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Boundedness of intrinsic square functions on total Fofana-Guliyev spaces

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Abstract. *Total Fofana-Guliyev spaces were recently introduced as a variant of Fofana spaces and are closely related to total Morrey spaces. In this paper, we discuss the boundedness of intrinsic square functions and their commutators generated by BMO-functions on these new spaces.*

Keywords. Morrey spaces, Fofana spaces, total Fofana-Guliyev spaces, intrinsic square functions, commutator

Mathematics Subject Classification (2010): 42B25, 47B38, 47B47

1 Introduction and main results

Let d be a fixed positive integer and $\mathbb{R}^d$ be d-dimensional Euclidean space equipped with the Lebesgue measure dx. For any subset E of $\mathbb{R}^d$, we denote by χ_E its characteristic function and by $|E|$ its Lebesgue measure when it is measurable. By $L^0(\mathbb{R}^d)$, we denote the complex vector space of equivalent classes (modulo equality Lebesgue almost everywhere) of Lebesgue measurable complex-valued functions on $\mathbb{R}^d$.

Let $0 < \gamma \leq 1$. we denote by $\mathcal{C}_\gamma$ the family of functions φ defined on $\mathbb{R}^d$ with support in the closed unit ball $\mathbb{B} = \{x \in \mathbb{R}^d : |x| \leq 1\}$ and vanishing integral, i.e, $\int_{\mathbb{R}^d} \varphi(x)dx = 0$, and such that for all $x, x^{'} \in \mathbb{R}^d$, $|\varphi(x) - \varphi(x^{'})| \leq |x - x^{'}|^\gamma$. For every $(x,t) \in \mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0,\infty)$, we write $\varphi_t(x) = t^{-d}\varphi(t^{-1}x)$ and $\Gamma(x)$ is the usual "cone of arperture one",

$$\Gamma(x) = \{(y,t) \in \mathbb{R}^{d+1}_+ : |x-y| < t\}.$$

Let $f \in L^1_{loc}(\mathbb{R}^d)$. We define the intrinsic square function of f (of order γ) by the formula

$$S_\gamma(f)(x) = \left[\int_{\Gamma(x)} \left(\sup_{\varphi \in \mathcal{C}_\gamma} |f * \varphi_t(y)|\right)^2 \frac{dydt}{t^{d+1}}\right]^{\frac{1}{2}}, \tag{1.1}$$

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the intrinsic Littlewood-Paley g-function $g_\gamma(f)$ by

$$g_\gamma(f)(x) = \left[\int_0^\infty \left(\sup_{\varphi\in\mathcal{C}_\gamma} |f*\varphi_t(x)|\right)^2 \frac{dt}{t}\right]^{\frac{1}{2}} \tag{1.2}$$

and the intrinsic Littlewood-Paley $g^*_{\mu,\gamma}$-function $g^*_{\mu,\gamma}(f)$ by

$$g^*_{\mu,\gamma}(f)(x) = \left[\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t+|x-y|}\right)^{\mu d} \left(\sup_{\varphi\in\mathcal{C}_\gamma} |f*\varphi_t(y)|\right)^2 \frac{dydt}{t^{d+1}}\right]^{\frac{1}{2}}. \tag{1.3}$$

Note that, the intrinsic square functions were first introduced by Wilson [24] in order to settle a conjecture proposed by Fefferman and Stein on the boundedness of the Lusin area function S from the weighted Lebesgue space $L^2_{M(v)}(\mathbb{R}^d)$ to $L^2_v(\mathbb{R}^d)$, where $0 \leq v \in L^1_{loc}(\mathbb{R}^d)$ and M denotes the Hardy-Littlewood maximal function. Later, he proved in [25] that these functions are bounded on the weighted Lebesgue spaces $L^q_w(\mathbb{R}^d)$ whenever $q \in (1,\infty)$ and $w \in \mathcal{A}_q(\mathbb{R}^d)$ (the class of Muckenhoupt weights). Therefore, many results on boundedness of intrinsic square functions have been obtained (see, for example, [4,16,17, 21–23] and the references therein). In [21], Wang established the boundedness of these operators and their commutators on weighted Morrey spaces $L^{q;\kappa}_w(\mathbb{R}^d)$ $(0 < \kappa < 1$ and $0 < q < 1$). Wang's work was later extended to various Morrey-type spaces (see [16]) and specifically to the weighted Fofana spaces $(L^q_\omega, L^p)^\alpha(\mathbb{R}^d)$ $(1 \leq q \leq \alpha \leq p \leq \infty)$ by Feuto in [4]. We recall that Fofana spaces were originally introduced by Fofana [9] in 1988 during the study of Fourier multipliers.

Let $1 \leq q,\, p \leq \infty$ and $r > 0$. For $f \in L^0(\mathbb{R}^d)$ we define

$$_r\left\|f\right\|_{q,p} := \left\|\left[\int_{\mathbb{R}^d} |f\chi_{B(y,r)}|^q(x)dx\right]^{\frac{1}{q}}\right\|_p,$$

with $\|\cdot\|_p$ denoting the usual norm of the Lebesgue space $L^p(\mathbb{R}^d)$, taken with respect to the variable y and $B(y,r) = \{x \in \mathbb{R}^d : |y-x| < r\}$. We adopt the usual convention $\frac{1}{\infty} = 0$. For $1 \leq q \leq \alpha \leq p \leq \infty$, the Fofana space $(L^q, L^p)^\alpha(\mathbb{R}^d)$ is defined as the set of all functions $f \in L^0(\mathbb{R}^d)$ satisfying $\|f\|_{q,p,\alpha} < \infty$, where

$$\left\|f\right\|_{q,p,\alpha} = \sup_{r>0} r^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\, {}_r\left\|f\right\|_{q,p}.$$

It is proved in [9] that, for $1 \leq q < \alpha$ fixed and $p \in [\alpha, \infty]$, the spaces $(L^q, L^p)^\alpha(\mathbb{R}^d)$ form a chain of distinct Banach spaces beginning with the Lebesgue space $L^\alpha(\mathbb{R}^d)$ and ending by the classical Morrey space $L^{q,\kappa}(\mathbb{R}^d) = (L^q, L^\infty)^\alpha(\mathbb{R}^d)$, where $\kappa = d(1-\frac{q}{\alpha})$. For more properties of Fofana spaces we refer to [2,5–8] and the references therein.

Note that, in recent years, Fofana spaces have enjoyed a lot of interest. Then, inspired by Guliyev's works (see [10–13]), Kpata and Nagacy [15] introduced new spaces of Fofana-type, known as total Fofana-Guliyev spaces, see also [1,14,18,20]. Let $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. The total Fofana-Guliyev space $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is defined as the set of all elements $f \in L^0(\mathbb{R}^d)$ such that $\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} < \infty$, where

$$\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \sup_{r>0} [r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})} [1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\, {}_r\left\|f\right\|_{q,p},$$

with $[r]_1 = \min\{1, r\}$, for all $r > 0$. $\left((L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d), \|\cdot\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}\right)$ is a complex Banach space (see [15]). In Section 2, we summarize basic results on total Fofana-Guliyev spaces. Nevertheless, let us mention that these spaces are generalizations of classical Fofana spaces and are exactly total Fofana spaces defined in [19]. Boundedness properties of classical operators, including the Hardy-Littlewood maximal operator and its commutator, fractional integral, and fractional maximal operators, have been obtained for $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$-spaces (see [15, 19]).

The aim of this paper is to prove that the intrinsic square functions, defined in (1.1), (1.2) and (1.3), and their corresponding commutators are bounded on $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.

Throughout this note, we will use the following abbreviation $A \lesssim B$ for the inequalities $A \leq CB$, where C is a positive constant. If $A \lesssim B$ and $B \lesssim A$, then we write $A \approx B$.

Our first result reads as follows.

Theorem 1.1 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *Then*

$$\|S_\gamma f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

Since $S_\gamma(f)$ and $g_\gamma(f)$ are pointwise comparable (see [24]), Theorem 1.1 obviously implies what follows. We omit the details.

Corollary 1.1 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *Then*

$$\|g_\gamma(f)\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

Our next result concerns the intrinsic Littlewood-Paley $g^*_{\mu,\gamma}$-function.

Theorem 1.2 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *If* $\mu > max\{q, 3\}$, *then*

$$\left\|g^*_{\mu,\gamma}(f)\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

We shall consider the commutators generated by a function $b \in L^1_{loc}(\mathbb{R}^d)$ and intrinsic square functions, which are defined by the following expressions. For all $x \in \mathbb{R}^d$,

$$[b, S_\gamma](f)(x) = \left[\int_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\gamma} \left|\int_{\mathbb{R}^d} (b(x) - b(z))\varphi_t(y - z) f(z) dz\right|^2 \frac{dydt}{t^{d+1}}\right]^{\frac{1}{2}},$$

$$[b, g_\gamma](f)(x) = \left[\int_0^\infty \sup_{\varphi \in \mathcal{C}_\gamma} \left|\int_{\mathbb{R}^d} (b(x) - b(z))\varphi_t(y - z) f(z) dz\right|^2 \frac{dt}{t}\right]^{\frac{1}{2}}$$

and

$$[b, g^*_{\mu,\gamma}](f)(x)$$
$$= \left[\int_{\mathbb{R}^{d+1}_+} \left(\frac{t}{t + |x - y|}\right)^{\mu d} \sup_{\varphi \in \mathcal{C}_\gamma} \left|\int_{\mathbb{R}^d} (b(x) - b(z))\varphi_t(y - z) f(z) dz\right|^2 \frac{dydt}{t^{d+1}}\right]^{\frac{1}{2}}.$$

Let $b \in L^1_{loc}(\mathbb{R}^d)$. b belongs to $BMO(\mathbb{R}^d)$ if $\|b\|_{BMO(\mathbb{R}^d)} < \infty$, where

$$\|b\|_{BMO(\mathbb{R}^d)} = \sup_{r>0, x \in \mathbb{R}^d} |B(x, r)|^{-1} \int_{B(x,r)} |b(y) - b_{B(x,r)}| dy,$$

with $b_{B(x,r)} = |B(x, r)|^{-1} \int_{B(x,r)} b(y) dy$.

For these commutators, we have the following boundedness properties.

Theorem 1.3 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *If* $b \in BMO(\mathbb{R}^d)$, *then*

$$\|[b, S_\gamma](f)\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

As an immediate consequence of Theorem 1.3, we obtain the following corollary. We omit the proof.

Corollary 1.2 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *If* $b \in BMO(\mathbb{R}^d)$, *then*

$$\|[b, g_\gamma](f)\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

Theorem 1.4 *Let* $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ *such that* $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. *If* $\mu > max\{q, 3\}$ *and* $b \in BMO(\mathbb{R}^d)$, *then*

$$\left\|[b, g^*_{\mu,\gamma}](f)\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d).$$

The remainder of the paper is organized as follows. In Section 2, we recall some properties of total Fofana-Guliyev spaces. Section 3 is devoted to the proofs of Theorem 1.1 and Theorem 1.2, and in the last section we give the proofs of Theorem 1.3 and Theorem 1.4.

2 Preliminary results on Total Fofana-Guliyev spaces

The propositions of this section are proved in [15].

Proposition 2.1 *Let* $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$. *Then*

$$(L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d) \hookrightarrow (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \leq \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

Proposition 2.2 *Let* $1 \leq q \leq \lambda \leq \alpha \leq p \leq \infty$. *Then*

$$(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) = (L^q, L^p)^{\alpha}(\mathbb{R}^d) \cap (L^q, L^p)^{\lambda}(\mathbb{R}^d)$$

and for $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$,

$$\|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} = \max\{\|f\|_{q,p,\alpha}, \|f\|_{q,p,\lambda}\}.$$

The result of Proposition 2.2 implies that for $1 \leq q \leq \alpha \leq p \leq \infty$, $(L^q, L^p)^{\alpha,\alpha}(\mathbb{R}^d) = (L^q, L^p)^{\alpha}(\mathbb{R}^d)$. So Total Fofana-Guliyev spaces are generalizations of classical Fofana spaces. We recall that for $\alpha \in \{q, p\}$, $(L^q, L^p)^{\alpha}(\mathbb{R}^d) = L^{\alpha}(\mathbb{R}^d)$ (see [9]).

The family of spaces $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$ is decreasing with respect to the q power and increasing with respect to the p power. More precisely, we have the following

Proposition 2.3 *Let* $1 \leq q_1 \leq q_2 \leq \alpha, \lambda \leq p_1 \leq p_2 \leq \infty$. *Then:*
(1) $\|f\|_{(L^{q_1},L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \leq \|f\|_{(L^{q_2},L^p)^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in L^0(\mathbb{R}^d)$
and consequently, $(L^{q_2}, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^{q_1}, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$;
(2) $\|f\|_{(L^q,L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)} \lesssim \|f\|_{(L^q,L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d)}, \quad f \in L^0(\mathbb{R}^d)$
and consequently, $(L^q, L^{p_1})^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^q, L^{p_2})^{\alpha,\lambda}(\mathbb{R}^d)$.

Total Fofana-Guliyev spaces are subspaces of total Morrey spaces since, for $1 \leq q \leq \alpha, \lambda \leq p \leq \infty$, part (2) of Proposition 2.3 asserts that $(L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d) \subset (L^q, L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)$. Also, it is easy to see that for $1 \leq q \leq \alpha, \lambda < \infty$, the space $(L^q, L^\infty)^{\alpha,\lambda}(\mathbb{R}^d)$ is the total Morrey space $L^{q,d(1-\frac{q}{\alpha}),d(1-\frac{q}{\lambda})}(\mathbb{R}^d)$ defined in [12].

3 Proofs of Theorems 1.1 and 1.2

The following result will be useful in the proof of Theorem 1.1. We use it in the case where $\omega = 1$.

Theorem 3.1 *[25] Let $0 < \gamma \leq 1$, $1 < q < \infty$ and $\omega \in A_q$. Then there exists a constant $C > 0$ independent of f such that*

$$\|S_\gamma f\|_{q,\omega} \leq C \|f\|_{q,\omega} .$$

Proof of Theorem 1.1. Let $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$. Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(y,2r)}$. Since S_γ is a sublinear operator, we have

$$\left\|S_\gamma(f)\chi_{B(y,r)}\right\|_q \leq \left\|S_\gamma(f_1)\chi_{B(y,r)}\right\|_q + \left\|S_\gamma(f_2)\chi_{B(y,r)}\right\|_q . \tag{3.1}$$

For the term in f_1, Theorem 3.1 imply

$$\left\|S_\gamma(f_1)\chi_{B(y,r)}\right\|_q \lesssim \left\|f\chi_{B(y,2r)}\right\|_q . \tag{3.2}$$

Next, for the term in f_2, proceeding as in the proof of Theorem 3.1 in [4], we obtain

$$\left\|S_\gamma(f_2)\chi_{B(y,r)}\right\|_q \lesssim \sum_{k=1}^{\infty} \frac{|B(y,r)|^{\frac{1}{q}}}{|B(y,2^{k+1}r)|^{\frac{1}{q}}} \left\|f\chi_{B(y,2^{k+1}r)}\right\|_q . \tag{3.3}$$

Therefore the L^p norm of both sides of (3.1) leads to

$${}_r\left\|S_\gamma(f)\right\|_{q,p} \lesssim {}_{2r}\|f\|_{q,p} + \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}}\, {}_{2^{k+1}r}\|f\|_{q,p} .$$

On the one hand, we have

$$\begin{aligned} {}_{2r}\|f\|_{q,p} &= \frac{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\, {}_{2r}\|f\|_{q,p} \\ &\leq [2r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} . \end{aligned}$$

Hence,

$${}_{2r}\|f\|_{q,p} \lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} . \tag{3.4}$$

On the other hand,

$$\begin{aligned} &\sum_{k=1}^{\infty} 2^{-\frac{kd}{q}}\, {}_{2^{k+1}r}\|f\|_{q,p} \\ &= \sum_{i=k}^{\infty} 2^{-\frac{kd}{q}} \frac{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\, {}_{2^{k+1}r}\|f\|_{q,p} \\ &\leq \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} [2^{k+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2^{k+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \\ &\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} (2^k)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})} . \end{aligned}$$

Since $-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p}<0$ then $\sum\limits_{k=1}^{\infty}(2^k)^{d(-\frac{1}{\alpha}-\frac{1}{\lambda}+\frac{1}{q}+\frac{2}{p})}<\infty$.
Therefore,

$$\begin{aligned}&\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\ {}_{2^{k+1}r}\left\|f\right\|_{q,p} \\ &\lesssim \left[r\right]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}\left[1/r\right]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.\end{aligned}\tag{3.5}$$

From (3.4) and (3.5), we deduce that

$$_r\left\|S_\gamma(f)\right\|_{q,p}\lesssim\left[r\right]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}\left[1/r\right]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.$$

It follows that

$$\left[r\right]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}\left[1/r\right]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\ {}_r\left\|S_\gamma(f)\right\|_{q,p}\lesssim\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.\tag{3.6}$$

We obtain the desired result by taking the supremum over all $r>0$ in the left hand side of (3.6).

Remark 3.1 For $\alpha=\lambda$ and $\omega=1$, Theorem 1.1 and Corollary 1.1 were established in [4, Theorem 3.1] and [4, Corollary 3.5], without the hypothesis $\frac{1}{q}+\frac{2}{p}<\frac{2}{\alpha}$.

For the proof of Theorem 1.2, we need the following varying-aperture versions of S_γ. For $0<\gamma\leq 1$ and $\beta>0$, we define $S_{\gamma,\beta}$ by

$$S_{\gamma,\beta}(f)(x)=\left[\int_{\varGamma_\beta(x)}\left(\sup_{\varphi\in\mathcal{C}_\gamma}|f*\varphi_t(y)|\right)^2\frac{dydt}{t^{d+1}}\right]^{\frac{1}{2}},$$

where for $x\in\mathbb{R}^d$ and $\varGamma_\beta(x)=\{(y,t)\in\mathbb{R}_+^{d+1}:|x-y|<\beta t\}$.
The following lemma will be useful for proving Theorem 1.2. We use it in the case $\omega=1$.

Lemma 3.1 *[21] Let $0<\gamma\leq 1$, $1<q<\infty$. Then for all non negative integer j, $S_{\gamma,2^j}$ is bounded on $L^q_\omega(\mathbb{R}^d)$. Moreover*

$$\|S_{\gamma,2^j}(f)\|_{q,\omega}\lesssim(2^{dj}+2^{\frac{djq}{2}})\|f\|_{q,\omega}.$$

Proof of Theorem 1.2. Let $0<\gamma\leq 1$, $1<q\leq\alpha,\lambda<p<\infty$ such that $\frac{1}{q}+\frac{2}{p}<\frac{1}{\alpha}+\frac{1}{\lambda}$. We suppose that $\mu>max\{q,3\}$ and $f\in(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
For $g^*_{\mu,\gamma}$, by the definition, we know that, for all $x\in\mathbb{R}^d$,

$$\begin{aligned}&[g^*_{\mu,\gamma}(f)(x)]^2\\ &=\int_0^\infty\int_{|x-y|<t}\left(\frac{t}{t+|x-y|}\right)^{\mu d}\left(\sup_{\varphi\in\mathcal{C}_\gamma}|f*\varphi_t(y)|\right)^2\frac{dydt}{t^{d+1}}\\ &+\sum_{j=1}^{\infty}\int_0^\infty\int_{2^{j-1}t\leq|x-y|<2^jt}\left(\frac{t}{t+|x-y|}\right)^{\mu d}\left(\sup_{\varphi\in\mathcal{C}_\gamma}|f*\varphi_t(y)|\right)^2\frac{dydt}{t^{d+1}}\\ &\lesssim[S_\gamma(f)(x)]^2+\sum_{j=1}^{\infty}2^{-j\mu d}[S_{\gamma,2^j}(f)(x)]^2.\end{aligned}$$

Thus, for all $x \in \mathbb{R}^d$ it holds true that

$$g^*_{\mu,\gamma}(f)(x) \lesssim S_\gamma(f)(x) + \sum_{j=1}^{\infty} 2^{-j\mu d/2} S_{\gamma,2^j}(f)(x). \tag{3.7}$$

Let $r > 0$. It follows that

$${}_r\left\|g^*_{\mu,\gamma}(f)\right\|_{q,p} \leq {}_r\left\|S_\gamma(f)\right\|_{q,p} + \sum_{j=1}^{\infty} 2^{-j\mu d/2}\, {}_r\left\|S_{\gamma,2^j}(f)\right\|_{q,p}. \tag{3.8}$$

The first term on the right can be controlled in this way.

$$\begin{aligned} {}_r\left\|S_\gamma(f)\right\|_{q,p} &= \frac{[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}\, {}_r\left\|S_\gamma(f)\right\|_{q,p} \\ &\leq [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \left\|S_\gamma(f)\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \end{aligned}$$

By Theorem 1.1, we have

$$\begin{aligned} &{}_r\left\|S_\gamma(f)\right\|_{q,p} \\ &\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}. \end{aligned} \tag{3.9}$$

Let $j \in \mathbb{N}^*$ and $y \in \mathbb{R}^d$. Let us estimate now ${}_r\left\|S_{\gamma,2^j}(f)\right\|_{q,p} = \left\|\left\|S_{\gamma,2^j}(f)\chi_{B(y,r)}\right\|_q\right\|_p$.
We proceed as in the proof of Theorem 1.1. So, for $f = f_1 + f_2$ with $f_1 = f\chi_{B(y,2r)}$, we have

$$\begin{aligned} &\left\|S_{\gamma,2^j}(f)\chi_{B(y,r)}\right\|_q \\ &\leq \left\|S_{\gamma,2^j}(f_1)\chi_{B(y,r)}\right\|_q + \left\|S_{\gamma,2^j}(f_2)\chi_{B(y,r)}\right\|_q. \end{aligned} \tag{3.10}$$

By applying Lemma 3.1, we get

$$\left\|S_{\gamma,2^j}(f_1)\chi_{B(y,r)}\right\|_q \lesssim (2^{dj} + 2^{\frac{djq}{2}}) \left\|f\chi_{B(y,2r)}\right\|_q. \tag{3.11}$$

We now estimate $\left\|S_{\gamma,2^j}(f_2)\chi_{B(y,r)}\right\|_q$. By proceeding as in the proof of Theorem 3.2 in [4] and by using Hölder's inequality, we have

$$\begin{aligned} &\left|S_{\gamma,2^j}(f_2)(x)\right| \\ &\lesssim 2^{3jd/2} \sum_{k=1}^{\infty} \left|B(y,2^{k+1}r)\right|^{-\frac{1}{q}} \left\|f\chi_{B(y,2^{k+1}r)}\right\|_q \end{aligned} \tag{3.12}$$

for all $x \in B(y,r)$. Thus, taking the $L^q(B(y,r))$-norm of both sides of the above estimation leads to

$$\left\|S_{\gamma,2^j}(f_2)\chi_{B(y,r)}\right\|_q \lesssim 2^{3jd/2} \sum_{k=1}^{\infty} 2^{-\frac{kd}{q}} \left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \tag{3.13}$$

Taking estimates (3.11) and (3.13) in (3.10), we get

$$\begin{aligned}\left\|S_{\gamma,2^j}(f)\chi_{B(y,r)}\right\|_q &\lesssim (2^{dj}+2^{\frac{djq}{2}})\left\|f\chi_{B(y,2r)}\right\|_q \\ &\quad + 2^{3jd/2}\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q ,\end{aligned}$$

so that the $L^p(\mathbb{R}^d)$-norms of both sides leads to

$$_r\left\|S_{\gamma,2^j}(f)\right\|_{q,p} \lesssim (2^{dj}+2^{\frac{djq}{2}})\,{}_{2r}\left\|f\right\|_{q,p} + 2^{3jd/2}\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\,{}_{2^{k+1}r}\left\|f\right\|_{q,p} .$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}&_r\left\|S_{\gamma,2^j}(f)\right\|_{q,p} \\ &\lesssim (2^{dj}+2^{\frac{djq}{2}}+2^{3jd/2})[r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .\end{aligned} \tag{3.14}$$

Taking estimates (3.9) and (3.14) in (3.8), we have

$$\begin{aligned}&_r\left\|g^*_{\mu,\gamma}(f)\right\|_{q,p} \\ &\lesssim \left(1+\sum_{j=1}^{\infty}2^{-j\mu d/2}(2^{dj}+2^{\frac{djq}{2}}+2^{3jd/2})\right) \\ &\quad \times [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} ,\end{aligned}$$

where the convergence of the series is due to the fact that $\mu > max\{q,3\}$.
It follows that

$$\begin{aligned}&[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}\,{}_r\left\|g^*_{\mu,\gamma}(f)\right\|_{q,p} \\ &\lesssim \left(1+\sum_{j=1}^{\infty}2^{-j\mu d/2}(2^{dj}+2^{jqd/2}+2^{3jd/2})\right)\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .\end{aligned}$$

We end the proof by taking the supremum over all $r > 0$.

Remark 3.2 Note that Theorem 1.2 in the case $\alpha = \lambda$ was established in [4, Theorem 3.2] for $\omega = 1$. However, the proof given in [4] does not use the hypothesis $\frac{1}{q}+\frac{2}{p}<\frac{2}{\alpha}$.

4 Proofs of Theorem 1.3 and Theorem 1.4.

Let b be a locally integrable function. It is proved in [3] that if $b \in BMO(\mathbb{R}^d)$, then for every $1 < q < \infty$, we have

$$\begin{aligned}&\|b\|_{BMO(\mathbb{R}^d)} \\ &\approx \sup_{r>0,x\in\mathbb{R}^d}\left(|B(x,r)|^{-1}\int_{B(x,r)}|b(y)-b_{B(x,r)}|^q dy\right)^{\frac{1}{q}} .\end{aligned} \tag{4.1}$$

The following lemma, proved in [16], will be very useful for the following theorem proofs. We use it in the case $\omega = 1$.

Lemma 4.1 *[21, Theorem 3.1]*
Let $0 < \gamma \leq 1$, $1 < q < \infty$. Then the commutators $[b, S_\gamma]$ and $[b, g^*_{\mu,\gamma}]$ are bounded on $L^q_\omega(\mathbb{R}^d)$ whenever $b \in BMO(\mathbb{R}^d)$.

Proof of Theorem 1.3. Let $0 < \gamma \leq 1, 1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q}+\frac{2}{p} < \frac{1}{\alpha}+\frac{1}{\lambda}$. We suppose that $b \in BMO(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(y,2r)}$.
By definition of $[b, S_\gamma](f)$, we have

$$
{}_r\left\|[b, S_\gamma](f)\right\|_{q,p} \leq {}_r\left\|[b, S_\gamma](f_1)\right\|_{q,p} + {}_r\left\|[b, S_\gamma](f_2)\right\|_{q,p}. \tag{4.2}
$$

For the term in f_1, we have

$$
\left\|[b, S_\gamma](f_1)\chi_{B(y,r)}\right\|_q \lesssim \left\|f\chi_{B(y,2r)}\right\|_q \tag{4.3}
$$

as an immediate consequence of the boundedness of Lemma 4.1. Then, taking the $L^p(\mathbb{R}^d)$-norm on both sides of the previous inequality and taking into consideration (3.4), we obtain

$$
\begin{aligned}
&{}_r\left\|[b, S_\gamma](f_1)\right\|_{q,p} \\
&\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})}\left\|f\right\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)}.
\end{aligned} \tag{4.4}
$$

Next, we turn to estimate ${}_r\left\|[b, S_\gamma](f_2)\right\|_{q,p}$. Let $x \in \mathbb{R}^d$. For $(u,t) \in \Gamma(x)$, we have

$$
\begin{aligned}
&\sup_{\varphi \in \mathcal{C}_\gamma}\left|\int_{\mathbb{R}^d}(b(x)-b(z))\varphi_t(u-z)f_2(z)dz\right| \\
&\leq |b(x)-b_{B(y,r)}|\sup_{\varphi \in \mathcal{C}_\gamma}|f_2 * \varphi_t(u)| + \sup_{\varphi \in \mathcal{C}_\gamma}|[(b-b_{B(y,r)})f_2] * \varphi_t(u)|
\end{aligned}
$$

so that the $L^2(\Gamma(x), \frac{dudt}{t^{d+1}})$-norm of both sides leads to

$$
\begin{aligned}
&|[b, S_\gamma](f_2)(x)| \\
&\leq |b(x)-b_{B(y,r)}|S_\gamma(f_2)(x) \\
&+ \left[\int_{\Gamma(x)}\left(\sup_{\varphi \in \mathcal{C}_\gamma}|[(b-b_{B(y,r)})f_2] * \varphi_t(u)|\right)^2\frac{dudt}{t^{d+1}}\right]^{\frac{1}{2}} \\
&= I_1(x) + I_2(x).
\end{aligned} \tag{4.5}
$$

We take $x \in B(y,r)$. Following the ideas of the proof of Theorem 3.1 in [4] and by using Hölder's inequality, we have

$$
|S_\gamma(f_2)(x)| \lesssim \sum_{k=1}^{\infty}|B(y, 2^{k+1}r)|^{-\frac{1}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \tag{4.6}
$$

Taking the $L^q(B(y,r))$-norm of I_1 and taking into account (4.1), we obtain

$$
\begin{aligned}
&\||b-b_{B(y,r)}|S_\gamma(f_2)\chi_{B(y,r)}\|_q \\
&\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q.
\end{aligned} \tag{4.7}
$$

On the other hand, by proceeding as in the proof of Theorem 3.3 in [4], we have

$$\begin{aligned}
I_2(x) &\lesssim \sum_{k=1}^{\infty}|B(y,2^{k+1}r)|^{-1}\int_{B(y,2^{k+1}r)\backslash B(y,2^k r)}|b(z)-b_{B(y,r)}||f(z)|dz\\
&\lesssim \sum_{k=1}^{\infty}|B(y,2^{k+1}r)|^{-1}\int_{B(y,2^{k+1}r)}|b(z)-b_{B(y,2^{k+1}r)}||f(z)|dz\\
&\quad+\sum_{k=1}^{\infty}\frac{|b_{B(y,2^{k+1}r)}-b_{B(y,r)}|}{|B(y,2^{k+1}r)|}\int_{B(y,2^{k+1}r)}|f(z)|dz\\
&:= J_1(x)+J_2(x)
\end{aligned}$$

for all $x\in B(y,r)$. Applying Hölder's inequality and using (4.1), we have

$$J_1(x)\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}|B(y,2^{k+1}r)|^{-\frac{1}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q$$

for all $x\in B(y,r)$, and the $L^q(B(y,r))$-norm of both sides leads to

$$\|J_1\chi_{B(y,r)}\|_q\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \tag{4.8}$$

For J_2, since $b\in BMO(\mathbb{R}^d)$, we have

$$|b_{B(y,2^{k+1}r)}-b_{B(y,r)}|\lesssim (k+1)\|b\|_{BMO(\mathbb{R}^d)}.$$

It follows that

$$\begin{aligned}
J_2(x) &\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+1)}{|B(y,2^{k+1}r)|}\int_{B(y,2^{k+1}r)\backslash B(y,2^k r)}|f(z)|dz\\
&\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+1)}{|B(y,2^{k+1}r)|^{\frac{1}{q}}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q
\end{aligned}$$

according to Hölder's inequality.
Hence,

$$\|J_2\chi_{B(y,r)}\|_q\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+1)}{2^{\frac{kd}{q}}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \tag{4.9}$$

Taking the $L^q(B(y,r))$-norm on both sides of (4.5) and taking into account inequalities (4.7), (4.8) and (4.9), we obtain

$$\|S_\gamma(f_2)\chi_{B(y,r)}\|_q\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+3)}{2^{\frac{kd}{q}}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q.$$

Therefore, the $L^p(\mathbb{R}^d)$-norm on both sides of the previous inequality, gives

$$\begin{aligned}
&{}_r\left\|[b,S_\gamma](f_2)\right\|_{q,p} \qquad (4.10)\\
&\lesssim \|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+3)}{2^{\frac{kd}{q}}}\,{}_{2^{k+1}r}\left\|f\right\|_{q,p}.
\end{aligned}$$

We have,

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \, {}_{2^{k+1}r} \|f\|_{q,p} \\
&= \sum_{i=k}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \frac{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}}{[2^{k+1}r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/2^{k+1}r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})}} \, {}_{2^{k+1}r} \|f\|_{q,p} \\
&\leq \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} [2^{k+1}r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/2^{k+1}r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \\
&\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{k+3}{2^{kd(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q}-\frac{2}{p})}}.
\end{aligned}
$$

Since the series $\sum\limits_{k=1}^{\infty} \frac{k+3}{2^{kd(\frac{1}{\alpha}+\frac{1}{\lambda}-\frac{1}{q}-\frac{2}{p})}}$ converges then inequality (4.10) becomes

$$
\begin{aligned}
&{}_r \|[b, S_\gamma](f_2)\|_{q,p} \\
&\lesssim \|b\|_{BMO(\mathbb{R}^d)} [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .
\end{aligned} \tag{4.11}
$$

Taking estimates (4.4) and (4.11) in (4.2), we obtain

$$
\begin{aligned}
{}_r \|[b, S_\gamma](f)\|_{q,p} \lesssim\ & (1 + \|b\|_{BMO(\mathbb{R}^d)}) \\
& \times [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&[r]_1^{d(\frac{1}{\alpha}-\frac{1}{q}-\frac{1}{p})}[1/r]_1^{d(-\frac{1}{\lambda}+\frac{1}{q}+\frac{1}{p})} \, {}_r \|[b, S_\gamma](f)\|_{q,p} \\
&\lesssim (1 + \|b\|_{BMO(\mathbb{R}^d)}) \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .
\end{aligned} \tag{4.12}
$$

We obtain the desired result by taking the supremum over all $r > 0$ in the left hand side of (4.12).

Proof of Theorem 1.4. [Proof of Theorem 1.4] Let $0 < \gamma \leq 1$, $1 < q \leq \alpha, \lambda < p < \infty$ such that $\frac{1}{q} + \frac{2}{p} < \frac{1}{\alpha} + \frac{1}{\lambda}$. We suppose that $\mu > max\{q, 3\}$, $b \in BMO(\mathbb{R}^d)$ and $f \in (L^q, L^p)^{\alpha,\lambda}(\mathbb{R}^d)$.
Let $r > 0$ and $y \in \mathbb{R}^d$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{B(y,2r)}$.
By definition of $[b, g^*_{\mu,\gamma}](f)$ and norm ${}_r \|\cdot\|_{q,p}$, we have

$$
\begin{aligned}
&{}_r \left\|[b, g^*_{\mu,\gamma}](f)\right\|_{q,p} \\
&\lesssim {}_r \left\|[b, g^*_{\mu,\gamma}](f_1)\right\|_{q,p} + {}_r \left\|[b, g^*_{\mu,\gamma}](f_2)\right\|_{q,p} .
\end{aligned} \tag{4.13}
$$

By proceeding as in the proof of Theorem 1.3, we get the following estimation for the term in f_1,

$$
\begin{aligned}
&{}_r \left\|[b, g^*_{\mu,\gamma}](f_1)\right\|_{q,p} \\
&\lesssim [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})}[1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} .
\end{aligned} \tag{4.14}
$$

Next, we turn to estimate ${}_r\left\|[b,g^*_{\mu,\gamma}](f_2)\right\|_{q,p}$. For $x\in B(y,r)$, we have

$$\begin{aligned}
&|[b,g^*_{\mu,\gamma}](f_2)(x)| &(4.15)\\
\leq\ &|b(x)-b_{B(y,r)}|g^*_{\mu,\gamma}(f_2)(x)\\
+\ &\left[\int_{\mathbb{R}^{d+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\mu d}\left(\sup_{\varphi\in\mathcal{C}_\gamma}|[(b-b_{B(y,r)})f_2]*\varphi_t(u)|\right)^2\frac{dudt}{t^{d+1}}\right]^{\frac{1}{2}}\\
:=\ &A(x)+B(x).
\end{aligned}$$

For any $x\in B(y,r)$, by using inequalities (3.7), (3.12), (4.6) and the fact that $\mu>3$, we conclude that

$$g^*_{\mu,\gamma}(f_2)(x)\lesssim\sum_{k=1}^{\infty}|B(y,2^{k+1}r)|^{-\frac{1}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q,$$

which further implies that

$$\|A\chi_{B(y,r)}\|_q\lesssim\|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}2^{-\frac{kd}{q}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \qquad (4.16)$$

On the other hand, by proceeding as in the proof of [16, Theorem 3.10], we have

$$B(x)\ \lesssim\ \sum_{j=0}^{\infty}2^{-j\mu d/2}I_j(x) \qquad (4.17)$$

for all $x\in B(y,r)$, where

$$\begin{aligned}
I_j(x)\ &\lesssim\ 2^{3jd/2}\sum_{k=1}^{\infty}|B(y,2^{k+1}r)|^{-1}\int_{B(y,2^{k+1}r)}|b(z)-b_{B(y,2^{k+1}r)}||f(z)|dz\\
&\quad+2^{3jd/2}\sum_{k=1}^{\infty}\frac{|b_{B(y,2^{k+1}r)}-b_{B(y,r)}|}{|B(y,2^{k+1}r)|}\int_{B(y,2^{k+1}r)}|f(z)|dz\\
&\lesssim\ 2^{3jd/2}(J_1(x)+J_2(x)).
\end{aligned}$$

It follows that

$$\|B\chi_{B(y,r)}\|_q\lesssim\sum_{j=0}^{\infty}2^{(-\mu+3)jd/2}\left(\|J_1\chi_{B(y,r)}\|_q+\|J_2\chi_{B(y,r)}\|_q\right).$$

Taking estimates (4.8) and (4.9) in the previous inequality and taking into account $\mu>3$, we obtain

$$\|B\chi_{B(y,r)}\|_q\lesssim\|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+2)}{2^{\frac{kd}{q}}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q. \qquad (4.18)$$

Next, taking the $L^q(B(y,r))$-norm on both sides of (4.15) and taking into account inequalities (4.16) and (4.18), we obtain

$$\|[b,g^*_{\mu,\gamma}](f_2)\chi_{B(y,r)}\|_q\lesssim\|b\|_{BMO(\mathbb{R}^d)}\sum_{k=1}^{\infty}\frac{(k+3)}{2^{\frac{kd}{q}}}\left\|f\chi_{B(y,2^{k+1}r)}\right\|_q.$$

Therefore, the $L^p(\mathbb{R}^d)$-norm on both sides of the previous inequality, gives

$$ {}_r\left\|[b, g^*_{\mu,\gamma}](f_2)\right\|_{q,p} \lesssim \|b\|_{BMO(\mathbb{R}^d)} \sum_{k=1}^{\infty} \frac{(k+3)}{2^{\frac{kd}{q}}} \, {}_{2^{k+1}r}\|f\|_{q,p} . $$

Using the same arguments as in proof in Theorem 1.3, we obtain

$$ \begin{aligned} & {}_r\left\|[b, g^*_{\mu,\gamma}](f_2)\right\|_{q,p} \\ & \lesssim \|b\|_{BMO(\mathbb{R}^d)} \, [r]_1^{d(-\frac{1}{\alpha}+\frac{1}{q}+\frac{1}{p})} [1/r]_1^{d(\frac{1}{\lambda}-\frac{1}{q}-\frac{1}{p})} \|f\|_{(L^q,L^p)^{\alpha,\lambda}(\mathbb{R}^d)} , \end{aligned} \tag{4.19} $$

which, combined with (4.14), completes the proof of Theorem 1.4.

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