the* $\alpha = 1$ *from (2.4), we get*

$$\left| \frac{\upsilon_2-\upsilon_1}{3}\left(\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2} + 2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)\right) - \int_{\upsilon_1}^{\upsilon_2} \mathbf{f}(x)dx \right| \tag{2.5}$$
$$\leq \frac{(\upsilon_2-\upsilon_1)^3}{48} \left\{ \frac{|\mathbf{f}''(\upsilon_1)| + |\mathbf{f}''(\upsilon_2)|}{3^{s+2}} \left(\frac{2-3^{3+s}}{s+3} - \frac{4\left(2-3^{2+s}\right)}{s+2} + \frac{3\left(2-3^{1+s}\right)}{s+1} \right) \right.$$
$$\left. + \frac{m\left(\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right| + \left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|\right)}{(2+s)(3+s)} \left(s + 4\left(\frac{2}{3}\right)^{2+s} \right) \right\}.$$

Proof. When $\alpha = 1$ for the integrals of right side of (2.4), we obtain

$$\int_0^1 |\mathbf{g}^*(\alpha,\xi)| h^s\left(\frac{1-\xi}{2}\right) d\xi = \int_0^1 |\mathbf{g}^*(1,\xi)| \xi^s d\xi = \int_0^1 \left|3\xi^2 - 4\xi + 1\right| \xi^s d\xi$$
$$= \int_0^{\frac{1}{3}} \left(3\xi^{2+s} - 4\xi^{1+s} + \xi^s\right) d\xi + \int_{\frac{1}{3}}^1 \left(-3\xi^{2+s} + 4\xi^{1+s} - \xi^s\right) d\xi$$
$$= \frac{1}{3^{2+s}(s+3)} - \frac{4}{3^{s+2}(s+2)} + \frac{1}{3^{s+1}(s+1)}$$
$$+ \frac{-3^{3+s}+1}{3^{2+s}(3+s)} + \frac{4\left(3^{2+s}-1\right)}{3^{2+s}(2+s)} - \frac{3^{1+s}-1}{3^{1+s}(1+s)}$$

Thus for the first integral from (2.4) we obtain

$$\int_0^1 |\mathbf{g}^*(\alpha,\xi)| h^s\left(\frac{1-\xi}{2}\right)d\xi \tag{2.6}$$
$$= \frac{1}{3^{s+2}}\left(\frac{2-3^{3+s}}{s+3} - \frac{4\left(2-3^{2+s}\right)}{s+2} + \frac{3\left(2-3^{1+s}\right)}{s+1}\right).$$

For the second integral, we have

$$\int_0^1 |\mathbf{g}^*(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi = \int_0^1 |\mathbf{g}^*(1,\xi)|\,(1-\xi)^s\,d\xi$$
$$= \int_0^1 \left|3\xi^2-4\xi+1\right|(1-\xi)^s\,d\xi = \int_0^1 |(\xi-1)(3\xi-1)|\,(1-\xi)^s\,d\xi$$
$$= \int_0^{\frac{1}{3}}\left[-(1-\xi)^{s+1}(3\xi-1)\right]d\xi + \int_{\frac{1}{3}}^1 (1-\xi)^{s+1}(3\xi-1)\,d\xi.$$

Let's calculate each of the integrals.

$$\int_0^{\frac{1}{3}}\left[-(1-\xi)^{s+1}(3\xi-1)\right]d\xi = -3\int_0^{\frac{1}{3}}(1-\xi)^{s+1}\xi d\xi + \int_0^{\frac{1}{3}}(1-\xi)^{s+1}d\xi \tag{2.7}$$
$$= -3\int_{\frac{2}{3}}^1 (1-z)\,z^{s+1}d\xi + \frac{1}{2+s}\left(1-\left(\frac{2}{3}\right)^{s+2}\right)$$
$$= -\frac{2}{s+2}\left(1-\left(\frac{2}{3}\right)^{s+2}\right) + \frac{3}{3+s}\left(1-\left(\frac{2}{3}\right)^{3+s}\right)$$

and

$$\int_{\frac{1}{3}}^1 (1-\xi)^{s+1}(3\xi-1)\,d\xi = 3\int_{\frac{1}{3}}^1 (1-\xi)^{s+1}\xi d\xi - \int_{\frac{1}{3}}^1 (1-\xi)^{s+1}d\xi, \tag{2.8}$$
$$= 3\int_0^{\frac{2}{3}} x^{s+1}(1-x)\,dx - \int_0^{\frac{2}{3}} x^{s+1}dx = -3\int_0^{\frac{2}{3}} x^{s+2}dx + 2\int_0^{\frac{2}{3}} x^{s+1}dx$$
$$= 2\left(\frac{2}{3}\right)^{2+s}\left[-\frac{1}{3+s}+\frac{1}{2+s}\right] = \frac{2}{(2+s)(3+s)}.\left(\frac{2}{3}\right)^{2+s}$$

By adding the results of integration's from (2.7) and (2.8), we obtain

$$\int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s\,d\xi \tag{2.9}$$
$$= -\frac{2}{s+2}\left(1-\left(\frac{2}{3}\right)^{s+2}\right) + \frac{3}{3+s}\left(1-\left(\frac{2}{3}\right)^{3+s}\right) + \frac{2}{(2+s)(3+s)}\left(\frac{2}{3}\right)^{2+s}$$
$$= \frac{1}{(2+s)(3+s)}\left(s+4\left(\frac{2}{3}\right)^{2+s}\right)$$

Inequality (2.5) follows from (2.6) and (2.9). End of proof.

Remark 2.3 For the $m = s = 1$ from (2.5), we get

$$\frac{\upsilon_2-\upsilon_1}{3}\left|\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2}+2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)-\int_{\upsilon_1}^{\upsilon_2}\mathbf{f}(x)dx\right|$$
$$\leq \frac{(\upsilon_2-\upsilon_1)^3}{81}\cdot\frac{|\mathbf{f}''(\upsilon_1)|+|\mathbf{f}''(\upsilon_2)|}{2}. \tag{2.10}$$

This estimate is available in the literature (see, for example, [21] or Remark 3.1 to [12]).

Theorem 2.2 *Let* $0 < m \leq 1$, $0 \leq \upsilon_1 < \upsilon_2$ *and* $\mathbf{f}$ *real function defined on the interval* $[\upsilon_1, \upsilon_2]$, *and* $\mathbf{f} \in C^2(\upsilon_1, \upsilon_2)$. *If* $\mathbf{f} \in L_1[\upsilon_1, \upsilon_2]$ *and* $|\mathbf{f}''|^q \in N^{s,2}_{h,m}[\upsilon_1, \upsilon_2]$ *with* $q > 1$ *and* $\alpha > 0$ *for some fixed* $s \in (0, 1]$, *then the inequality*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \leq \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}\left(\int_0^1|\mathbf{g}(\alpha,\xi)|^p\,d\xi\right)^{\frac{1}{p}}\left((\mathbf{H}_1)^{\frac{1}{q}}+(\mathbf{H}_2)^{\frac{1}{q}}\right) \tag{2.11}$$

holds $\forall\alpha > 0, \frac{1}{p}+\frac{1}{q} = 1$. *Here* $H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1) = \int_0^1 u^{\alpha-1}w\left(\frac{\upsilon_2-\upsilon_1}{2}u\right)du$,

$$\mathbf{g}(\alpha,\xi) = \frac{-3\xi^{\alpha+1}+2(\alpha+1)\xi+(1-2\alpha)}{3\alpha(\alpha+1)},$$
$$\mathbf{H}_1 = \left|\mathbf{f}''(\upsilon_1)\right|^q\int_0^1 h^s\left(\frac{1-\xi}{2}\right)d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi,$$
$$\mathbf{H}_2 = \left|\mathbf{f}''(\upsilon_2)\right|^q\int_0^1 h^s\left(\frac{1-\xi}{2}\right)d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi.$$

Proof. By using Hölder inequality and the fact that $|\mathbf{f}''|^q$ belongs to the class $N^{s,2}_{h,m}[\upsilon_1, \upsilon_2]$, for the $|\mathbf{I}_1|$, from (2.3), we obtain

$$|\mathbf{I}_1| \leq \left(\int_0^1|\mathbf{g}(\alpha,\xi)|^p\,d\xi\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1+\frac{1+\xi}{2}\upsilon_2\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1|\mathbf{g}(\alpha,\xi)|^p\,d\xi\right)^{\frac{1}{p}}$$
$$\times\left[\left|\mathbf{f}''(\upsilon_1)\right|^q\int_0^1 h^s\left(\frac{1-\xi}{2}\right)d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}$$

Similarly, for the $|\mathbf{I}_2|$, we can obtain

$$|\mathbf{I}_2| \leq \left(\int_0^1|\mathbf{g}(\alpha,\xi)|^p\,d\xi\right)^{\frac{1}{p}}$$
$$\times\left[\left|\mathbf{f}''(\upsilon_2)\right|^q\int_0^1 h^s\left(\frac{1-\xi}{2}\right)d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}.$$

Thus, for the $|\mathbf{I}_1| + |\mathbf{I}_2|$, we get

$$|\mathbf{I}_1| + |\mathbf{I}_2| \le \left(\int_0^1 |\mathbf{g}(\alpha,\xi)|^p \, d\xi\right)^{\frac{1}{p}}$$
$$\times \left\{ \left(\left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 h^s\left(\frac{1-\xi}{2}\right) d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 \left(1 - h\left(\frac{1-\xi}{2}\right)\right)^s d\xi \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 h^s\left(\frac{1-\xi}{2}\right) d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 \left(1 - h\left(\frac{1-\xi}{2}\right)\right)^s d\xi \right)^{\frac{1}{q}} \right\}.$$

Finally, multiplying both sides of the last inequality by $\frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}$ and taking into account the adopted notation we obtain the required inequality (2.11).

Remark 2.4 For $h(\xi) = \xi$ and $s = m = 1$, from (2.11) we have
1) If $w(z\xi) = e^{-\lambda z \xi}$, $\alpha > 0$, then we obtain Theorem 3.2 of [12];
2) If we take $w \equiv 1$, then for $\alpha > 0$ we obtain Theorem 4 of [21];
3) For $\alpha = 0$ and $0 < H_{0,z}(\xi) = \int_0^\xi \frac{w(zu)}{u} du < \infty$, we get Theorem 5 from [43].

Corollary 2.3 *If we take $h(\xi) = |1-2\xi|$ and $w \equiv 1$, then from Theorem 2.2, we get*
$\mathbf{1}$. *for the $(s,m)-$convex function:*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \le \frac{(\upsilon_2-\upsilon_1)^2}{24(\alpha+1)} \left(\int_0^1 |\mathbf{g}^*(\alpha,\xi)|^p \, d\xi\right)^{\frac{1}{p}} \left((\mathbf{H}_1^*)^{\frac{1}{q}} + (\mathbf{H}_2^*)^{\frac{1}{q}} \right), \tag{2.12}$$

where

$$\mathbf{g}^*(\alpha,\xi) = -3\xi^{\alpha+1} + 2(\alpha+1)\xi + (1-2\alpha),$$
$$\mathbf{H}_1^* = \frac{|\mathbf{f}''(\upsilon_1)|^q + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q}{s+1} \quad \textit{and} \quad \mathbf{H}_2^* = \frac{|\mathbf{f}''(\upsilon_2)|^q + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q}{s+1},$$

$\mathbf{2}$. *For the $\alpha = 1$ from (2.12):*

$$\frac{\upsilon_2-\upsilon_1}{3}\left|\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2} + 2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) - \int_{\upsilon_1}^{\upsilon_2} \mathbf{f}(x)dx\right| \tag{2.13}$$
$$\le \frac{(\upsilon_2-\upsilon_1)^3}{48}\left(\int_0^1 \left|3\xi^2 - 4\xi + 1\right|^p d\xi\right)^{\frac{1}{p}} \left((\mathbf{H}_1^*)^{\frac{1}{q}} + (\mathbf{H}_2^*)^{\frac{1}{q}} \right),$$

$\mathbf{3}$. *For the $m = s = 1$ from (2.13):*

$$\frac{\upsilon_2-\upsilon_1}{3}\left|\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2} + 2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) - \int_{\upsilon_1}^{\upsilon_2} \mathbf{f}(x)dx\right| \tag{2.14}$$
$$\le \frac{(\upsilon_2-\upsilon_1)^3}{48}\left(\int_0^1 \left|3\xi^2 - 4\xi + 1\right|^p d\xi\right)^{\frac{1}{p}} \left(\left|\mathbf{f}''(\upsilon_1)\right|^q + \left|\mathbf{f}''(\upsilon_2)\right|^q \right)^{\frac{1}{q}}.$$

Proof. $\mathbf{1}$. For $w \equiv 1$, from Corollary 2.1, we have auxiliary functions:

$$H_{\alpha,z}(\xi) = \frac{\xi^\alpha}{\alpha}, \quad M(\xi) = \frac{\xi^{\alpha+1}}{\alpha(\alpha+1)} \text{ and } \mathbf{g}(\alpha,\xi) = \frac{-3\xi^{\alpha+1} + 2(\alpha+1)\xi + (1-2\alpha)}{3\alpha(\alpha+1)}.$$

Taking into account these functions with $h(\xi) = |1-2\xi|$, inequality (2.12) obviously follows from (2.11);

2. For $\alpha = 1$ we get $\mathbf{g}^*(1,\xi) = -3\xi^2 + 4\xi - 1$ and hence (2.13);

3. When $m = s = 1$ since $\mathbf{H}_1^* = \mathbf{H}_2^*$, then for right–side (2.13), we get

$$\frac{(\upsilon_2-\upsilon_1)^2}{48}\left(\int_0^1 \left|3\xi^2-4\xi+1\right|^p d\xi\right)^{\frac{1}{p}}\left((\mathbf{H}_1^*)^{\frac{1}{q}}+(\mathbf{H}_2^*)^{\frac{1}{q}}\right)$$
$$=\frac{(\upsilon_2-\upsilon_1)^2}{48}\left(\int_0^1 \left|3\xi^2-4\xi+1\right|^p d\xi\right)^{\frac{1}{p}}\left(\left|\mathbf{f}''(\upsilon_1)\right|^q+\left|\mathbf{f}''(\upsilon_2)\right|^q\right)^{\frac{1}{q}}.$$

End of proof.

Theorem 2.3 *Let* $0 < m \leq 1$, $0 \leq \upsilon_1 < \upsilon_2$ *and* $\mathbf{f}$ *real function defined on the interval* $[\upsilon_1,\upsilon_2]$, *and* $\mathbf{f} \in C^2(\upsilon_1,\upsilon_2)$. *If* $\mathbf{f} \in L_1[\upsilon_1,\upsilon_2]$ *and* $|\mathbf{f}''|^q \in N_{h,m}^{s,2}[\upsilon_1,\upsilon_2]$ *with* $q \geq 1$ *and* $\alpha > 0$ *for some fixed* $s \in (0,1]$, *then the inequality*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \leq \frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}\left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\,d\xi\right)^{1-\frac{1}{q}}\left((\mathbf{H}_3)^{\frac{1}{q}}+(\mathbf{H}_4)^{\frac{1}{q}}\right) \qquad (2.15)$$

holds $\forall\alpha > 0$, $\frac{1}{p}+\frac{1}{q}=1$. *Here* $H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1) = \int_0^1 u^{\alpha-1} w\left(\frac{\upsilon_2-\upsilon_1}{2}u\right) du$,

$$\mathbf{H}_3 = \left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi$$
$$+\, m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi,$$
$$\mathbf{H}_4 = \left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi$$
$$+\, m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi.$$

Proof. By using power mean inequality and the fact that $|\mathbf{f}''|^q$ belongs to the class $N_{h,m}^{s,2}[\upsilon_1,\upsilon_2]$, for the $|\mathbf{I}_1|$, from (2.3), we obtain

$$|\mathbf{I}_1| \leq \left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\,d\xi\right)^{1-\frac{1}{q}}\left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\left|\mathbf{f}''\left(\frac{1-\xi}{2}\upsilon_1+\frac{1+\xi}{2}\upsilon_2\right)\right|^q d\xi\right)^{\frac{1}{q}}$$
$$\leq \left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\,d\xi\right)^{1-\frac{1}{q}}\left[\left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi\right.$$
$$\left.+m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}.$$

Similarly, for the $|\mathbf{I}_2|$, we can obtain

$$|\mathbf{I}_2| \leq \left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\,d\xi\right)^{1-\frac{1}{q}}\left[\left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi\right.$$
$$\left.+m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}.$$

Thus, for the $|\mathbf{I}_1| + |\mathbf{I}_2|$, we get

$$\begin{aligned}|\mathbf{I}_1| + |\mathbf{I}_2| \leq \left(\int_0^1 |\mathbf{g}(\alpha,\xi)|\, d\xi\right)^{1-\frac{1}{q}} &\left\{\left[\left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi\right.\right.\\
&\left. +m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)| \left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}\\
&+\left[\left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi\right.\\
&\left.\left. +m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)| \left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

And, by multiplying both sides of the last inequality by $\frac{(\upsilon_2-\upsilon_1)^2}{8H_{\alpha,\frac{\upsilon_2-\upsilon_1}{2}}(1)}$, and taking into account the adopted notation we obtain the required inequality (2.15).

Remark 2.5 For $h(\xi)=\xi$ and $s=m=1$, from (2.15) we have

1) If $w(z\xi)=e^{-\lambda z\xi}$, $\alpha>0$, then we obtain Theorem 3.3 of [12];

2) If we take $w\equiv 1$, then for $\alpha>0$ we obtain Theorem 5 of [21];

3) For $\alpha=0$ and $0<H_{0,z}(\xi)=\int_0^\xi \frac{w(zu)}{u}du<\infty$, we get Theorem 6 from [43].

Corollary 2.4 *If we take $h(\xi)=|1-2\xi|$ and $w\equiv 1$, then from Theorem 2.3 for the $(s,m)-$convex function, we get*

$$|\mathbf{G}(\mathbf{f},\alpha,w)| \leq \frac{(\upsilon_2-\upsilon_1)^2}{24(\alpha+1)}\left(\int_0^1 |\mathbf{g}^*(\alpha,\xi)|\, d\xi\right)^{1-\frac{1}{q}}\left[(\mathbf{H}_3^*)^{\frac{1}{q}}+(\mathbf{H}_4^*)^{\frac{1}{q}}\right], \qquad (2.16)$$

where

$$\begin{aligned}\mathbf{g}^*(\alpha,\xi) &= -3\xi^{\alpha+1}+2(\alpha+1)\xi+(1-2\alpha),\\
\mathbf{H}_3^* &= \left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,\xi^s d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s\, d\xi,\\
\mathbf{H}_4^* &= \left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,\xi^s d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s\, d\xi.\end{aligned}$$

Proof. When $w\equiv 1$ for auxiliary functions from Corollary 2.1 we have:

$$H_{\alpha,z}(\xi)=\frac{\xi^\alpha}{\alpha},\quad M(\xi)=\frac{\xi^{\alpha+1}}{\alpha(\alpha+1)} \text{ and } \mathbf{g}(\alpha,\xi)=\frac{-3\xi^{\alpha+1}+2(\alpha+1)\xi+(1-2\alpha)}{3\alpha(\alpha+1)}$$

If $h(\xi)=|1-2\xi|$, then $h\left(\frac{1-\xi}{2}\right)=\xi$ and taken account for auxiliary functions, for the right side of (2.15), we get

$$\begin{aligned}
\mathbf{H}_3 &= \left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)|\, h^s\left(\frac{1-\xi}{2}\right) d\xi \\
&\qquad + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}(\alpha,\xi)| \left(1-h\left(\frac{1-\xi}{2}\right)\right)^s d\xi \\
&= \left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 \frac{|\mathbf{g}^*(\alpha,\xi)|}{3\alpha(\alpha+1)}\xi^s d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 \frac{|\mathbf{g}^*(\alpha,\xi)|}{3\alpha(\alpha+1)}(1-\xi)^s d\xi \\
&= \frac{1}{3\alpha(\alpha+1)}\left[\left|\mathbf{f}''(\upsilon_1)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,\xi^s d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s d\xi\right] \\
&= \frac{1}{3\alpha(\alpha+1)}\cdot \mathbf{H}_3^*
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{H}_4 &= \frac{1}{3\alpha(\alpha+1)}\left[\left|\mathbf{f}''(\upsilon_2)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,\xi^s d\xi + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s d\xi\right] \\
&= \frac{1}{3\alpha(\alpha+1)}\cdot \mathbf{H}_4^*.
\end{aligned}$$

Thus

$$\left((\mathbf{H}_3)^{\frac{1}{q}}+(\mathbf{H}_4)^{\frac{1}{q}}\right)=\left[\frac{1}{3\alpha(\alpha+1)}\right]^{\frac{1}{q}}\left((\mathbf{H}_3^*)^{\frac{1}{q}}+(\mathbf{H}_4^*)^{\frac{1}{q}}\right).$$

End of proof.

Corollary 2.5 *For the $\alpha=1$ from (2.16), we obtain*

$$\begin{aligned}
&\frac{\upsilon_2-\upsilon_1}{3}\left|\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2}+2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right)-\int_{\upsilon_1}^{\upsilon_2}\mathbf{f}(x)dx\right| \qquad (2.17)\\
&\leq \frac{(\upsilon_2-\upsilon_1)^3}{48}\left(\frac{8}{27}\right)^{1-\frac{1}{q}}\left((\mathbf{H}_5^*)^{\frac{1}{q}}+(\mathbf{H}_6^*)^{\frac{1}{q}}\right),
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{H}_5^* &= \frac{\left|\mathbf{f}''(\upsilon_1)\right|^q}{3^{s+2}}\left(\frac{2-3^{3+s}}{s+3}-\frac{4\left(2-3^{2+s}\right)}{s+2}+\frac{3\left(2-3^{1+s}\right)}{s+1}\right) \\
&\qquad + m\left|\mathbf{f}''\left(\frac{\upsilon_2}{m}\right)\right|^q \frac{\left(s+4\left(\frac{2}{3}\right)^{2+s}\right)}{(2+s)(3+s)}, \\
\mathbf{H}_6^* &= \frac{\left|\mathbf{f}''(\upsilon_2)\right|^q}{3^{s+2}}\left(\frac{2-3^{3+s}}{s+3}-\frac{4\left(2-3^{2+s}\right)}{s+2}+\frac{3\left(2-3^{1+s}\right)}{s+1}\right) \\
&\qquad + m\left|\mathbf{f}''\left(\frac{\upsilon_1}{m}\right)\right|^q \frac{\left(s+4\left(\frac{2}{3}\right)^{2+s}\right)}{(2+s)(3+s)}.
\end{aligned}$$

Proof. Indeed, for $\alpha = 1$ we have

a) $\left(\int_0^1 |\mathbf{g}^*(\alpha,\xi)|\, d\xi\right)^{1-\frac{1}{q}} = \left(\int_0^1 \left|-3\xi^2+4\xi-1\right| d\xi\right)^{1-\frac{1}{q}} = \left(\frac{8}{27}\right)^{1-\frac{1}{q}}$,

b) The integrals below were calculated above in Corollary 2.1:

$$1)\int_0^1 |\mathbf{g}^*(\alpha,\xi)| h^s\left(\frac{1-\xi}{2}\right) d\xi = \int_0^1 \left|3\xi^2-4\xi+1\right| \xi^s d\xi$$
$$= \int_0^{\frac{1}{3}} \left(3\xi^2-4\xi+1\right)\xi^s d\xi + \int_{\frac{1}{3}}^1 \left(-3\xi^2+4\xi-1\right)\xi^s d\xi$$
$$= \frac{1}{3^{s+2}}\left(\frac{2-3^{3+s}}{s+3} - \frac{4\left(2-3^{2+s}\right)}{s+2} + \frac{3\left(2-3^{1+s}\right)}{s+1}\right),$$

$$2)\int_0^1 |\mathbf{g}^*(\alpha,\xi)|\,(1-\xi)^s\, d\xi = \int_0^1 \left|3\xi^2-4\xi+1\right| (1-\xi)^s\, d\xi$$
$$= \int_0^{\frac{1}{3}} \left(3\xi^2-4\xi+1\right)(1-\xi)^s\, d\xi + \int_{\frac{1}{3}}^1 \left(-3\xi^2+4\xi-1\right)(1-\xi)^s\, d\xi$$
$$= \frac{1}{(2+s)(3+s)}\left(s+4\left(\frac{2}{3}\right)^{2+s}\right).$$

With the above in mind, the proof is complete.

Remark 2.6 For the $m = s = 1$ from (2.17), we get

$$\frac{\upsilon_2-\upsilon_1}{3}\left|\frac{\mathbf{f}(\upsilon_1)+\mathbf{f}(\upsilon_2)}{2} + 2\mathbf{f}\left(\frac{\upsilon_1+\upsilon_2}{2}\right) - \int_{\upsilon_1}^{\upsilon_2}\mathbf{f}(x)dx\right| \tag{2.18}$$
$$\le \frac{(\upsilon_2-\upsilon_1)^3}{162\cdot 8^{\frac{1}{q}}}\left((\mathbf{H}_7^*)^{\frac{1}{q}} + (\mathbf{H}_8^*)^{\frac{1}{q}}\right),$$

where

$$\mathbf{H}_7^* = \frac{37\left|\mathbf{f}''(\upsilon_1)\right|^q + 59\left|\mathbf{f}''(\upsilon_2)\right|^q}{12} \quad \text{and} \quad \mathbf{H}_8^* = \frac{37\left|\mathbf{f}''(\upsilon_2)\right|^q + 59\left|\mathbf{f}''(\upsilon_1)\right|^q}{12}.$$

Remark 2.7 For $q = 1$, from (2.18) we have (2.10).

3 Conclusions

In this paper, various extensions and generalizations of the classical Simpson's inequality have been established, in the context of weighted integral operators. Throughout our work, we have seen how various results reported in the literature are particular cases of ours, which shows the breadth of strength of these. However, we did not want to conclude without pointing out two more aspects regarding the breadth of our results. Firstly, referring to the integral operator used, given that the weight function can include several known cases, we can add that if $w(z\xi) = e^{-\lambda z\xi}$ and $\alpha > 0$, then Lemma 2.1 becomes Lemma 2.1 of [12]. If $\alpha = 0$ and $0 < H_{0,z}(\xi) = \int_0^{\xi}\frac{w(zu)}{u}du < \infty$, then we get Lemma 2 from [43]. If we take $w \equiv 1$, then we obtain Lemma 1 of [21]. Evidently, a significant portion of the conclusions from their research can similarly be derived from ours, taking into consideration convex functions. Another important point is the concept of convexity used, which includes a number of well-known convexity classes. This means that our results cover most of the results published so far.

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