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On stability of a basis consisting of an perturbed exponential system in the weighted grand Lebesgue space

Migdad I. Ismailov · **Tural X. Hagverdi**⋆

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Abstract. *In this paper, we investigate the stability of an perturbed exponential system* $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$ *in a separable subspace of the weighted grand Lebesgue space* $N_{p),\omega}(-\pi,\pi)$. *Analogues of classical Levinson-type theorems are obtained for the completeness of the system in space* $L_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$ *and using this result, the stability of the perturbed exponential system is established.*

Keywords. exponential system, weighted grand Lebesgue space, q-Hilbert system, separated system, Levinson theorem.

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1 Introduction

The study of exponential systems plays a central role in harmonic analysis and the theory of function spaces. Among them, perturbed exponential systems, which generalize classical exponential systems, have attracted considerable interest due to their connection with eigenfunctions of differential operators subject to integral boundary conditions. The investigation of exponential systems, commonly known as the theory of nonharmonic Fourier series (see [1–5]), traces its roots to the classical works in this direction include those by Paley and Wiener [6], N. Levinson [1], who explored the basis properties of such systems in various functional frameworks. One of the well-known early results in this area states that the trigonometric system $\left\{e^{inx}\right\}_{n=-\infty}^{+\infty}$ possesses a stable basis property in $L_2(-\pi,\pi)$. More precisely, the system $\left\{e^{i\lambda_n x}\right\}_{n=-\infty}^{+\infty}$ forms a Riesz basis in $L_2(-\pi,\pi)$ provided that $|\lambda_n - n| \le L < 1/4$. The optimality of the constant $\frac{1}{4}$ was established by M. I. Kadec [7], and later by R. M. Redheffer and R. M. Young [8].

In recent years, the investigation of such systems in generalized function spaces, such as grand Lebesgue spaces (see [9–16]) and weighted grand Lebesgue spaces, has gained

⋆ Corresponding author

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M.I. Ismailov
Azerbaijan State Oil and Industry University
E-mail: migdad-ismailov@rambler.ru

T.X. Hagverdi
Institute of Mathematics and Mechanics, The Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: turalhagverdi@gmail.com

significant momentum. These spaces provide a flexible framework for analyzing systems with variable integrability properties and are particularly effective in addressing questions of stability and completeness. In this direction, special attention has been paid to the stability of bases generated by perturbed exponential systems, both in classical L_p (see [17]), Morrey-Lebesgue spaces (see [18]) and in grand Lebesgue spaces where the non-uniform integrability structure plays a crucial role in the preservation of basis properties (see also [19–21]).

Now, let us consider a certain sequence of real numbers $\{\alpha_n\}_{-\infty}^{\infty}$ of real numbers.We examine the following exponential system

$$\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}. \tag{1.1}$$

In this paper, we focus on the stability of a basis formed by a perturbed exponential system within a separable subspace $N_{p),\omega}(-\pi,\pi)$ of the weighted grand Lebesgue space $L_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$. We establish analogues of classical Levinson-type theorems, providing criteria for the completeness and minimality of the system in this setting. The stability of system (1.1) in space $N_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$, is studied.

2 Preliminaries

Let $\omega(x)$ be a weight function defined on $[-\pi,\pi]$. We denote by $L_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$, the weighted grand Lebesgue space consisting of measurable functions defined on $[-\pi,\pi]$ with finite norm

$$\|f\|_{p),\omega} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |f(t)\omega(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

We denote by $A_p(-\pi,\pi)$, $1 < p < +\infty$, the Muckenhoupt weight class consisting of periodic functions $\omega(x)$ with period 2π, satisfying the following condition

$$\sup_{I\subset[-\pi,\pi]} \left(\frac{1}{|I|}\int_I \omega^p(t)dt\right)\left(\frac{1}{|I|}\int_I \omega^{-\frac{p}{p-1}}(t)dt\right)^{p-1} < +\infty,$$

where the supremum is taken over all finite intervals I in $[-\pi,\pi]$.

Since the space $L_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$, is non-separable, we consider its separable subspace $N_{p),\omega}(-\pi,\pi)$ of functions $f \in L_{p),\omega}(-\pi,\pi)$:

$$\|f(\cdot+\delta) - f(\cdot)\|_{L_{p),\omega}(-\pi,\pi)} \to 0, \ \ \delta \to 0.$$

We will need the following theorem regarding the density of the set $C^\infty[-\pi,\pi]$ in the space $N_{p),\omega}(-\pi;\pi)$.

Theorem 2.1 ([22]) *Let there is $\varepsilon \in (0,p-1)$ such that $\omega^{-1} \in L_{\frac{1}{p-\varepsilon-1}}$. Then set of infinitely differentiable functions $C^\infty[-\pi,\pi]$ is dense in $N_{p),\omega}(-\pi,\pi)$.*

Let X be a b.f.s. over $(M;\mu)$ with norm $\|\cdot\|_X$. We will also use certain concepts and results from the theory of Banach function spaces (see [23]). The necessary facts from this theory are stated below.

Definition 2.1 ([23]) *A function f in a b.f.s. (Banach function space) X is said to have continuous norm in X if $\|f\chi_{E_n}\|_X \to 0$, for every sequence $\{E_n\}_{n=1}^{\infty}$ satisfying $E_n \to \emptyset$ $\mu-$* a.e. *The set of all functions in X of absolutely continuous norm is denoted by X_a.*

Theorem 2.2 ([23]) *The dual X^* of a b.f.s. X is canonically isometrically isomorphic to the associate space $X^{'}$ if and only if X has absolutely continuous norm,where*

$$X' = \left\{ g \in L_1 : \|g\|_{X'} = \sup_{\|f\|_X \le 1} \int_M |f(t)g(t)|\, dt < +\infty \right\}.$$

Proposition 2.1 ([23]) *A function $f \in X$ has absolutely continuous norm if and only if $\|f\chi_{E_n}\|_X \downarrow 0$ for every sequence $\{E_n\}_{n\in N}$ satisfying $E_n \downarrow \emptyset \mu - a.e.$*

We also need the following theorems.

Theorem 2.3 (Hausdroff-Young) *Let $1 \le p \le 2,\ \frac{1}{p}+\frac{1}{q} = 1,\ \ f \in L^p(-\pi,\pi)$, and define its Fourier coefficients by $\hat{f}\,(n) = c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,(x)e^{-inx}dx,\ \ n \in \mathbb{Z}$. Then Hausdorff–Young inequality states that*

$$\|\ \{c_n\}_{n\in\mathbb{Z}}\|_{\ell_q} \le C_p\ \|f\|_{L_p(-\pi,\pi)},$$

where

$$\|\ \{c_n\}\|_{\ell_q} = \left(\sum_{n\in\mathbb{Z}} |c_n|^q\right)^{1/q}$$

and C_p is a constant depending only on p .

We will also use the following analogue of the Paley-Wiener theorem in Banach spaces

Theorem 2.4 *Let X be a Banach space and let $\{x_n\}_{n=1}^{\infty} \in X$, be a (Schauder) basis of X. Let $\{y_n\}_{n=1}^{\infty} \in X$, be another sequence such that there exists a constant $\theta \in [0,1)$ satisfying*

$$\left\|\sum_{n=1}^{m} a_n(y_n - x_n)\right\| \le \theta \left\|\sum_{n=1}^{m} a_n x_n\right\|$$

for all finite scalar sequences $(a_1,\ldots,a_m)$.
Then $\{y_n\}$ is also a Schauder basis of X. *Moreover, $\{x_n\}$ and $\{y_n\}$ are equivalent bases.*

3 A Necessary Condition for the Sequence $\{\alpha_n\}$ to be Separated

Definition 3.1 ([24]) *A system $\{f_n\}_{n\in\mathbb{N}} \subset L_{p),\omega}(-\pi,\pi),\ \ 1 < p < +\infty$, is called a q-Hilbert system, if there exists C>0, such that for any finite system $\{c_n\}_{n\in\mathbb{Z}}$ of complex numbers the following inequality holds*

$$(\textstyle\sum_n\ |c_n|^q)^{\frac{1}{q}} \le C\|\sum_n\ c_n f_n\|_{L_{p),\omega}},$$

where $\frac{1}{q}+\frac{1}{p} = 1$.

Definition 3.2 ([25]) *A sequence $\{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is called separated if*

$$\inf_{\mathrm{n\neq m}}\ |\lambda_n - \lambda_m| > 0,$$

holds.

Lemma 3.1 *Let $\{\alpha_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers. If for $\omega \neq 0$ and $\omega \in L_{p)}(-\pi,\pi)$ the system $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$ is a q-Hilbert system in the space $L_{p),\omega}(-\pi,\pi)$, $\quad 1 < p < +\infty$, then the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ is separated.*

Proof. By the definition of q-Hilbert system, we have:

$$\left(\sum_{k=-\infty}^{\infty} |c_k|^q\right)^{\frac{1}{q}} \leq C \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi}\left|\sum_{k=-\infty}^{\infty} c_k f_k \omega\right|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}.$$

If we choose the coefficients as $c_n = 1, c_m = -1,\ c_k = 0, k \neq n \neq m$ and $f_k = e^{i\alpha_k t}$ then we obtain

$$2^{\frac{1}{q}} \leq C \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} \left|\left(e^{i\alpha_n t} - e^{i\alpha_{\mathrm{m}} t}\right)\omega\right|^{p-\varepsilon} dt\right)^{\frac{1}{p-\varepsilon}}$$

$$\Rightarrow 2^{\frac{1}{q}} \leq C\left\|e^{i\alpha_n t} - e^{i\alpha_{\mathrm{m}} t}\right\|_{p),\omega}. \tag{3.1}$$

On the other hand

$$\left|e^{i\alpha_n t} - e^{i\alpha_m t}\right| = 2\left|\sin\left(\frac{\alpha_n - \alpha_m}{2} t\right)\right| \leq |\alpha_n - \alpha_m|\,|t| \leq \pi\,|\alpha_n - \alpha_m|\,.$$

It follows from this that

$$\left\|e^{i\alpha_n t} - e^{i\alpha_m t}\right\|_{L_{p),\omega}(-\pi,\pi)} \leq \pi\,|(\alpha_n - \alpha_m)|\,\|\omega\|_{L_{p)}(-\pi,\pi)}.$$

Then, from relation (3.1), we obtain

$$2^{\frac{1}{q}} \leq \left\|e^{i\alpha_n t} - e^{i\alpha_m t}\right\|_{L_{p),\omega}(-\pi,\pi)} \leq \pi\,|(\alpha_n - \alpha_m)|\,\|\omega\|_{L_{p)}(-\pi,\pi)}$$

$$\Rightarrow |\alpha_n - \alpha_m| \geq \frac{2^{\frac{1}{q}}}{\pi\|\omega\|_{L_{p)}(-\pi,\pi)}} \Rightarrow \inf_{n\neq m}\ |\alpha_n - \alpha_m| > 0.$$

The theorem is thus proved.

4 Analogues of Levinson-Type Theorems in the Space $N_{p),\omega}(-\pi;\pi)$

We establish the following analogues of Levinson's theorems [26].

Theorem 4.1 *Let α_k , $k = 1, 2, \ldots$ be a sequence of complex number. A necessary and sufficient condition for the exponential system $\left\{e^{i\alpha_k t}\right\}_{k\in N}$ to be incomplete in the space $N_{p),\omega}(-\pi;\pi)$ is the existence of a non-zero entire function $F(z)$ of exponential type such that*

$$F(\alpha_n) = 0 \textit{ for all } n,$$

where,

$$F(z) = \int_{-\pi}^{\pi} f(x)\, e^{izx} dx,\ \ f(x) \in N'_{p),\omega}(-\pi,\pi)\,,\ f \neq 0.$$

Proof. Suppose that the system $\left\{e^{i\alpha_k t}\right\}_{k\in N}$ is not complete in the space. Then there exists a non-zero functional $L \in N'_{p),\omega}(-\pi,\pi)$ such that

$$L\left(e^{i\alpha_k x}\right) = 0 \text{ for all } k.$$

Let us show that the spaces $N'_{p),\omega}(-\pi,\pi)$ and $N^*_{p),\omega}(-\pi,\pi)$ are isometrically isomorphic. According to Theorem 2.2 it is sufficient to prove that the space $N_{p),\omega}(-\pi,\pi)$, $1 < p < +\infty$, admits an absolutely continuous norm.

By Theorem 2.1, we have

$$\overline{C^\infty([-\pi,\pi])} = N_{p),\omega}(-\pi,\pi).$$

Then for every $\varepsilon > 0$ there exists $f_0 \in C[-\pi,\pi]$ such that

$$\|f - f_0\|_{L_{p),\omega}} < \varepsilon.$$

Suppose that $\{E_n\}_{n\in\mathbb{N}} \subset (-\pi,\pi)$ is a sequence of Lebesgue measurable sets such that $E_n \to \varnothing$, $\mu - a.e.$, i.e. $\chi_{E_n} \to\to 0$, $\mu - a.e.$We need to show

$$\|f\chi_{E_n}\|_{L_{p),\omega}} \to 0.$$

Let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned} \|f\chi_{E_n}\|_{L_p,\omega} &= \|(f - f_0)\chi_{E_n} + f_0\chi_{E_n}\|_{L_p,\omega} \\ &\le \|(f - f_0)\chi_{E_n}\|_{L_p,\omega} + \|f_0\chi_{E_n}\|_{L_p,\omega} \\ &< \varepsilon + \|f_0\chi_{E_n}\|_{L_p,\omega}. \end{aligned} \tag{4.1}$$

Let $M = \|f_0\|_{L_\infty(-\pi,\pi)}$. Then we obtain

$$\|f_0\chi_{E_n}\|_{L_{p),\omega}} \le M\|\chi_{E_n}\|_{L_{p),\omega}} = M\, sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{-\pi}^{\pi} |\chi_{E_n}\omega(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}}$$
$$\le M\, sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\int_{E_n} |\omega(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \le M\, sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} ||\omega(t)||_\infty |E_n|^{\frac{1}{p-\varepsilon}},$$

where $|\cdot|$ denotes Lebesgue measure.

Since $\lim_{\mathrm{n}} E_n = \bigcap_n E_n = \varnothing$, a.e. we have

$$\lim_{\mathrm{n}} |E_n| = \left|\lim_{\mathrm{n}} E_n\right| = 0.$$

From (4.1) it follows that

$$\|f\chi_{E_n}\|_{L_{p),\omega}} \to 0 \text{ as } n \to \infty.$$

Thus, by Proposition 2.1, we conclude that the space $N_{p),\omega}(-\pi,\pi)$ possesses an absolutely continuous norm. From Theorem 2.2, it follows that

$$N'_{p),\omega}(-\pi,\pi) \cong N^*_{p),\omega}(-\pi,\pi),$$

which means that there exists a function $g \in N'_{p),\omega}(-\pi,\pi)$ such that for every $f \in N_{p),\omega}(-\pi,\pi)$, we have

$$L(f) = \int_{-\pi}^{\pi} f(x)g(x)dx. \tag{4.2}$$

Substituting $f(x)=e^{i\alpha_n x}$ into (4.2), we obtain:

$$L\left(e^{i\alpha_n x}\right)=\int_{-\pi}^{\pi} e^{i\alpha_n x} g(x)dx=0.$$

It thus becomes clear that $F(z)=\int_{-\pi}^{\pi} f(x)e^{izx}dx$ is an entire function and hence

$$F(\alpha_n)=0 \quad \text{for all } n.$$

The theorem is thus proved.

Theorem 4.2 *Suppose that the function $F(z)$ is represented in the form* (4.2) *and $g \in N'_{p),\omega}(-\pi,\pi), 1<p<+\infty, F(\alpha_0)=0$ and β any complex number then*

$$F_1(\alpha)=\frac{\alpha-\beta}{\alpha-\alpha_0}F(\alpha),$$

is also represented in the form (4.2).

Proof. As in the classical Levinson theorem, let us define the new $h(x)$ function in the following way

$$h(x)=g(x)+i\left(\beta-\alpha_0\right)e^{-i\alpha_0 x}\int_{-\pi}^{x} e^{i\alpha_0 y}g(y)dy. \tag{4.3}$$

If we multiply both sides of equation (4.3) by $e^{i\alpha x}$ and integrate, we obtain

$$\int_{-\pi}^{\pi} e^{i\alpha x}h(x)\,dx=\int_{-\pi}^{\pi} e^{i\alpha x}g(x)\,dx+i(\beta-\alpha_0)\int_{-\pi}^{\pi} e^{i(\alpha-\alpha_0)x}\left(\int_{-\pi}^{x} e^{i\alpha_0 y}g(y)\,dy\right)dx$$

$$=F(\alpha)+i(\beta-\alpha_0)\int_{-\pi}^{\pi}\int_{y}^{\pi} e^{i(\alpha-\alpha_0)x}e^{i\alpha_0 y}g(y)\,dx\,dy.$$

By Fubini's theorem, changing the order of integration yields

$$\int_{-\pi}^{\pi} e^{i\alpha x}h(x)\,dx=F(\alpha)+\frac{\beta-\alpha_0}{\alpha-\alpha_0}\int_{-\pi}^{\pi}\left(e^{i(\alpha-\alpha_0)\pi}-e^{i(\alpha-\alpha_0)y}\right)e^{i\alpha_0 y}g(y)\,dy$$

$$=F(\alpha)+\frac{\beta-\alpha_0}{\alpha-\alpha_0}\int_{-\pi}^{\pi} e^{i\alpha y}g(y)\,dy$$

$$=F(\alpha)-\frac{\beta-\alpha_0}{\alpha-\alpha_0}F(\alpha)=\frac{\alpha-\beta}{\alpha-\alpha_0}\,F(\alpha).$$

Thus

$$F_1(\alpha)=\int_{-\pi}^{\pi} e^{i\alpha x}h(x)dx.$$

The theorem is proved.

Corollary 4.1 *Suppose that the system $\left\{e^{i\alpha_k x}\right\}_{k\in Z}$ is complete in the space $N_{p),\omega}(-\pi,\pi)$. Then, if we remove any n functions from this system and replace them with any other n exponential functions $\left\{e^{i\beta_k x}\right\}_{k=1}^{n}$ (here $\beta_k\neq\alpha_k, \forall k=\overline{1,n}$) then the newly obtained system will be complete in space $N_{p),\omega}(-\pi,\pi)$.*

5 Stability of the System $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$ in the Space $N_{p),\omega}(-\pi,\pi)$

Let us consider the following theorem concerning the stability of an exponential system with respect to the sequence $\{\alpha_n\}_{n\in\mathbb{Z}}$ in the space $N_{p),\omega}(-\pi,\pi)$.

Theorem 5.1 *Let* $1 < p < +\infty$, $\omega\epsilon A_p$, $r\epsilon(1,\min(p,q))$, $r < p_1 < p$, $\omega^{-1} \in L_{\frac{p_1 r}{p_1 - r}}$, α_n, β_n $\epsilon\mathbb{R}$, $n \in \mathbb{Z}$, *such that* , $\alpha_i{\neq}\alpha_j$, $\beta_i{\neq}\beta_j$, *for* $i{\neq}j$, *and*

$$\sum_{n=-\infty}^{+\infty} |\alpha_n - \beta_n|^r < +\infty. \tag{5.1}$$

If the system $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$ *forms a basis isomorphic to the system* $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ *in space* $N_{p),\omega}(-\pi,\pi)$, *then* $\left\{e^{i\beta_n x}\right\}_{n\in\mathbb{Z}}$ *forms a basis isomorphic to the system* $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ *in* $N_{p),\omega}(-\pi,\pi)$

Proof. Let's denote

$$e_n(x) = e^{inx} \ , \varphi_n(x) = e^{i\alpha_n x}$$

and

$$\psi_n(x) = e^{i\beta_n x} \ , n \in \mathbb{Z}, \ x \in [-\pi,\pi] .$$

According to the Lemma 3.1,

$$\|\varphi_n - \ \psi_n\|_{p),\omega} \le M\,|\alpha_n - \beta_n| \, , n \in \mathbb{Z}, \tag{5.2}$$

here $M = \pi\|\omega\|_{p)}$. Let fix number $m \in \mathbb{Z}_+$ and consider the system of functions as following

$$f_n = \begin{cases} \varphi_n, & |n| < m \\ \psi_n, & |n| \ge m \end{cases} \ , \ n \in \mathbb{Z} \ .$$

Applying inequality (5.2) together with Hölder's inequality, we obtain the following for any finite complex sequence $\{c_n\}$

$$\begin{aligned} \left\|\sum_n c_n(f_n - \varphi_n)\right\|_{L_{p),\omega}} &\le \sum_n |c_n| \, \|f_n - \varphi_n\|_{L_{p),\omega}} \\ &\le \sum_n |c_n| \, |\alpha_n - \beta_n| \\ &\le \left(\sum_n |c_n|^{r'}\right)^{\frac{1}{r'}} \left(\sum_n (\alpha_n - \beta_n)^r\right)^{\frac{1}{r}} . \end{aligned} \tag{5.3}$$

Since $1 < r < 2$ and $\omega\epsilon A_p$ the system $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ becomes basis in $N_{p),\omega}(-\pi,\pi)$ (see result [22]) so we can apply here by the Theorem 2.4 about Hausdroff-Young inequality

$$\left(\sum_n |c_n|^{r'}\right)^{\frac{1}{r'}} \le \left\|\sum_n c_n e_n\right\|_r . \tag{5.4}$$

Since $\omega^{-1} \in L_{\frac{p_1 r}{p_1 - r}}$ for $\forall f \in L_{p),\omega}$ by using Hölder's inequality we obtain

$$\|f\|_r = \left(\int_{-\pi}^{\pi} |f\omega|^r \cdot \omega^{-r} dt\right)^{\frac{1}{r}}$$

$$\leq \left(\int_{-\pi}^{\pi}|f\omega|^{p_1}dt\right)^{\frac{1}{p_1}}\left(\int_{-\pi}^{\pi}\omega^{-\frac{p_1 r}{p_1-r}}dt\right)^{\frac{p_1-r}{p_1 r}} \leq K\|f\|_{p),\omega}. \tag{5.5}$$

Taking inequality (5.5) into account in inequality (5.4), we obtain

$$\left(\sum_n |c_n|^{r'}\right)^{\frac{1}{r'}} \leq K\left\|\sum_n c_n e_n\right\|_{p),\omega}. \tag{5.6}$$

Since the systems $\{\varphi_n\}$ and $\{\psi_n\}$ are isomorphic bases, there exists a constant $L > 0$ such that

$$\left\|\sum_n c_n e_n\right\|_{p),\omega} \leq L\left\|\sum_n c_n \varphi_n\right\|_{p),\omega} \tag{5.7}$$

Thus, taking inequality (5.7) into account in inequality (5.6), and using inequality (5.5), we have

$$\|\textstyle\sum_n c_n(f_n-\varphi_n)\|_{p),\omega} \leq M_1\left(\sum_{|n|\geq m}(\alpha_n-\beta_n)^r\right)^{\frac{1}{r}}\|\sum_n c_n\varphi_n\|_{p),\omega}.$$

Since the series $\sum_{n=-\infty}^{+\infty}|\alpha_n-\beta_n|^r$ converges, for a sufficiently large $m \in \mathbb{Z}_+$ we can choose it so that $\left(\sum_{|n|\geq m}|\alpha_n-\beta_n|^r\right)^{\frac{1}{r}} < \frac{1}{M_1}$, holds. Thus, for an arbitrarily chosen $m \in \mathbb{Z}_+$:

$$\theta = M_1\left(\sum_{|n|\geq m}|\alpha_n-\beta_n|^r\right)^{\frac{1}{r}} < 1,$$

and

$$\|\textstyle\sum_n c_n(f_n-\varphi_n)\|_{p),\omega} \leq \theta\|\sum_n c_n\varphi_n\|_{p),\omega}$$

holds.

Hence, by the Paley–Wiener theorem, the system $\{f_n\}$ is an isomorphic basis to the system $\{\varphi_n\}$ in $N_{p),\omega}$. Since the systems $\{f_n\}$ and $\{\psi_n\}$ differ from each other countable number functions, more precisely system $\{\psi_n\}$ consist of functions

$$e^{i\mu_{-m+1}x}, e^{i\mu_{-m+2}x}, \ldots, e^{i\mu_{m-2}x}, e^{i\mu_{m-1}x},$$

which are different functions in system $\{f_n\}$, it follows that according to the theorem, the fact that the system $\{\varphi_n\}$ is an isomorphic basis to the system $\{f_n\}$ is equivalent to its completeness. On the other hand, according to Theorem 2.2, which is an analogue of Levinson's theorem in $N_{p),\omega}$ space, if in the system $\{f_n\}$ the elements with indices $|n| < m$ are replaced, respectively, by the elements $\{\psi_n\}$, then the resulting system, $\{\psi_n\}$ will be complete in $N_{p),\omega}$. Therefore, the system $\{\psi_n\}$ is an isomorphic basis to the system $\{\varphi_n\}$ in $N_{p),\omega}$. The theorem is proved.

Corollary 5.1 *Let* $1 < p < +\infty$, $1 < p_0 < p$, $\omega \in A_p \cap A_{p_0}$, *and*

$$r \in \left(1, min\left(\frac{p}{p-1}, \frac{pp_0}{p_0+p(p_0-1)}\right)\right),$$

$\{\alpha_n\}, \{\beta_n\} \subset \mathbb{R}$ *is a sequence of distinct numbers and* $\sum_{n\in\mathbb{Z}}|\alpha_n-\beta_n|^r < +\infty$. *Then if the system* $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$ *is an isomorphic basis to the system* $\left\{e^{inx}\right\}_{n\in\mathbb{Z}}$ *in space* $N_{p),\omega}$ *then the system* $\left\{e^{i\mu_n x}\right\}_{n\in\mathbb{Z}}$ *is an isomorphic basis to the* $\left\{e^{i\alpha_n x}\right\}_{n\in\mathbb{Z}}$.

Proof. From the condition
$$r < \frac{pp_0}{p_0 + p(p_0 - 1)},$$
we obtain
$$\frac{rp_0}{p_0 + r - rp_0} < p.$$
Hence, there exists a number p_1 such that
$$\frac{rp_0}{p_0 + r - rp_0} < p_1 < p.$$
Choosing such a p_1, we have
$$\frac{rp_1}{p_1 - r} < \frac{p_0}{p_0 - 1}.$$
Thus, since $\omega^{-1} \in L_{\frac{p_0}{p_0-1}}$, it follows that $\omega^{-1} \in L_{\frac{rp_1}{p_1-r}}$. By Theorem 5.1, if the system $\left\{e^{i\alpha_n x}\right\}$ is an isomorphic basis to $\left\{e^{inx}\right\}$ in $N_{p),\omega}$, then the system $\left\{e^{i\mu_n x}\right\}$ is also an isomorphic basis to $\left\{e^{inx}\right\}$ in $N_{p),\omega}$.

The result is proved.

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