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A note on singular integrals along higher dimensional subvarieties

Ahmad Al-Salman⋆**, Badriya Al-Azri**

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Abstract. Suppose that $\Omega \in L^1(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ is a homogeneous function of degree zero in the sense (1.5) and satisfying the cancellation property (1.4). Under certain convexity assumptions on the mapping $\theta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, we prove that the singular integral operator

$$T_{\theta,\Omega} f(x, x_{n+1}) = p.v. \int_{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}} f(x - u - v, x_{n+1} - \theta(|u|, |v|)) \frac{\Omega(u', v')}{|u|^n\, |v|^n}\, dudv$$

is bounded on $L^p(\mathbb{R}^n \times \mathbb{R}^n), 1 < p < \infty$ provided the kernel function Ω is in $L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$.

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1 Introduction and Statement of Results

For $n \geq 2$, let $\mathbb{S}^{n-1}$ be the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $d\sigma$ be the induced normalized Lebesgue measure on $\mathbb{S}^{n-1}$. For $y \neq 0$, let $y' = |y|^{-1} y \in \mathbb{S}^{n-1}$ and $\Omega \in L^1(\mathbb{S}^{n-1})$ be a homogeneous function of degree zero on $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(y') d\sigma(y') = 0. \tag{1.1}$$

Let $\phi : [0, \infty) \to \mathbb{R}$ be a smooth function. Consider the singular integral operator $S_{\varphi,\Omega}$ given by

$$S_{\varphi,\Omega} f(x, x_{n+1}) = \text{p.v.} \int_{\mathbb{R}^{n+1}} f(x - y, x_{n+1} - \phi(|y|)) \frac{\Omega(y')}{|y|^n} dy. \tag{1.2}$$

⋆ Corresponding author

Ahmad Al-Salman
Sultan Qaboos University, College of Science, Department Mathematics, Muscat, Sultanate of Oman
Department of Mathematics, Yarmouk University, Irbid, Jordan
E-mail: alsalman@squ.edu.om, alsalman@yu.edu.jo

Badriya Al-Azri
Sultan Qaboos University, College of Science, Department Mathematics, Muscat, Sultanate of Oman
E-mail: ab8500703@gmail.com

If $\phi(t) \equiv 0$, then the operator $S_{\varphi,\Omega}$ is the well known classical Calderón-Zygmund singular integral operator S_Ω given by

$$S_\Omega f(x) = \text{p.v.} \int\limits_{\mathbb{R}^n} f(x-y)\frac{\Omega(y')}{|y|^n}dy. \tag{1.3}$$

In their fundamental papers [9] and [8], Calderón and Zygmund proved that the operator S_Ω is bounded on L^p $(1 < p < \infty)$ provided that $\Omega \in L(\log L)(\mathbb{S}^{n-1})$. Moreover, it was shown in [8] that $L(\log L)(\mathbb{S}^{n-1})$ is the most desirable size condition in the sense that S_Ω can fail to boundedness on L^p if Ω is assumed to be in $L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1})\backslash L(\log L)(\mathbb{S}^{n-1})$ for some $\varepsilon > 0$. For further results concerning the operator S_Ω, we cite, among others, [1], [2], [6], [7], [10], [11], [12], [15], [16], [17] and references to their in.

In 1996, Kim, Winger, Wright and Ziesler [13] studied the $L^p(\mathbb{R}^{n+1})$ boundedness of $S_{\varphi,\Omega}$ when $\Omega \in C^\infty(\mathbb{S}^{n-1})$ for $(1 < p < \infty)$. In [7], Al-Salman and Pan established $L^p(\mathbb{R}^{n+1})$ $(1 < p < \infty)$ the boundedness of $S_{\varphi,\Omega}$ under the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1})$. For more results on this topic, we advise readers to consult [14], [11], [12], among other.

Let $\theta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a smooth mapping. Suppose that $\Omega \in L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ is satisfied

$$\int_{\mathbb{S}^{n-1}} \Omega(u',.)d\sigma(u') = \int_{\mathbb{S}^{n-1}} \Omega(.,v')d\sigma(v') = 0 \tag{1.4}$$

and

$$\Omega(tu, sv) = \Omega(u, v) \tag{1.5}$$

for any $t, s > 0$. Consider the singular integral operator

$$T_{\theta,\Omega} f(x, x_{n+1}) = p.v. \int_{\mathbb{R}^{n+1}\times\mathbb{R}^{n+1}} f(x-u-v, x_{n+1}-\theta(|u|,|v|))\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,dudv. \tag{1.6}$$

In order to state our results in this paper, we cite the following remarks:

(i) When $\theta = 0$, then the corresponding operator $T_{\theta,\Omega}$ reduces to the operator

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n\times\mathbb{R}^n} f(x-u-v)\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,dudv; \tag{1.7}$$

which was introduced in [3]. In [3], Al-Salman proved that the operator T_Ω is bounded on $L^p(\mathbb{R}^n), 1 < p < \infty$, provided that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$, i.e.,

$$\int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} |\Omega(u,v)|(\log\,2 + |\Omega(u,v)|)^2\,d\sigma(u)\,d\sigma(v) < \infty. \tag{1.8}$$

It is worth pointing that,

$$L(\log^+ L)^s(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}) \subset L(\log^+ L)^r(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}) \quad \text{whenever} \quad r < s$$

and

$$L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}) \subsetneqq L(\log^+ L)^r(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}) \subsetneqq L^1(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$$

whenever $q > 1$ and $r \geq 1$.In addition, it was pointed out in [3] that the condition $\Omega \in L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ is nearly optimal. Namely, there exists an Ω in $L(\log L)^{2-\varepsilon}(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ for some $\varepsilon > 0$ such that $T_{\gamma,\Omega}$ is not bounded on $L^p(\mathbb{R}^n)$.

(ii) When θ is separable in the sense that $\theta(s,t) = \varphi_1(s)+\varphi_2(t)$ and $\Omega(x,y) = \Omega_1(x)\,\Omega_2(y)$ for some $\Omega_1, \Omega_2 \in L^1(\mathbb{S}^{n-1})$ where φ_1 and φ_2 are suitable real valued functions, then the special operator $T_{\varphi_1,\varphi_2,\Omega} = T_{\theta,\Omega}$ is a composition of two singular integral operators. Namely,

$$T_{\varphi_1,\varphi_2,\Omega}(f)(x, x_{n+1}) = S_{\varphi_1,\Omega_1} \circ S_{\varphi_2,\Omega_2}(f)(x, x_{n+1}) \tag{1.9}$$

where S_{φ_1,Ω_1} is the operator given by (1.2) with φ replaced by φ_1 and Ω replaced by Ω_1. Similarly, the operator S_{φ_2,Ω_2}.

(iii) By Theorem 1.2 in [7], it follows that if $\Omega_1,\ \Omega_2 \in L(\log L)(\mathbb{S}^{n-1})$ and φ_1 and φ_2 are $\mathcal{C}^2$, convex, and increasing functions satisfying $\varphi_1(0) = \varphi_2(0) = 0$, then the operators S_{φ_1,Ω_1} and S_{φ_2,Ω_2} are bounded on $L^p(\mathbb{R}^{n+1})$ for all $p \in (1, \infty)$. Hence, by this and (1.9), we deduce that the operator $T_{\varphi_1,\varphi_2,\Omega}$ in (1.9) is bounded on L^p for all $1 < p < \infty$.

(iv) For general Ω and θ, it is not visible if the operator $T_{\theta,\Omega}$ can be written as a composition of two operators of convolution type.

In light of above remarks, it is natural to investigate the L^p boundedness of the operator $T_{\theta,\Omega}$ for various functions θ under the condition $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Our results are the following:

Theorem 1.1. *Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ satisfies (1.4)-(1.5). If $\theta(s,t) = \varphi_1(s) + \varphi_2(t)$ where φ_1 and φ_2 are $\mathcal{C}^2$, convex, and increasing functions satisfying $\varphi_1(0) = \varphi_2(0) = 0$, then $T_{\theta,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$.*

Theorem 1.2. *Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ satisfies (1.4)-(1.5). If $\theta(s,t) = \varphi_1(s) + \varphi_2(t)$ where (i) φ_1 is $\mathcal{C}^2$, convex, and increasing function with $\varphi_1(0) = 0$ and (ii) φ_2 is a polynomial mapping with $\varphi_2(0) = 0$, then $T_{\theta,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$. The L^p may depend on the degree of the polynomial φ_2 but it is independent of its coefficients.*

Theorem 1.3. *Suppose that $\Omega \in L(\log L)^2(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ satisfies (1.4)-(1.5). If $\theta(s,t) = \varphi_1(s) + \varphi_2(t)$ where φ_1 and φ_2 are polynomial mappings with $\varphi_1(0) = \varphi_2(0) = 0$, then $T_{\theta,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for $1 < p < \infty$. The L^p may depend on the degrees of the polynomials φ_1 and φ_2 but it is independent of their coefficients.*

Throughout this paper, the letter C will stand for a constant that may vary at each occurrence but it is independent of the essential variables.

2 Preliminary Lemmas

For $j \in \mathbb{Z}$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ for some $q > 1$ and satisfying (1.4)-(1.5), let $\alpha(q,\Omega) = \log_2(e + \|\Omega\|_q)$ and let $I_{j,q}$ be the interval $[2^{\alpha(q,\Omega)j}, 2^{\alpha(q,\Omega)(j+1)})$ in $\mathbb{R}$. Let $\theta(t,s) = \varphi_1(s) + \varphi_2(t)$ where $\varphi_1,\, \varphi_2 : \mathbb{R} \to \mathbb{R}$ are suitable real valued functions. Define the sequence $\{\mathbb{M}_{q,\theta_{,j,k}} : j, k \in \mathbb{Z}\}$ of multipliers on $\mathbb{R}^{n+1}$ by

$$\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta) = \int\int_{(|u|,|v|)\in I_{j,q}\times I_{k,q}} e^{-i((u+v)\cdot\xi + \theta(|u|,|v|)\eta)} \frac{\Omega(u',v')}{|u|^n\,|v|^n}\, du\, dv \tag{2.1}$$

where $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$.

For $j \in \mathbb{Z}$ and $\Omega \in L^q(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$ for some $q > 1$, we let $e_{j,q} : \mathbb{R} \to \mathbb{C}$ be the mapping defined by

$$e_{j,q}(t) = e^{-i2^{\alpha(q,\Omega)j}t}. \tag{2.2}$$

Now, we have the following lemma:

Lemma 2.1. *Let $\mathbb{M}_{q,\varphi_1,\varphi_{2,j,k}}$ be as in (2.1). Let*

$$C_{q,\Omega} = (\alpha(q,\Omega))^2 \left\|\Omega\right\|_1^{2-\frac{2}{\alpha(q,\Omega)}}$$

and let θ, φ_1, and $\varphi_2(t)$ be as above. Then

$$\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \leq \left|2^{\alpha(q,\Omega)j}\xi\right|^{-\frac{1}{4q'\alpha(q,\Omega)}} \left|2^{\alpha(q,\Omega)k}\xi\right|^{-\frac{1}{4q'\alpha(q,\Omega)}} CC_{q,\Omega}, \tag{2.3}$$

$$\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \leq \left|2^{\alpha(q,\Omega)j}\xi\right|^{-\frac{1}{4\alpha(q,\Omega)}} \left|2^{\alpha(q,\Omega)(k+1)}\xi\right|^{\frac{1}{\alpha(q,\Omega)}} CC_{q,\Omega}, \tag{2.4}$$

$$\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \leq \left|2^{\alpha(q,\Omega)(j+1)}\xi\right|^{\frac{1}{\alpha(q,\Omega)}} \left|2^{\alpha(q,\Omega)k}\xi\right|^{-\frac{1}{4\alpha(q,\Omega)}} CC_{q,\Omega}, \tag{2.5}$$

$$\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \leq (\alpha(q,\Omega))^2 \left\|\Omega\right\|_1 \left|2^{\alpha(q,\Omega)(j+1)}\xi\right| \left|2^{\alpha(q,\Omega)(k+1)}\xi\right| C \tag{2.6}$$

for all $j, k \in \mathbb{Z}$, $\xi \in \mathbb{R}^n$, $\eta \in \mathbb{R}$, and mappings φ_1 and φ_2. The constant C is independent of the essential variables.

Proof. We start by the proof of (2.3). We write $\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)$ as

$$\begin{aligned}
&\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right|^2 \\
&\leq \left(\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} \left|\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \Omega(u',v')e_{j,q}(r\xi\cdot u')\, e_{k,q}(s\xi\cdot v')\, d\sigma(u')d\sigma(v')\right| \frac{drds}{rs}\right)^2 \\
&\leq \left(\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} \left|\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \Omega(u',v')e_{j,q}(r\xi\cdot u')\, e_{k,q}(s\xi\cdot v')\, d\sigma(u')d\sigma(v')\right| \frac{drds}{rs}\right)^2 \\
&\leq \left\|\Omega\right\|_q^4 \left(\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \left|\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot u')e_{k,q}(s\xi\cdot v')\,\frac{drds}{rs}\right|^{q'} d\sigma(u',v',z',w')\right)^{\frac{1}{q'}}
\end{aligned} \tag{2.7}$$

where

$$d\sigma(u',v',z',w') = d\sigma(u')d\sigma(v')d\sigma(z')d\sigma(w').$$

Now,

$$\begin{aligned}
&\left|\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w'))e_{k,q}(s\xi\cdot(v'-z'))\frac{drds}{rs}\right|^{q'} \\
&= \left|\int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w'))\frac{dr}{r}\right|^{q'} \left|\int_1^{2^{\alpha(q,\Omega)}} e_{k,q}(s\xi\cdot(v'-z'))\frac{ds}{s}\right|^{q'}.
\end{aligned}$$

Thus, by integration by parts, we have

$$\left|\int_1^{2^{\|\Omega\|_q}}\int_1^{2^{\|\Omega\|_q}} e_{j,q}(r\xi\cdot(u'-w'))e_{k,q}(s\xi\cdot(v'-z'))\frac{drds}{rs}\right|^{q'}$$
$$\leq \left|2^{\alpha(q,\Omega)j}\xi\cdot(u'-w')\right|^{-q'}\left|2^{\alpha(q,\Omega)k}s\xi\cdot(v'-z')\right|^{-q'}. \tag{2.8}$$

On the other hand, we have

$$\left|\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w'))e_{k,q}(s\xi\cdot(v'-z'))\frac{drds}{rs}\right|^{q'}$$
$$\leq \alpha(q,\Omega)^{2q'}. \tag{2.9}$$

Thus, by (2.8) and (2.9), we have

$$\left|\int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w'))e_{k,q}(s\xi\cdot(v'-z'))\frac{drds}{rs}\right|^{q'}$$
$$\leq \alpha(q,\Omega)^{2q'(1-\frac{1}{2q'})}\left|2^{\alpha(q,\Omega)j}\xi\cdot(u'-w')\right|^{-\frac{1}{2}}\left|2^{\alpha(q,\Omega)k}s\xi\cdot(v'-z')\right|^{-\frac{1}{2}}. \tag{2.10}$$

By (2.10), (2.7), and the fact that

$$\sup_{\xi\in\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\left|\xi\cdot(u'-v')\right|^{-\delta}d\sigma(u')d\sigma(v')=C_\delta<\infty \tag{2.11}$$

for $0<\delta<1$, we have

$$\left|\mathbb{M}_{q,\varphi_1,\varphi_{2,j,k}}(\xi,\eta)\right|\leq\left|2^{\alpha(q,\Omega)j}\xi\right|^{-\frac{1}{4q'}}\left|2^{\alpha(q,\Omega)k}\xi\right|^{-\frac{1}{4q'}}CC_{q,\Omega} \tag{2.12}$$

where C is independent of the essential variables. By (2.12) and the observation that

$$\left|\mathbb{M}_{q,\varphi_1,\varphi_{2,j,k}}(\xi,\eta)\right|\leq\alpha(q,\Omega)^2\left\|\Omega\right\|_1^2. \tag{2.13}$$

By (2.12) and (2.13), we get

$$\left|\mathbb{M}_{q,\varphi_1,\varphi_{2,j,k}}(\xi,\eta)\right|$$
$$\leq C_{q,\Omega}\left(\left\|\Omega\right\|_q^2(\alpha(q,\Omega))^{2(1-\frac{1}{2q'})}C\right)^{\frac{1}{\alpha(q,\Omega)}}\left|2^{\alpha(q,\Omega)(j+k)}\left|\xi\right|^2\right|^{-\frac{1}{4q'\alpha(q,\Omega)}}. \tag{2.14}$$

By (2.14) and the observation

$$C_{q,\Omega}\left(\left\|\Omega\right\|_q^2(\alpha(q,\Omega))^{2(1-\frac{1}{2q'})}C\right)^{\frac{1}{\alpha(q,\Omega)}}\leq(\alpha(q,\Omega))^2\left\|\Omega\right\|_1^{2-\frac{2}{\alpha(q,\Omega)}}CC_{q,\Omega}, \tag{2.15}$$

we obtain (2.3).

Nest, we verify (2.4). Notice that by the cancellation property (1.4), we have

$$\int\int_{(|u|,|v|)\in I_{j,q}\times I_{k,q}} e^{-i((u\cdot\xi+\theta(|u|,|v|)\eta)}\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,du\,dv=0.$$

Thus,

$$\begin{aligned}
&\left|\mathbb{M}_{q,\theta,j,k}(\xi,\eta)\right| \\
&=\left|\mathbb{M}_{q,\theta,j,k}(\xi,\eta)-\int\int_{(|u|,|v|)\in I_{j,q}\times I_{k,q}} e^{-i((u\cdot\xi+\theta(|u|,|v|)\,\eta)}\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,du\,dv\right| \\
&\leq \int_1^{2^{\alpha(q,\Omega)}}\int_1^{2^{\alpha(q,\Omega)}}\int_{\mathbb{S}^{n-1}}\left|\left(e_{k,q}(s\xi\cdot v')-1\right)\right|\left|\int_{\mathbb{S}^{n-1}}\Omega(u',v')e_{j,q}(r\xi\cdot u')\,d\sigma(u')\right|d\sigma(v')\frac{drds}{rs} \\
&\leq C_{q,\Omega}\left|2^{\alpha(q,\Omega)(k+1)}\xi\right|\int_1^{2^{\alpha(q,\Omega)}}\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{S}^{n-1}}\Omega(u',v')e_{j,q}(r\xi\cdot u')d\sigma(u')\right|d\sigma(v')\frac{dr}{r}. \qquad (2.16)
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_1^{2^{\alpha(q,\Omega)}}\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{S}^{n-1}}\Omega(u',v')e_{j,q}(r\xi\cdot u')d\sigma(u')\right|d\sigma(v')\frac{dr}{r} \\
&\leq \int_1^{2^{\alpha(q,\Omega)}}\left(\left|\mathbb{S}^{n-1}\right|\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{S}^{n-1}}\Omega(u',v')e_{j,q}(r\xi\cdot u')d\sigma(u')\right|^2d\sigma(v')\right)^{\frac{1}{2}}\frac{dr}{r} \\
&\leq (\alpha(q,\Omega))^{\frac{1}{2}}\left(\int_1^{2^{\alpha(q,\Omega)}}\left|\mathbb{S}^{n-1}\right|\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{S}^{n-1}}\Omega(u',v')e_{j,q}(r\xi\cdot u')d\sigma(u')\right|^2d\sigma(v')\frac{dr}{r}\right)^{\frac{1}{2}} \\
&\leq \left(\left|\mathbb{S}^{n-1}\right|\alpha(q,\Omega)\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\left|I_{j,q,\Omega}(u',w')\right|d\sigma(u',v',z',w')\right)^{\frac{1}{2}} \qquad (2.17)
\end{aligned}$$

where

$$I_{j,q,\Omega}(u',w')=\Omega(u',v')\Omega(w',v')\left|\int_1^{2^{\alpha(q,\Omega)}}e_{j,q}(r\xi\cdot(u'-w'))\frac{dr}{r}\right|.$$

By integration by parts, we have

$$\left|\int_1^{2^{\alpha(q,\Omega)}}e_{j,q}(r\xi\cdot(u'-w'))\,\frac{dr}{r}\right|\leq\alpha(q,\Omega)\left|2^{\alpha(q,\Omega)j}\xi\cdot(u'-w')\right|^{-1}$$

which when combined with the estimate

$$\left|\int_{1}^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w^{'}))\frac{dr}{r}\right| \leq \alpha(q,\Omega) \tag{2.18}$$

implies that

$$\left|\int_{1}^{2^{\alpha(q,\Omega)}} e_{j,q}(r\xi\cdot(u'-w^{'}))\,\frac{dr}{r}\right| \leq \alpha(q,\Omega)\left|2^{\alpha(q,\Omega)j}\xi\cdot(u'-w')\right|^{-\frac{1}{2q'}}. \tag{2.19}$$

By (2.16), (2.17), (2.19), (2.11), and Hölder's inequality, we have

$$\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \leq (\alpha(q,\Omega))^2\left|\mathbb{S}^{n-1}\right|^{\frac{1}{2}}\|\Omega\|_q\left|2^{\alpha(q,\Omega)(k+1)}\xi\right|\left|2^{\alpha(q,\Omega)j}\xi\right|^{-\frac{1}{4}}. \tag{2.20}$$

By (2.13) and (2.20), we obtain (2.4). By symmetry, we can obtain (2.5). To see (2.6), we make use of the observation

$$\int\int_{(|u|,|v|)\in I_{j,q}\times I_{k,q}} e^{-i\theta(|u|,|v|)\eta}\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,du\,dv = 0.$$

In fact,

$$\begin{aligned}
&\left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta)\right| \\
&= \left|\mathbb{M}_{q,\theta_{,j,k}}(\xi,\eta) - \int\int_{(|u|,|v|)\in I_{j,q}\times I_{k,q}} e^{-i\theta(|u|,|v|)\,\eta}\frac{\Omega(u',v')}{|u|^n\,|v|^n}\,du\,dv\right| \\
&\leq \int_{1}^{2^{\alpha(q,\Omega)}}\int_{1}^{2^{\alpha(q,\Omega)}}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\left|\Omega(u',v')\right|\left|e_{k,q}(s\xi\cdot v')-1\right|\left|e_{j,q}(r\xi\cdot u')-1\right|d\sigma(u')d\sigma(v')\frac{drds}{rs} \\
&\leq (\alpha(q,\Omega))^2\,\|\Omega\|_1\left|e^{-i2^{\alpha(q,\Omega)(j+1)}}\xi\right|\left|e^{-i2^{\alpha(q,\Omega)(k+1)}}\xi\right|.
\end{aligned}$$

This completes the proof of Lemma 2.1.

Now, let $\overline{\mu}_{\Omega}^{(\varphi_1,\varphi_2)}$ be the maximal function given by

$$\overline{\mu}_{\Omega,q}^{(\varphi_1,\varphi_2)}(f)(x,x_{n+1}) = \sup_{j,k}\left|\int_{|u|\in I_{j,q}}\int_{|v|\in I_{k,q}}|f(x-u-v,x_{n+1}-\theta(|u|\,,|v|))|\,\frac{|\Omega(u',v')|}{|u|^n\,|v|^n}\right|. \tag{2.21}$$

Lemma 2.2. *Suppose that $\Omega\in L^q(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ for some $q>1$. Let φ_1 and φ_2 be as in Theorem 1.1 or Theorem 1.2 or Theorem 1.3. Then*

$$\|\overline{\mu}_{\Omega,q}^{(\varphi_1,\varphi_2)}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C\,(\alpha(q,\Omega))^2\,\|\Omega\|_{L^1}\|f\|_{L^p(\mathbb{R}^{n+1})} \tag{2.22}$$

for all $1<p<\infty$ and C is a constant independent of of the essential variables.

Proof. Notice that

$$\overline{\mu}_{\Omega,q}^{(\varphi_1,\varphi_2)}(f)(x,x_{n+1}) \leq (\alpha(q,\Omega))^2 \|\Omega\|_1 \overline{\mathcal{M}}_{\Omega,q}^{(\varphi_1,\varphi_2)}(f)(x,x_{n+1}) \tag{2.23}$$

where

$$\overline{\mathcal{M}}^{(\theta)}(f)(x,x_{n+1}) = \sup_{j,k\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \int_{2^k}^{2^{k+1}} \left|f(x-ru'-sv', x_{n+1}-\theta(r,s))\right| \frac{drds}{rs}. \tag{2.24}$$

It can be shown that

$$\overline{\mathcal{M}}^{(\theta)}(f)(x,x_{n+1})(f)(x,x_{n+1}) \leq \overline{\mathcal{M}}^{\varphi_1} \circ \overline{\mathcal{M}}^{\varphi_2}(f)(x,x_{n+1}) \tag{2.25}$$

where

$$\overline{\mathcal{M}}^{\varphi_1}(f)(x,x_{n+1}) = \sup_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \left|f(x-ru', x_{n+1}-\varphi_1(r))\right| \frac{dr}{r} \tag{2.26}$$

and

$$\overline{\mathcal{M}}^{\varphi_2}(f)(x,x_{n+1}) = \sup_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \left|f(x--sv', x_{n+1}-\varphi_2(s))\right| \frac{ds}{s}. \tag{2.27}$$

Under the given assumptions on φ_1 and φ_2, it follows by Theorem 2.4 in [4] that

$$\|\overline{\mathcal{M}}^{\varphi_1}(f)\|_{L^p(\mathbb{R}^{n+1})} \leq C\,\|f\|_{L^p(\mathbb{R}^{n+1})} \tag{2.28}$$

and

$$\|\overline{\mathcal{M}}^{\varphi_2}\|_{L^p(\mathbb{R}^{n+1})} \leq C\,\|f\|_{L^p(\mathbb{R}^{n+1})} \tag{2.29}$$

for all $1 < p < \infty$ with constant C independent of u' and v'. Hence, by (2.23), (2.28), and (2.29), we obtain (2.22).This completes the proof of Lemma 2.2.

3 Proof of Main result

Proof of Theorem 1.1. We start by choosing a sequence $\{A_m : m \in \mathbb{N}\}$ of functions on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ and a sequence $\{\lambda_m : m \in \mathbb{N}\} \subset \mathbb{R}$ such that

$$\int_{\mathbb{S}^{n-1}} A_m(u',.)d\sigma(u') = \int_{\mathbb{S}^{n-1}} A_m(.,v')d\sigma(v') = 0, \tag{3.1}$$

$$A_m(ru', sv^{'}) = A_m(u', v^{'}), r, s > 0, \tag{3.2}$$

$$\|A_m\|_1 \leq 4, \ \ \|A_m\|_2 \leq 2^{2m+2}, \tag{3.3}$$

$$\Omega(x,y) = \sum_{m=1}^{\infty} \lambda_m A_m(x,y), \tag{3.4}$$

$$\sum_{m=1}^{\infty} (m+2)^2 \lambda_m \leq \|\Omega\|_{L(\log L)^2(\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})}. \tag{3.5}$$

By (3.4), it follows that

$$T_{\theta,\Omega}(f)(x,x_{n+1)} = \sum_{m=1}^{\infty} \lambda_m T_{\theta,A_m}(f)(x,x_{n+1}), \tag{3.6}$$

where T_{θ,A_m} is given by (1.6) with Ω is replaced by A_m. Let $\{\sigma_{m,j,k} : j,k \in \mathbb{Z}\}$ be the sequence of measures such that

$$\hat{\sigma}_{m,j,k}(\xi,\eta) = \mathbb{M}_{2,\theta_{,j,k}}(\xi,\eta) \tag{3.7}$$

where $\mathbb{M}_{2,\theta,j,k}$ is given by (2.1) with $q = 2$ and $\alpha(q, A_m) = \log_2(e + \|A_m\|_2)$. Thus, the operator T_{θ,A_m} is decomposed as follows:

$$T_{\theta,A_m}(f)(x,x_{n+1}) = \sum_{j,k\in\mathbb{Z}} \sigma_{m,j,k} * f(x,x_{n+1}). \tag{3.8}$$

Thus, by (3.6), (3.8), and Minkowski's inequality, it suffices to prove that

$$\|T_{\theta,A_m}\|_{L^p(\mathbb{R}^{n+1})} \le (m+2)^2 C \|f\|_{L^p(\mathbb{R}^{n+1})} \tag{3.9}$$

where C is a constant independent of m. Now, by an argument similar to that used in [7], we let $\{\varpi_j{}^1\}_{-\infty}^{\infty}$ and $\{\varpi_k{}^2\}_{-\infty}^{\infty}$ be smooth partitions of unity on $(0,\infty)$ in the sense that

$$supp(\varpi_j^{(1)}) \subseteq \left\{t : 2^{-2\alpha(2,A_m)(j+1)} < t < 2^{-2\alpha(2,A_m)(j-1)}\right\}, \tag{3.10}$$

$$supp(\varpi_k^{(2)}) \subseteq \left\{t : 2^{-2\alpha(2,A_m)(k+1)} < t < 2^{-2\alpha(2,A_m)(k-1)}\right\}, \tag{3.11}$$

$$0 \le \varpi_j^{(1)}, \varpi_k^{(2)} \le 1; \tag{3.12}$$

$$\sum_{j\in\mathbb{Z}} \varpi_j^{(1)})(t) = \sum_{k\in\mathbb{Z}} \varpi_k^{(2)})(t) = 1; \tag{3.13}$$

$$\left|\frac{d^r \varpi_j^{(1)}}{dt^r}(t)\right|, \ \left|\frac{d^r \varpi_k^{(2)}}{dt^r}(u)\right| \le \frac{C_r}{t^r} \tag{3.14}$$

where C_r is independent of m. Defined the multiplier operator $\mathcal{F}_{j,k}^{(m)}$ in $\mathbb{R}^{n+1}$ by

$$(\widehat{\mathcal{F}_{j,k}^{(m)} f})(\xi,\eta) = \varpi_j^{(1)}(|2^{\alpha(2,A_m)j}\xi|)\, \varpi_k^{(2)}(|2^{\alpha(2,A_m)k}\xi|)\, \hat{f}(\xi,\eta).$$

Thus, for $r, s \in \mathbb{Z}$, we have

$$(\widehat{\mathcal{F}_{j+r,k+s}^{(m)} \mathcal{F}_{j+r,k+s}^{(m)} f})(\xi,\eta) = \hat{f}(\xi,\eta) \left(\varpi_{j+r}^{(1)}(|2^{\alpha(2,A_m)j}\xi|)\right)^2 \left(\varpi_{k+s}^{(2)}(|2^{\alpha(2,A_m)k}\xi|)\right)^2.$$

Therefore, by (3.13), we get

$$\sum_{r,s\in\mathbb{Z}} (\mathcal{F}_{j+r,k+s}^{(m)} \mathcal{F}_{j+r,k+s}^{(m)} f)(x,x_{n+1}) = f(x,x_{n+1}). \tag{3.15}$$

By (3.15), we decompose the operator T_{θ,A_m} as follows

$$T_{\theta,A_m} f(x,x_{n+1}) = \sum_{j,k\in\mathbb{Z}} \sigma_{m,j,k} * f(x,x_{n+1}) = \sum_{j,k\in\mathbb{Z}} \sigma_{m,j,k} * \left(\sum_{r,s\in\mathbb{Z}} \mathcal{F}_{j+r,k+s}^{(m)} \mathcal{F}_{j+r,k+s}^{(m)} f\right).$$

Now, let

$$\mathcal{N}_{r,s}^{(m)}(f)=\sum_{j,k\in\mathbb{Z}}\mathcal{F}_{j+r,k+s}^{(m)}\left(\sigma_{m,j,k}*\mathcal{F}_{j+r,k+s}^{(m)}f\right). \tag{3.16}$$

Hence, by (3.16), T_{θ,A_m} reduces to

$$T_{\theta,A_m}f(x,x_{n+1})=\sum_{r,s\in\mathbb{Z}}\mathcal{N}_{r,s}^{(m)}(f)(x,x_{n+1}). \tag{3.17}$$

Now, for $p>2$, we have

$$\begin{aligned}\|\mathcal{N}_{r,s}^{(m)}f\|_p&\le C_p\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\sigma_{m,j,k}*\mathcal{F}_{j+r,k+s}^{(m)}f\right|^2\right)^{\frac{1}{2}}\right\|_p\\&\le C_p\,(\alpha(2,A_m))^2\,\|A_m\|_1\left\|\left(\sum_{j,k\in\mathbb{Z}}\left|\mathcal{F}_{j+r,k+s}^{(m)}f\right|^2\right)^{\frac{1}{2}}\right\|_p\\&\le C_p\,(\alpha(2,A_m))^2\,\|A_m\|_1\,\|f\|_p \end{aligned}\tag{3.18}$$

for all $r,s\in\mathbb{Z}$ and C_p is a constant independent of m. Here, the first and the last inequalities follow by Littlewood-Paley theory while the second inequality follows by (2.22)
Next, we consider the L^2- norm. Notice that by Plancherel's theorem and the properties of the portions $\{\varpi_j{}^1\}_{-\infty}^{\infty}$ and $\{\varpi_k{}^2\}_{-\infty}^{\infty}$, we have

$$\|\mathcal{N}_{r,s}^{(m)}f\|_{L^2}^2\le\sum_{j,k\in\mathbb{Z}}\int_{\Lambda_{j+r,k+s}}|f(\xi,\eta)|^2\,|\mathbb{M}_{2,\theta,j,k}(\xi,\eta)|^2\,d\xi\,d\eta, \tag{3.19}$$

where

$$\Lambda_{j+r,k+s}=\Lambda_{j+r}^{(m)}\cap\Lambda_{k+s}^{(m)},$$

$$\Lambda_{j+r}^{(m)}=\left\{\xi:2^{-2\alpha(2,A_m)(j+r+1)}<|2^{\alpha(2,A_m)j}\,\xi|<2^{-2\alpha(2,A_m)(j+r+1)}\right\},$$

and

$$\Lambda_{k+s}^{(m)}=\left\{\xi:2^{-2\alpha(2,A_m)(k+s+1)}<|2^{\alpha(2,A_m)k}\,\xi|<2^{-2\alpha(2,A_m)(k+s+1)}\right\}.$$

By Lemma 2.1, (2.13), and the fact that $\|A_m\|_1\le 4$, we get

$$\|\mathcal{N}_{r,s}^{(m)}f\|_{L^2}\le(\alpha(2,A_m))^2\,C\,2^{-\tilde{\alpha}_l|r|}\,2^{-\tilde{\beta}_l|s|}\,\|f\|_{L^2} \tag{3.20}$$

for some real $\tilde{\alpha}_l>0$ and $\tilde{\beta}_l>0$. Hence, by interpolation between (3.18) and (3.20) along with the fact that $\alpha(2,A_m)\le 2(m+2)$, we get (3.9) for $p\ge 2$. The case for $1<p<2$ follows by duality. This completes the proof of Theorem 1.1.

Proof of Theorems 1.2 and 1.3. The proofs follows by similar argument as that for the proof of Theorem 1.1 with minor modifications. We omit the details.

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