

Maximal operator in total Morrey-Guliyev spaces defined on Carleson curves

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Abstract. *In this paper we study the maximal operator M_Γ in the total Morrey-Guliyev space $L_{p,\lambda,\mu}(\Gamma)$ defined on Carleson curves Γ . We prove that for $1 < p < \infty$ the maximal operator M_Γ is bounded on the total Morrey-Guliyev space $M_{p,\lambda,\mu}(\Gamma)$, and from the total Morrey-Guliyev space $M_{1,\lambda,\mu}(\Gamma)$ to $WL_{1,\lambda,\mu}(\Gamma)$.*

Keywords. Carleson curves, Morrey spaces, modified Morrey spaces, total Morrey-Guliyev spaces.

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1 Introduction

Morrey spaces, introduced by Morrey [29], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [13], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ , which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, t > 0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \|f\|_{L_p(B(x,t))},$$

where $B(x, t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \leq \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, see also [1, 2, 8, 24, 25, 27, 28, 30–33, 36–38]. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\tilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [3, 4, 7, 12, 14–16, 22, 23, 31].

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Let $\Gamma = \{t \in \mathbb{C} : t = t(s), 0 \leq s \leq l \leq \infty\}$ be a rectifiable Jordan curve in the complex plane with arc-length measure $\nu(t) = s$, here $l = \nu\Gamma = \text{lengths of } \Gamma$.

We denote

$$\Gamma(t, r) = \Gamma \cap B(t, r), \quad t \in \Gamma, \quad r > 0,$$

where $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$.

A rectifiable Jordan curve Γ is called a Carleson curve (regular curve) if the condition

$$\nu\Gamma(t, r) \leq c_0 r$$

holds for all $t \in \Gamma$ and $r > 0$, where the constant $c_0 > 0$ does not depend on t and r .

It is well known that maximal operator play an important role in harmonic analysis (see [35]). In this paper, in the framework of this analysis in the setting of Carleson curve, we study the boundedness of the maximal operator M_Γ defined on Carleson curves Γ on total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\Gamma)$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we prove $L_{p,\lambda,\mu}(\Gamma)$ -boundedness of the maximal operator M_Γ .

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries in the setting on Carleson curves and some auxiliary lemmas

Definition 2.1 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[r]_1 = \min\{1, r\}$, $r > 0$. We denote by $L_{p,\lambda}(\Gamma)$ the Morrey space [26], by $\tilde{L}_{p,\lambda}(\Gamma)$ the modified Morrey space [9], and by $L_{p,\lambda,\mu}(\Gamma)$ the total Morrey-Guliyev space [13], associated with the Carleson curves the set of all classes of locally integrable functions f with the finite norms

$$\begin{aligned} \|f\|_{L_{p,\lambda}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}, \\ \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))}, \\ \|f\|_{L_{p,\lambda,\mu}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{L_p(\Gamma(t,r))}, \end{aligned}$$

respectively.

Definition 2.2 Let $0 < p < \infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. We define the weak Morrey space $L_{p,\lambda}(\Gamma)$ [26] (\equiv weak D_ν -Morrey space), the weak modified Morrey space $\tilde{L}_{p,\lambda}(\Gamma)$ [9], and the weak total Morrey-Guliyev space $L_{p,\lambda,\mu}(\Gamma)$ [13], associated with the Carleson curves the set of all classes of locally integrable functions f with the finite norms

$$\begin{aligned} \|f\|_{WL_{p,\lambda}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))}, \\ \|f\|_{W\tilde{L}_{p,\lambda}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{WL_p(\Gamma(t,r))}, \\ \|f\|_{WL_{p,\lambda,\mu}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|f\|_{WL_p(\Gamma(t,r))}, \end{aligned}$$

respectively.

Note that

$$\begin{aligned} L_{p,0,0}(\Gamma) &= \tilde{L}_{p,0}(\Gamma) = L_{p,0}(\Gamma) = L_p(\Gamma), \\ WL_{p,0,0}(\Gamma) &= W\tilde{L}_{p,0}(\Gamma) = WL_{p,0}(\Gamma) = WL_p(\Gamma), \\ L_{p,\lambda,\lambda}(\Gamma) &= L_{p,\lambda}(\Gamma), \quad L_{p,\lambda,0}(\Gamma) = \tilde{L}_{p,\lambda}(\Gamma), \\ \|f\|_{WL_{p,\lambda,\mu}(\Gamma)} &\leq \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \quad \text{and therefore } L_{p,\lambda,\mu}(\Gamma) \subset WL_{p,\lambda,\mu}(\Gamma) \end{aligned}$$

and

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\lambda}(\Gamma), \quad \mu \leq \lambda \quad \text{and} \quad \|f\|_{L_{p,\lambda}(\Gamma)} \leq \|f\|_{L_{p,\lambda,\mu}(\Gamma)}, \quad (2.1)$$

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{p,\mu}(\Gamma), \quad \mu \leq \lambda \quad \text{and} \quad \|f\|_{L_{p,\mu}(\Gamma)} \leq \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \quad (2.2)$$

$$\tilde{L}_{p,\lambda}(\Gamma) \subset_{\succ} L_p(\Gamma) \quad \text{and} \quad \|f\|_{L_p(\Gamma)} \leq \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)}$$

and if $\lambda < 0$ or $\lambda > 1$, then $L_{p,\lambda}(\Gamma) = \tilde{L}_{p,\lambda}(\Gamma) = WL_{p,\lambda}(\Gamma) = W\tilde{L}_{p,\lambda}(\Gamma) = \Theta$. Here $\Theta \equiv \Theta(\Gamma)$ is the set of all functions on Γ that are equivalent to 0 on Γ .

Maximal operators and potential operators in various spaces defined on Carleson curves have been widely studied by many authors (see, for example [5, 6, 17–21, 34]). In Morrey spaces defined on quasimetric measure spaces, in particular Morrey spaces $L_{p,\lambda}(\Gamma)$ defined on Carleson curves Γ . Samko [34] studied the boundedness of the maximal operator M_Γ defined by

$$M_\Gamma f(t) = \sup_{r>0} (\nu\Gamma(t, r))^{-1} \int_{\Gamma(t,r)} |f(\tau)| d\nu(\tau)$$

and proved the following:

Theorem A. [34] *Let Γ be a Carleson curve, $1 \leq p < \infty$ and $0 \leq \lambda < 1$.*

- 1) *If $1 < p < \infty$, then M_Γ is bounded from $L_{p,\lambda}(\Gamma)$ to $L_{p,\lambda}(\Gamma)$.*
- 2) *If $p = 1$, then M_Γ is bounded from $L_{1,\lambda}(\Gamma)$ to $WL_{1,\lambda}(\Gamma)$.*

Theorem B. [26] *Let Γ be a Carleson curve, $1 \leq p < \infty$ and $0 \leq \lambda < 1$.*

- 1) *If $1 < p < \infty$, then M_Γ is bounded from $\tilde{L}_{p,\lambda}(\Gamma)$ to $\tilde{L}_{p,\lambda}(\Gamma)$.*
- 2) *If $p = 1$, then M_Γ is bounded from $\tilde{L}_{1,\lambda}(\Gamma)$ to $W\tilde{L}_{1,\lambda}(\Gamma)$.*

To prove our theorems we need the following lemmas.

Lemma 2.1 *Let Γ be a Carleson curve, $1 \leq p < \infty$, $0 \leq \mu \leq \lambda \leq 1$. Then*

$$L_{p,\lambda,\mu}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\Gamma)} = \max \left\{ \|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)} \right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\Gamma)$. Then from (2.1) and (2.2) we have that $f \in L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$

and $\max \left\{ \|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)} \right\} \leq \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Let now $f \in L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$. Then

$$\begin{aligned} \|f\|_{L_{p,\lambda,\mu}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} \left([r]_1^{-\lambda} [1/r]_1^\lambda \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} \\ &= \max \left\{ \sup_{t \in \Gamma, 0 < r \leq 1} \left(r^{-\lambda} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p}, \sup_{t \in \Gamma, r > 1} \left(r^{-\mu} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} \right\} \\ &\leq \max \left\{ \|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)} \right\}. \end{aligned}$$

Therefore, $f \in L_{p,\lambda,\mu}(\Gamma)$ and the embedding $L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma) \subset_{\succ} L_{p,\lambda,\mu}(\Gamma)$ is valid.

Thus $L_{p,\lambda,\mu}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_{p,\mu}(\Gamma)$ and $\max \left\{ \|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_{p,\mu}(\Gamma)} \right\} = \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$.

Corollary 2.1 *Let Γ be a Carleson curve, $1 \leq p < \infty$, $0 \leq \lambda \leq 1$. Then*

$$\tilde{L}_{p,\lambda}(\Gamma) = L_{p,\lambda}(\Gamma) \cap L_p(\Gamma)$$

and

$$\|f\|_{\tilde{L}_{p,\lambda}(\Gamma)} = \max \left\{ \|f\|_{L_{p,\lambda}(\Gamma)}, \|f\|_{L_p(\Gamma)} \right\}.$$

Remark 2.1 If $1 \leq p < \infty$, and $\mu < 0$ or $\lambda > 1$, then

$$L_{p,\lambda,\mu}(\Gamma) = WL_{p,\lambda,\mu}(\Gamma) = \Theta(\Gamma).$$

The following statement can be proved analogously.

Lemma 2.2 *Let Γ be a Carleson curve, $1 \leq p < \infty$, $0 \leq \mu \leq \lambda \leq 1$. Then*

$$W\tilde{L}_{p,\lambda}(\Gamma) = WL_{p,\lambda}(\Gamma) \cap WL_{p,\mu}(\Gamma)$$

and

$$\|f\|_{W\tilde{L}_{p,\lambda}(\Gamma)} = \max \left\{ \|f\|_{WL_{p,\lambda}(\Gamma)}, \|f\|_{WL_{p,\mu}(\Gamma)} \right\}.$$

Lemma 2.3 *Let Γ be a Carleson curve, $1 \leq p < \infty$, $0 \leq \lambda \leq 1$ and $0 \leq \mu \leq 1$, then*

$$L_{p,1,\mu}(\Gamma) \subset_{\succ} L_{\infty}(\Gamma) \subset_{\succ} L_{p,\lambda,1}(\Gamma)$$

and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)} \leq c_0^{\frac{1}{p}} \|f\|_{L_{\infty}(\Gamma)} \leq \|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Proof. Let $f \in L_{\infty}(\Gamma)$. Then for all $t \in \Gamma$ and $0 < r \leq 1$

$$r^{-\frac{\lambda}{p}} \|f\|_{L_p(\Gamma(t,r))} \leq c_0^{\frac{1}{p}} \|f\|_{L_{\infty}(\Gamma)}, \quad 0 \leq \lambda \leq 1$$

and for all $t \in \Gamma$ and $r \geq 1$

$$r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \leq c_0^{\frac{1}{p}} \|f\|_{L_{\infty}(\Gamma)}.$$

Thus $f \in L_{p,\lambda,1}(\Gamma)$ and

$$\|f\|_{L_{p,\lambda,1}(\Gamma)} \leq c_0^{\frac{1}{p}} \|f\|_{L_{\infty}(\Gamma)}.$$

Let $f \in L_{p,1,\mu}(\Gamma)$. By the Lebesgue's differentiation theorem we have

$$\lim_{t \rightarrow 0} |\Gamma(t,r)|^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} = |f(t)| \quad \text{for a.e. } t \in \Gamma.$$

Then for a.e. $t \in \Gamma$

$$\begin{aligned} |f(t)| &= |\Gamma(t,r)|^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \\ &\leq c_0^{-\frac{1}{p}} \sup_{t \in \Gamma, 0 < r \leq 1} r^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,r))} \\ &\leq c_0^{-\frac{1}{p}} \|f\|_{L_{p,1,\mu}(\Gamma)}. \end{aligned}$$

Thus $f \in L_{\infty}(\Gamma)$ and

$$\|f\|_{L_{\infty}(\Gamma)} \leq c_0^{-\frac{1}{p}} \|f\|_{L_{p,1,\mu}(\Gamma)}.$$

Corollary 2.2 [26] *Let Γ be a Carleson curve and $1 \leq p < \infty$. Then*

$$L_{p,1}(\Gamma) = L_\infty(\Gamma)$$

and

$$\|f\|_{L_\infty(\Gamma)} \leq \|f\|_{L_{p,1}(\Gamma)} \leq c_0^{1/p} \|f\|_{L_\infty(\Gamma)}.$$

Lemma 2.4 [8] *Let Γ be a Carleson curve, $0 \leq \lambda < 1$, $0 \leq \mu < 1$, $0 \leq \alpha < 1 - \lambda$ and $0 \leq \beta < 1 - \mu$, then for $\frac{1-\lambda}{\alpha} \leq p \leq \frac{1-\mu}{\beta}$*

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{1,1-\alpha,1-\beta}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ the inequality

$$\|f\|_{L_{1,1-\alpha,1-\beta}(\Gamma)} \leq c_0^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

From Lemma 2.4 we obtain the following results.

Corollary 2.3 *Let Γ be a Carleson curve, $0 \leq \mu \leq \lambda < 1$, $0 \leq \alpha < 1 - \lambda$, then for $\frac{1-\lambda}{\alpha} \leq p \leq \frac{1-\mu}{\alpha}$*

$$L_{p,\lambda,\mu}(\Gamma) \subset_{\succ} L_{1,1-\alpha}(\Gamma)$$

and for $f \in L_{p,\lambda,\mu}(\Gamma)$ the inequality

$$\|f\|_{L_{1,1-\alpha}(\Gamma)} \leq c_0^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}$$

holds.

Corollary 2.4 [26] *Let Γ be a Carleson curve, $1 \leq p < \infty$ and $0 \leq \lambda < 1$. If $p = \frac{1-\lambda}{\alpha}$, then*

$$L_{p,\lambda}(\Gamma) \subset L_{1,1-\alpha}(\Gamma) \quad \text{and} \quad \|f\|_{L_{1,1-\alpha}(\Gamma)} \leq c_0^{1/p'} \|f\|_{L_{p,\lambda}(\Gamma)},$$

where $1/p + 1/p' = 1$.

Corollary 2.5 *Let Γ be a Carleson curve, $0 < \alpha < 1$, $0 \leq \lambda \leq 1 - \alpha$. Then for $\frac{1-\lambda}{\alpha} \leq p \leq \frac{1}{\alpha}$*

$$\tilde{L}_{p,\lambda}(\Gamma) \subset_{\succ} L_{1,1-\alpha}(\Gamma) \quad \text{and} \quad \|f\|_{L_{1,1-\alpha}(\Gamma)} \leq c_0^{1/p'} \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)}.$$

For the $0 \leq \alpha < 1$ we define the following fractional maximal functions on Γ

$$M_{\Gamma,p}^\alpha f(t) \equiv (M_\Gamma^\alpha |f|^p)^{1/p}(t) = \sup_{r>0} \left(|\Gamma(t,r)|^{-1+\alpha} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p}.$$

In the case $\alpha = 0$ we denote $M_{\Gamma,p}^0 f$ by $M_{\Gamma,p} f$ and in the case $p = 1$ we denote $M_{\Gamma,1}^\alpha f$ by $M_\Gamma^\alpha f$.

Lemma 2.5 *Let $1 \leq p < \infty$, $0 \leq \alpha < 1$ and $f \in L_{p,1-\alpha}(\Gamma)$. Then $M_{\Gamma,p}^\alpha f \in L_\infty(\Gamma)$ and*

$$\|M_{\Gamma,p}^\alpha f\|_{L_\infty(\Gamma)} = \|f\|_{L_{p,1-\alpha}(\Gamma)}.$$

Proof.

$$\|M_{\Gamma,p}^\alpha f\|_{L_\infty(\Gamma)} = \sup_{t \in \Gamma, r > 0} \left(r^{\alpha-1} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} = \|f\|_{L_{p,1-\alpha}(\Gamma)}.$$

Lemma 2.6 *Let $1 \leq p < \infty$, $0 \leq \alpha < 1$ and $f \in \tilde{L}_{p,1-\alpha}(\Gamma)$. Then $M_{\Gamma,p}^\alpha f \in L_\infty(\Gamma)$ and*

$$\|M_{\Gamma,p}^\alpha f\|_{L_\infty(\Gamma)} \leq \|f\|_{\tilde{L}_{p,1-\alpha}(\Gamma)}.$$

Proof.

$$\begin{aligned} \|M_{\Gamma,p}^\alpha f\|_{L_\infty(\Gamma)} &= \sup_{t \in \Gamma, r > 0} \left(r^{\alpha-1} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} \\ &= \sup_{t \in \Gamma, r > 0} (r^{-1}[r]_1)^{\frac{1-\alpha}{p}} \left([r]_1^{\alpha-1} \int_{\Gamma(t,r)} |f(\tau)|^p d\nu(\tau) \right)^{1/p} \\ &\leq \|f\|_{\tilde{L}_{p,1-\alpha}} \sup_{r > 0} (r^{-1}[r]_1)^{\frac{1-\alpha}{p}} \\ &= \|f\|_{\tilde{L}_{p,1-\alpha}(\Gamma)}. \end{aligned}$$

In the case $\alpha = 0$ from Lemma 2.5 for $M_{\Gamma,p}f$ the following property is valid.

Corollary 2.6 *Let $1 \leq p < \infty$ and $f \in L_\infty(\Gamma)$. Then $M_{\Gamma,p}f \in L_\infty(\Gamma)$ and*

$$\|M_{\Gamma,p}f\|_{L_\infty(\Gamma)} = \|f\|_{L_\infty(\Gamma)}.$$

In the case $p = 1$ from Corollary 2.4 and Lemma 2.5 for $M_\Gamma^\alpha f$ the following property is valid.

Corollary 2.7 *Let Γ be a Carleson curve, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1 - \alpha$ and $f \in L_{\frac{1-\lambda}{\alpha},\lambda}(\Gamma)$. Then $M_\Gamma^\alpha f \in L_\infty(\Gamma)$ and*

$$\|M_\Gamma^\alpha f\|_{L_\infty(\Gamma)} = \|f\|_{L_{1,1-\alpha}(\Gamma)} \leq c_0^{1-\frac{\alpha}{1-\lambda}} \|f\|_{L_{\frac{1-\lambda}{\alpha},\lambda}(\Gamma)}.$$

From Corollary 2.5 and Lemma 2.6 for $M_\Gamma^\alpha f$ the following property is valid.

Corollary 2.8 *Let Γ be a Carleson curve, $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1 - \alpha$ and $f \in \tilde{L}_{p,\lambda}(\Gamma)$. Then $M_\Gamma^\alpha f \in L_\infty(\Gamma)$ for $\frac{1-\lambda}{\alpha} \leq p \leq \frac{1}{\alpha}$ and*

$$\|M_\Gamma^\alpha f\|_{L_\infty(\Gamma)} = \|f\|_{L_{1,1-\alpha}(\Gamma)} \leq c_0^{1/p'} \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)}.$$

3 $L_{p,\lambda,\mu}(\Gamma)$ -boundedness of the maximal operator M_Γ

In this section we study the $L_{p,\lambda,\mu}(\Gamma)$ -boundedness of the maximal operator M_Γ .

The following Guliyev type local estimates are valid (see also [10, 11]).

Lemma 3.1 [4] *Let Γ be a Carleson curve and $1 \leq p < \infty$. If $p > 1$, then the inequality*

$$\|M_\Gamma f\|_{L_p(\Gamma(t,r))} \lesssim r^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,\tau))} \quad (3.1)$$

holds for all $f \in L_p^{\text{loc}}(\Gamma)$.

Moreover if $p = 1$, then the inequality

$$\|M_\nu f\|_{WL_1(\Gamma(t,r))} \lesssim r \sup_{\tau > 2r} \tau^{-1} \|f\|_{L_1(\Gamma(t,\tau))} \quad (3.2)$$

holds for all $f \in L_1^{\text{loc}}(\Gamma)$.

Applying Lemma 3.1, one obtains the following result.

Theorem 3.1 *Let Γ be a Carleson curve, $0 \leq \lambda < 1$ and $0 \leq \mu < 1$.*

1. If $f \in L_{1,\lambda,\mu}(\Gamma)$, then $M_\Gamma f \in WL_{1,\lambda,\mu}(\Gamma)$ and

$$\|M_\Gamma f\|_{WL_{1,\lambda,\mu}(\Gamma)} \leq C_{1,\lambda,\mu} \|f\|_{L_{1,\lambda,\mu}(\Gamma)}, \quad (3.3)$$

where $C_{1,\lambda,\mu}$ is independent of f .

2. If $f \in L_{p,\lambda,\mu}(\Gamma)$, $1 < p < \infty$, then $M_\Gamma f \in L_{p,\lambda,\mu}(\Gamma)$ and

$$\|M_\Gamma f\|_{L_{p,\lambda,\mu}(\Gamma)} \leq C_{p,\lambda,\mu} \|f\|_{L_{p,\lambda,\mu}(\Gamma)}, \quad (3.4)$$

where $C_{p,\lambda,\mu}$ depends only on p, λ and μ .

Proof. Let $p = 1$. From the inequality (3.2) we get

$$\begin{aligned} \|M_\Gamma f\|_{WL_{1,\lambda,\mu}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\lambda} [1/r]_1^\mu \|M_\Gamma f\|_{WL_1(\Gamma(t,r))} \\ &\lesssim \sup_{t \in \Gamma, r > 0} [r]_1^{-\lambda} [1/r]_1^\mu r \sup_{\tau > 2r} \tau^{-1} \|f\|_{L_1(\Gamma(t,\tau))} \\ &\lesssim \|f\|_{L_{1,\lambda,\mu}(\Gamma)} \sup_{t \in \Gamma, r > 0} [r]_1^{-\lambda} [1/r]_1^\mu r \sup_{\tau > r} \tau^{-1} [\tau]_1^\lambda [1/\tau]_1^{-\mu} \\ &= \|f\|_{L_{1,\lambda,\mu}(\Gamma)} \sup_{t \in \Gamma, r > 0} [r]_1^{1-\lambda} [1/r]_1^{\mu-1} \sup_{\tau > r} [\tau]_1^{\lambda-1} [1/\tau]_1^{1-\mu} \\ &\lesssim \|f\|_{L_{1,\lambda,\mu}(\Gamma)} \end{aligned}$$

which implies that the operator M_Γ is bounded from $L_{1,\lambda,\mu}(\Gamma)$ to $WL_{1,\lambda,\mu}(\Gamma)$.

Let $1 < p < \infty$. From the inequality (3.1) we get

$$\begin{aligned} \|M_\Gamma f\|_{L_{p,\lambda,\mu}(\Gamma)} &= \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} \|M_\Gamma f\|_{L_p(\Gamma(t,r))} \\ &\lesssim \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} r^{\frac{1}{p}} \sup_{\tau > 2r} \tau^{-\frac{1}{p}} \|f\|_{L_p(\Gamma(t,\tau))} \\ &\lesssim \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \sup_{t \in \Gamma, r > 0} [r]_1^{-\frac{\lambda}{p}} [1/r]_1^{\frac{\mu}{p}} r^{\frac{1}{p}} \sup_{\tau > r} \tau^{-\frac{1}{p}} [\tau]_1^{\frac{\lambda}{p}} [1/\tau]_1^{-\frac{\mu}{p}} \\ &= \|f\|_{L_{p,\lambda,\mu}(\Gamma)} \sup_{t \in \Gamma, r > 0} [r]_1^{\frac{1-\lambda}{p}} [1/t]_1^{\frac{\mu-1}{p}} \sup_{\tau > r} [\tau]_1^{\frac{\lambda-1}{p}} [1/\tau]_1^{\frac{1-\mu}{p}} \\ &\lesssim \|f\|_{L_{p,\lambda,\mu}(\Gamma)}, \end{aligned}$$

which implies that the operator M_Γ is bounded in $L_{p,\lambda,\mu}(\Gamma)$.

From Theorem 3.1 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

Corollary 3.1 [34] 1. If $f \in L_{1,\lambda}(\Gamma)$ and $0 \leq \lambda < 1$, then $M_\Gamma f \in WL_{1,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{WL_{1,\lambda}(\Gamma)} \leq C_{1,\lambda} \|f\|_{L_{1,\lambda}(\Gamma)},$$

where $C_{1,\lambda}$ is independent of f .

2. If $f \in L_{p,\lambda}(\Gamma)$, $1 < p < \infty$ and $0 \leq \lambda < 1$, then $M_\Gamma f \in L_{p,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{L_{p,\lambda}(\Gamma)} \leq C_{p,\lambda} \|f\|_{L_{p,\lambda}(\Gamma)},$$

where $C_{p,\lambda}$ depends only on p and λ .

Corollary 3.2 [9, Theorem 1] 1. If $f \in \tilde{L}_{1,\lambda}(\Gamma)$ and $0 \leq \lambda < 1$, then $M_\Gamma f \in W\tilde{L}_{1,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{W\tilde{L}_{1,\lambda}(\Gamma)} \leq C_{1,\lambda} \|f\|_{\tilde{L}_{1,\lambda}(\Gamma)},$$

where $C_{1,\lambda}$ is independent of f .

2. If $f \in \tilde{L}_{p,\lambda}(\Gamma)$, $1 < p < \infty$ and $0 \leq \lambda < n$, then $M_\Gamma f \in \tilde{L}_{p,\lambda}(\Gamma)$ and

$$\|M_\Gamma f\|_{\tilde{L}_{p,\lambda}(\Gamma)} \leq C_{p,\lambda} \|f\|_{\tilde{L}_{p,\lambda}(\Gamma)},$$

where $C_{p,\lambda}$ depends only on p and λ .

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References

1. Abasova, G.A., Omarova, M.N.: *Commutator of anisotropic maximal function with BMO functions on anisotropic total Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **43** (1), 3-15 (2023).
2. Abasova, G.A., Omarova, M.N.: *Corrigendum to: "Commutator of anisotropic maximal function with BMO functions on total anisotropic Morrey spaces"*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **44** (4) Mathematics, 3-4, (2024).
3. Akbulut, A., Isayev, F.A., Omarova, M.N.: *Commutator of parabolic fractional integral operators with parabolic Lipschitz functions on parabolic total Morrey-Guliyev spaces*, Azerb. J. Math. **16** (1), 287-300 (2026).
4. Armutcu, H., Eroglu, A., Isayev, F.: *Characterizations for the fractional maximal operators on Carleson curves in local generalized Morrey spaces*, Tbilisi Mathematical Journal **13** (1), 23-38 (2020).
5. Böttcher, A., Karlovich, Yu.I.: *Carleson Curves, Muckenhoupt Weights, and Toeplitz Operators*, Basel, Boston, Berlin: Birkhäuser Verlag (1997).
6. Böttcher, A., Karlovich, Yu.I.: *Toeplitz operators with PC symbols on general Carleson Jordan curves with arbitrary Muckenhoupt weights*, Trans. Amer. Math. Soc. **351**, 3143-3196 (1999).
7. Celik, S., Akbulut, A., Omarova, M.N.: *Characterizations of anisotropic Lipschitz functions via the commutators of anisotropic maximal function in total anisotropic Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci **45** (1) Mathematics, 25-37 (2025).
8. Celik, S.: *Some embeddings into total Morrey-Guliyev spaces on Carleson curves*, Proceedings of IAM **15** (1), 1-14 (2026).

9. Dadashova, I.B., Aykol, C., Cakir, Z., Serbetci, A.: *Potential operators in modified Morrey spaces defined on Carleson curves*, Trans. A. Razmadze Math. Inst. **172** (1), 15-29 (2018).
10. Guliyev, V.S.: Integral operators on function spaces on the homogeneous groups and on domains in \mathbb{R}^n , (Russian) *Doctor's degree dissertation, Moscow, Mat. Inst. Steklov*, 1-329 (1994).
11. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp. (2009)
12. V.S. Guliyev, J.J. Hasanov, Y. Zeren, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. **5** (4), 491-506 (2011).
13. Guliyev, V.S.: *Maximal commutator and commutator of maximal function on total Morrey spaces*, J. Math. Inequal., **16** (4), 1509-1524 (2022).
14. Guliyev, V.S.: *Characterizations of Lipschitz functions via the commutators of maximal function in total Morrey spaces*, Math. Meth. Appl. Sci. **47** (11), 8669-8682 (2024).
15. Guliyev, V.S., Akbulut, A., Isayev, F.A., Serbetci, A.: *Commutators of maximal function with BMO functions on total mixed Morrey spaces*, Journal of Contemporary Applied Mathematics **16** (1), 119-138 (2026).
16. Guliyev, V.S.: *Hardy-Littlewood-Sobolev inequality in total Morrey spaces on stratified Lie groups*, Appl. Comput. Math. **25** (1), 180-196 (2026).
17. Karlovich, A.Yu.: *Maximal operators on variable Lebesgue spaces with weights related to oscillations of Carleson curves*, Math. Nachr. **283** (1), 85-93 (2010).
18. Kokilashvili, V.: *Fractional integrals on curves*, (Russian) Dokl. Akad. Nauk SSSR, **305** (1), 33-35 (1989).
19. Kokilashvili, V., Samko, S.: *Sobolev theorem for potentials on Carleson curves in variable Lebesgue spaces*, Mem. Differential Equations Math. Phys. **33**, 157-158 (2004).
20. Kokilashvili, V., Meskhi, A.: *Boundedness of maximal and singular operators in Morrey spaces with variable exponent*, Arm. J. Math. (Electronic), **1** (1), 18-28 (2008).
21. Kokilashvili, V., Samko, S.: *Boundedness of maximal operators and potential operators on Carleson curves in Lebesgue spaces with variable exponent*, Acta Math. Sin. (English Ser.), **24** (11), 1775-1800 (2008).
22. Hasanov S.G.: *Multi-sublinear rough maximal operator on product Morrey and product modified Morrey spaces*, Journal of Contemporary Applied Mathematics, **4** (2), 57-65 (2014).
23. Hasanov S.G.: *Multi-sublinear rough fractional maximal operator on product modified Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **41** (1), 77-87 (2015).
24. Hasanov S.G.: *Multi-sublinear maximal operators on product total Morrey-Guliyev spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci **46** (1) Mathematics, 1-8 (2026).
25. Hasanov S.G.: *Multi-sublinear rough maximal operator on product total Morrey-Guliyev spaces*, Journal of Contemporary Applied Mathematics, **17** (1), 1-12 (2027).
26. Mamedkhanov, J.I., Dadashova, I.B.: *Some properties of the potential operators in Morrey spaces defined on Carleson curves*, Complex Var. Elliptic Equ. **55** (8-10), 937-945 (2010).
27. Mammadov, Y.Y., Guliyev, V.S., Muslimova, F.A.: *Commutator of the maximal function in total Morrey spaces for the Dunkl operator on the real line*, Azerb. J. Math. **15** (2), 85-104 (2025).
28. Mammadov, Y.Y., Guliyev, V.S., Akbulut, A., Omarova, M.N.: *Commutator of the maximal function in total Morrey spaces in the Dunkl setting*, TWMS Jour. Pure Appl. Math. **16** (1), 1-20 (2026).
29. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, 126-166 (1938).

30. Muslimova, F.A.: *Boundedness of the fractional maximal commutator in total Morrey-Guliyev spaces for the Dunkl operator on the real line*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **46** (1) Mathematics, 1-8 (2026).
31. Omarova, M.N.: *Commutators of parabolic fractional maximal operators on parabolic total Morrey spaces*, Math. Meth. Appl. Sci. **48** (11), 11037-11044 (2025).
32. Omarova, M.N.: *Commutators of anisotropic maximal operators with BMO functions on anisotropic total Morrey spaces*, Azerb. J. Math. **15** (2), 150-162 (2025).
33. Serbetci, A.: *Boundedness of the B-maximal operator in B-total Morrey-Guliyev spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **45** (4) Mathematics, 145-154 (2025).
34. Samko, N.: *Weighted Hardy and singular operators in Morrey spaces*, J. Math. Anal. Appl. **350** (1), 56-72 2009.
35. Stein, E.M.: *Harmonic analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, New Jersey (1993).
36. Xue, F.: *Commutators for the fractional maximal and sharp functions on total Morrey spaces*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **25** (2), 103-110 (2024).
37. Wei, M., Li, S., Liu, X., Hou, Y.: *Three geometric constants for total Morrey spaces*, Math. Inequal. Appl. **28** (2), 271-277 (2025).
38. Wei, M., Hou, Y., Yan, D.: *Sharp bounds for the m-linear Hardy operator on central total Morrey spaces*, Mediterr. J. Math. **22**, Paper No. 47 (2025).