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Ricci bi-conformal vector fields on tangent bundle TM with respect to some lift metrics

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Abstract. *Consider TM as the tangent bundle of a (pseudo-) Riemannian manifold M, endowed with certain lift metrics. This paper aims to determine the necessary and sufficient conditions under which the tangent bundle TM admits a Ricci bi-conformal vector field with respect to these lift metrics.*

Keywords. Ricci bi-conformal vector field, Sasaki metric, synectic lift metric, tangent bundle.

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1 Introduction

Killing and conformal vector fields constitute fundamental infinitesimal transformations on a Riemannian manifold (M, g), frequently employed in the analysis of symmetries and conserved quantities in physical systems. These vector fields play a crucial role in the classification of solutions to Einsteins field equations. Specifically, Killing vector fields correspond to isometric deformations, with the isometry being represented by a Killing vector field that determines the directions along which infinitesimal transformations preserve the metric, thereby maintaining the underlying spacetime structure. A vector field V is classified as a Killing vector field if the Lie derivative of the metric tensor g with respect to V vanishes:

$$L_V g = 0.$$

Furthermore, if there exists a scalar function ψ such that $L_V g = 2\psi g$, then V is termed a conformal vector field. More generally, a conformal vector field generates a conformal transformation of the metric, meaning that the Lie derivative of g with respect to V remains proportional to g itself. Conformal vector fields have broad applications in kinematics, geometry, and dynamical systems, as they preserve fundamental structures in physical theories, including Maxwells equations and light cone structures [10].

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A natural extension of conformal vector fields is provided by Kerr-Schild vector fields [3], which satisfy the conditions

$$L_V g = \alpha(\rho \otimes \rho), \quad L_V \rho = \beta\rho,$$

where ρ is a null 1-form field, and α, β are smooth functions on (M, g). The generalized Kerr-Schild vector field is further defined by

$$L_V g = \alpha g + \beta(\rho \otimes \rho), \quad L_V \rho = \gamma\rho,$$

where α, β, γ are smooth functions on (M, g). These vector fields, which exhibit a more intricate structure than classical Killing or conformal fields, have been instrumental in deriving exact solutions to the vacuum Einstein field equations.

Analogously, bi-conformal transformations have been introduced as a generalization of conformal transformations by Garca-Parrado and Senovilla [5]. A bi-conformal vector field is a vector field on the tangent bundle TM that induces conformal transformations with respect to two distinct metrics on TM, denoted by g and h. Such a vector field V satisfies the coupled system

$$L_V g = \alpha g + \beta h, \quad L_V h = \alpha h + \beta g,$$

where L_V represents the Lie derivative along V, and the functions α, β, referred to as symmetry gauges, play a role analogous to the function ψ in the definition of conformal vector fields. Moreover, h is a symmetric square root of g, satisfying $h_{ik}h^k_j = g_{ij}$. Bi-conformal vector fields provide insights into the interplay between different metrics on TM, thereby contributing to the geometric analysis of the tangent bundle. These structures extend the framework of conformal mappings and are particularly relevant for studying transformations that preserve geometric structures across multiple metrics. In certain physical models, including mechanics and general relativity, tangent bundles equipped with multiple metrics serve as models for phase spaces or extended geometric structures. Bi-conformal vector fields describe transformations that preserve these dual structures.

Subsequently, De et al. [4] introduced the concept of Ricci bi-conformal vector fields by considering the symmetric tensor h as the Ricci tensor R. A vector field V is termed a Ricci bi-conformal vector field if it satisfies the conditions

$$(L_V g)(\xi_1, \xi_2) = \alpha g(\xi_1, \xi_2) + \beta R(\xi_1, \xi_2), \tag{1.1}$$

$$(L_V R)(\xi_1, \xi_2) = \alpha R(\xi_1, \xi_2) + \beta g(\xi_1, \xi_2), \tag{1.2}$$

for arbitrary vector fields ξ_1 and ξ_2, where R denotes the Ricci tensor of (M, g), and α, β are smooth functions.

Motivated by recent advancements in the study of Ricci bi-conformal vector fields (see [2,4,11]), the present research investigates the necessary and sufficient conditions for a vector field on the tangent bundle TM to be a Ricci bi-conformal vector field with respect to specific lift metrics.

2 Preliminaries

2.1 The tangent bundle

Consider an $n-$dimensional differentiable manifold M. The tangent bundle TM of the manifold M is defined as:

$$TM = \bigcup_{P \in M} T_P M,$$

where $T_P M$ denotes the tangent space of M at P. Let us choose a local coordinate system $\{U, x^h\}$ within M, and use Cartesian coordinates (y^h) in each tangent space $T_P M$ at a point

$P \in M$. These Cartesian coordinates are established using the natural basis $\left\{\frac{\partial}{\partial x^h} \mid_P\right\}$.With this setup, we can define a local coordinate system in TM denoted as $\{\pi^{-1}(U), x^h, y^h\}$. Here, π represents the natural projection function defined as: $\pi : TM \longmapsto M$, and P stands for an arbitrary point belonging to U. Moreover, the coordinate system (x^h, y^h) is referred to as the induced coordinates on $\pi^{-1}(U)$, which comes from the original coordinate system $\{U, x^h\}$ within M.

Given a torsion-free linear connection ∇ on M, we can introduce on each induced coordinate neighborhood $\pi^{-1}(U)$ a frame field which is very useful in our computation. It is called the adapted frame on $\pi^{-1}(U)$ and consists of the following $2n$ linearly independent vector fields $\{E_\beta\} = \left\{E_j, E_{\overline{j}}\right\}$ given by

$$E_j = \partial_j - y^s \Gamma^h_{sj} \partial_{\overline{h}},\ E_{\overline{j}} = \partial_{\overline{j}},$$

where $\partial_h = \frac{\partial}{\partial x^h}$.

Lemma 2.1 *The Lie brackets of the adapted frame of TM satisfy the following identities:*

$$\begin{aligned}
[E_j, E_i] &= y^b R_{ijb}^{\ \ \ a} E_{\overline{a}},\\
[E_j, E_{\overline{i}}] &= \Gamma^a_{ji} E_{\overline{a}},\\
\left[E_{\overline{j}}, E_{\overline{i}}\right] &= 0,
\end{aligned}$$

where $R_{ijb}^{\ \ \ a}$ denotes the components of the curvature tensor of M [15].

A vector field $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ on TM within the adapted frame $\{E_\beta\}$ is identified as a fibre-preserving vector field if the components v^h solely depend on the variable $\left(x^h\right)$.

2.2 The Synectic lift metric

Given a pseudo-Riemannian metric g on M, another well-known classical pseudo-Riemannian metric on TM is the metric $\overline{g}$ defined by Talantova and Shirokov [12]

$$\begin{aligned}
&\overline{g}\left(X^H, Y^H\right) = a\left(X, Y\right),\\
&\overline{g}\left(X^H, Y^V\right) = \overline{g}\left(X^V, Y^H\right) = g\left(X, Y\right),\\
&\overline{g}\left(X^V, Y^V\right) = 0
\end{aligned}$$

for all vector fields X, Y on M. Here $a = (a_{ij})$ is a symmetric tensor field of type $(0,2)$ on M. The metric $\overline{g}$, called the synectic lift of the Riemannian metric g, has components:

$$\overline{g} = \left(\overline{g}_{\alpha\beta}\right) = \begin{pmatrix} a_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix} \tag{2.1}$$

with respect to the adapted frame on TM.

Lemma 2.2 *[6] The Lie derivative of $\overline{g}$ with respect to the fibre-preserving vector field $\widetilde{X}$ is given as follows:*

$$\begin{aligned}
L_{\widetilde{X}}\overline{g} = &\left\{L_V a_{ij} - 2g_{im}\left(y^b v^c R_{jcb}^{\ \ \ m} - v^{\overline{b}} \Gamma^m_{bj} - E_j\left(v^{\overline{m}}\right)\right)\right\} dx^i dx^j \\
&+2\left\{L_V g_{ij} - g_{im}\left(\nabla_j v^m - E_{\overline{j}}\left(v^{\overline{m}}\right)\right)\right\} dx^i \delta y^j,
\end{aligned} \tag{2.2}$$

where $\widetilde{X} = V^a E_a + V^{\overline{a}} E_{\overline{a}}$.

Lemma 2.3 *The components $\overline{R}_{\alpha\beta}$ of the Ricci tensor of $\overline{g}$ are as follows:*

$$\overline{R} = \left(\overline{R}_{\alpha\beta}\right) = \begin{pmatrix} 2R_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.3}$$

Also the Lie derivative of $\overline{R}_{\alpha\beta}$ with respect to the fibre-preserving vector field $\widetilde{X}$ is given by

$$L_{\widetilde{X}}\overline{R} = 2\left\{L_V R_{ij}\right\} dx^i dx^j. \tag{2.4}$$

2.3 The Sasaki metric

Given a pseudo-Riemannian metric g on M, another well-known classical pseudo-Riemannian metric on TM is the Sasaki metric. The Sasaki metric ${}^S g$ is defined by [9]

$$\begin{aligned} {}^S g\left(X^H, Y^H\right) &= g\left(X, Y\right), \\ {}^S g\left(X^H, Y^V\right) &= g\left(X^V, Y^H\right) = 0, \\ {}^S g\left(X^V, Y^V\right) &= g\left(X, Y\right) \end{aligned}$$

for all vector fields X, Y on M (see also [1]). The Sasaki metric ${}^S g$ has components:

$${}^S g = \left({}^S g_{\alpha\beta}\right) = \begin{pmatrix} g_{ij} & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{2.5}$$

with respect to the adapted frame on TM.

Lemma 2.4 *[14] The Lie derivative of ${}^S g$ with respect to the fibre-preserving vector field $\widetilde{X}$ is given as follows:*

$$\begin{aligned} L_{\widetilde{X}}{}^S g &= \left\{L_V g_{ij}\right\} dx^i dx^j - 2g_{mj}\left\{y^b v^c R_{icb}^{\ \ \ m} - v^{\overline{b}}\Gamma_{bi}^m - E_i\left(v^{\overline{m}}\right)\right\} dx^i \delta y^j \\ &\quad + \left\{L_V g_{ij} - 2g_{mj}\left(\nabla_i v^m - E_{\overline{i}}\left(v^{\overline{m}}\right)\right)\right\} \delta y^i \delta y^j. \end{aligned} \tag{2.6}$$

where $\widetilde{X} = V^a E_a + V^{\overline{a}} E_{\overline{a}}$ (see also [8,13]).

Lemma 2.5 *[7] The components ${}^s R_{\alpha\beta}$ of the Ricci tensor of ${}^S g$ are as follows:*

$$\begin{array}{ll} i) & {}^S R_{ij} = R_{ij} + \frac{1}{4} y^s y^p (R_{mis}^{\ \ \ \ h} R_{phj}^{\ \ \ \ m} + R_{msi}^{\ \ \ \ h} R_{jhp}^{\ \ \ \ m}) \\ ii) & {}^S R_{\overline{i}j} = \frac{1}{2} y^s\left(\nabla_s R_{ij} - \nabla_i R_{sj}\right) \\ iii) & {}^S R_{i\overline{j}} = \frac{1}{2} y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right) \\ iv) & {}^S R_{\overline{i}\overline{j}} = -\frac{1}{4} y^s y^p R_{ish}^{\ \ \ \ m} R_{jpm}^{\ \ \ \ h}. \end{array} \tag{2.7}$$

Also, the Lie derivative of ${}^S R_{\alpha\beta}$ with respect to the fibre-preserving vector field $\widetilde{X}$ is given by

$$\begin{aligned} L_{\widetilde{X}}{}^S R_{\alpha\beta} = \frac{1}{2}\{ & 2L_V R_{ij} + y^a y^t\left[\left(\nabla_i A_a^h\right)\left(\nabla_t R_{hj} - \nabla_h R_{tj}\right)\right] \\ & + y^a y^t\left[\left(\nabla_j A_a^h\right)\left(\nabla_t R_{hi} - \nabla_h R_{ti}\right)\right] \\ & + v^h y^a y^t\left[R_{hia}^{\ \ \ \ m}\left(\nabla_t R_{mj} - \nabla_m R_{tj}\right) + R_{hja}^{\ \ \ \ m}\left(\nabla_t R_{mi} - \nabla_m R_{ti}\right)\right]\} dx^i dx^j \\ & + \frac{1}{2}\{v^h y^t\left[\nabla_h\left(\nabla_t R_{ij} - \nabla_i R_{tj}\right)\right] + y^t\left(\nabla_j v^h\right)\left(\nabla_t R_{ih} - \nabla_i R_{th}\right) \\ & + y^t A_t^h\left(\nabla_h R_{ij} - \nabla_i R_{hj}\right) + y^t A_i^h\left(\nabla_t R_{hj} - \nabla_h R_{tj}\right) \\ & + B^h\left(\nabla_h R_{ij} - \nabla_i R_{hj}\right)\} dx^j \delta y^i. \end{aligned} \tag{2.8}$$

3 Main Results

3.1 Ricci bi-conformal vector fields with respect to synectic lift metric on $(TM,\ \overline{g})$

We will now investigate the necessary and sufficient conditions for a vector field on the tangent bundle TM to be a fibre-preserving Ricci bi-conformal vector field with respect to the synectic lift metric.

Theorem 3.1 *Let $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ be a vector field on $(TM, \overline{g})$ with respect to the adapted frame $\{E_\beta\}$. The scalar function α on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). Then $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field with respect to the synectic lift if and only if the following conditions are satisfied:*

$$\begin{aligned} i.&\ \ \beta = 0,\\ ii.&\ \ L_V R_{ij} = \alpha R_{ij},\\ iii.&\ \ v^{\overline{h}} = y^a A_a^h + B^h,\\ iv.&\ \ v^c R_{ca} = \nabla_j A_a^j,\\ v.&\ \ L_V a_{ij} + 2\left(\nabla_j B_i\right) = \alpha a_{ij},\\ vi.&\ \ L_V \Gamma_{km}^m = n \nabla_k \alpha. \end{aligned}$$

Proof. Referring to the relation (1.2), the following equation is obtain

$$\left(L_{\widetilde{X}} \overline{R}\right)(\xi_1, \xi_2) = \alpha \overline{R}(\xi_1, \xi_2) + \beta \overline{g}(\xi_1, \xi_2). \tag{3.1}$$

Using the expressions (2.4), (2.3) and (2.1) in (3.1), we write

$$2\left(L_V R_{ij}\right) dx^i dx^j = 2\alpha R_{ij} dx^i dx^j + \beta\left(a_{ij} dx^i dx^j + 2 g_{ij} dx^i \delta y^j\right).$$

From this equality, we conclude

$$2\beta g_{ij} = 0 \Rightarrow \beta = 0.$$

and

$$L_V R_{ij} = \alpha R_{ij}.$$

Using relation (1.1), we derive the equation:

$$\left(L_{\widetilde{X}} \overline{g}\right)(\xi_1, \xi_2) = \alpha \overline{g}(\xi_1, \xi_2) + \beta \overline{R}(\xi_1, \xi_2). \tag{3.2}$$

Using the expressions (2.2), (2.1) and (2.3) in (3.2), we have

$$\begin{aligned} &\alpha\left(a_{ij} dx^i dx^j + 2 g_{ij} dx^i \delta y^j\right) + 2\beta R_{ij} dx^i dx^j \\ &= 2\left\{L_V g_{ij} - g_{im}\left(\nabla_j v^m - E_{\overline{j}}\left(v^{\overline{m}}\right)\right)\right\} dx^i \delta y^j \\ &+ \left\{L_V a_{ij} - 2 g_{im}\left(y^b v^c R_{jcb}^{\ \ m} - v^{\overline{b}} \Gamma_{bj}^m - E_j\left(v^{\overline{m}}\right)\right)\right\} dx^i dx^j \end{aligned}$$

from which we get

$$L_V g_{ij} - g_{im}\left(\nabla_j v^m - E_{\overline{j}}\left(v^{\overline{m}}\right)\right) = \alpha g_{ij}, \tag{3.3}$$

and

$$\left\{L_V a_{ij} - 2 g_{im}\left(y^b v^c R_{jcb}^{\ \ m} - v^{\overline{b}} \Gamma_{bj}^m - E_j\left(v^{\overline{m}}\right)\right)\right\} = \alpha a_{ij} + 2\beta R_{ij}.$$

If the value $\beta = 0$ is used in the last equation, we have

$$\left\{L_V a_{ij} - 2g_{im}\left(y^b v^c R_{jcb}^{\ \ \ m} - v^{\overline{b}} \Gamma_{bj}^m - E_j\left(v^{\overline{m}}\right)\right)\right\} = \alpha a_{ij}. \tag{3.4}$$

Applying $E_{\overline{k}}$ to both sides of Equation (3.3), we have

$$g_{im} E_{\overline{k}} E_{\overline{j}}\left(v^{\overline{m}}\right) = E_{\overline{k}}\left(\alpha\right) g_{ij} \tag{3.5}$$

if the indices k and j are interchanged, from which we get

$$E_{\overline{k}}\left(\alpha\right) g_{ij} = E_{\overline{j}}\left(\alpha\right) g_{ik}.$$

It follows that

$$\left(n-1\right) E_{\overline{k}}\left(\alpha\right) = 0 \Rightarrow E_{\overline{k}}\alpha = 0.$$

This shows that the scalar function α on TM depends only on the variable $\left(x^h\right)$ with respect to the induced coordinates $\left(x^h, y^h\right)$. If the last equality is used in Equation (3.5), the following equation is obtained

$$v^{\overline{h}} = y^a A_a^h + B^h, \tag{3.6}$$

where A_a^h and B^h are certain functions which depend only on the variable $\left(x^h\right)$. Furthermore one can show that A_a^h and B^h are the components of a $(1,1)$ tensor field and a contravariant vector field on M, respectively. Substituting (3.6) into (3.4), we have

$$L_V a_{ij} - 2g_{im}\left(y^b v^c R_{jcb}^{\ \ \ m} - \left(y^a A_a^b + B^b\right)\Gamma_{bj}^m - \left(\partial_j - y^s \Gamma_{sj}^h \partial_{\overline{h}}\right)\left(y^a A_a^m + B^m\right)\right) = \alpha a_{ij}.$$

Further simplification yields:

$$L_V a_{ij} - 2g_{im}\left(y^a\left(v^c R_{jca}^{\ \ \ m} - \nabla_j A_a^m\right) - \nabla_j B^m\right) = \alpha a_{ij},$$

from which we conclude:

$$v^c R_{ca} = \nabla_j A_a^j, \tag{3.7}$$

and

$$L_V a_{ij} + 2\left(\nabla_j B_i\right) = \alpha a_{ij}.$$

Substituting (3.6) into (3.3), we have

$$L_V g_{ij} - \alpha g_{ij} - \nabla_j v_i + g_{im} A_j^m = 0. \tag{3.8}$$

Applying the covariant derivative ∇_k to both sides of (3.8)

$$\begin{aligned} g_{im}\left(\nabla_k A_j^m\right) &= \nabla_k\left(\alpha g_{ij} + g_{im}\nabla_j v^m - L_V g_{ij}\right) \\ &= (\nabla_k \alpha) g_{ij} + \nabla_k \nabla_j v_m - \nabla_k\left(L_V g_{ij}\right) \\ &= (\nabla_k \alpha) g_{ij} + \nabla_k \nabla_j v_i - \nabla_k\left[v^m\left(\nabla_m g_{ij}\right) + \left(\nabla_i v_j\right) + \left(\nabla_j v_i\right)\right] \\ &= (\nabla_k \alpha) g_{ij} + \nabla_k \nabla_j v_i - \nabla_k\left(\nabla_i v_j\right) \end{aligned}$$

Using the relation $L_V \Gamma_{ki}^m = \nabla_k \nabla_i v^m + R_{aki}^{\ \ \ m} v^a$, we can obtain

$$g_{im}\left(\nabla_k A_j^m\right) = (\nabla_k \alpha) g_{ij} - \left(L_V \Gamma_{ki}^m - R_{aki}^{\ \ \ m} v^a\right)$$

From the eq.(3.7), we get

$$g_{im}\left(v^a R_{akj}^{\ \ \ m}\right) = (\nabla_k \alpha) g_{ij} - \left(L_V \Gamma_{ki}^m - R_{aki}^{\ \ \ m} v^a\right) g_{mj}.$$

Contracting both sides of the last equation with g^{ij}, we have

$$L_V \Gamma_{km}^m = n\nabla_k \alpha.$$

Conversely, if B^h, A_j^h,α and β are taken so that they satisfy *(i)-(vii)*, reserving the above steps, we see that $\widetilde{X} = v^h E_h + \left(y^a A_a^h + B^h\right) E_{\overline{h}}$ is a fibre-preserving Ricci bi-conformal vector field on TM with respect to the synectic lift. This completes the proof.

Let g be a Riemannian metric of M with components g_{ij}, then we see that

$$\widetilde{g} = 2g_{ij}dx^i\delta y^j$$

is non-singular and can be regarded as pseudo-Riemannian metric on TM. The metric $\widetilde{g}$, which is called complete lift metric has components

$$\widetilde{g} = (\widetilde{g}_{\alpha\beta}) = \begin{pmatrix} 0 & g_{ij} \\ g_{ij} & 0 \end{pmatrix}$$

with respect to the adapted frame on TM. This metric is obtained by setting the symmetric $(0, 2)$-type tensor $a = (a_{ij})$ to zero in the synectic lift metric. As a direct consequence of Theorem 3.1, we derive the following result.

Corollary 3.1 *Let $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ be a vector field on $(TM, \widetilde{g})$ with respect to the adapted frame $\{E_\beta\}$. The scalar function α on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). Then $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field with respect to the complete lift metric $\widetilde{g}$ if and only if the following conditions are satisfied:*

$$\begin{aligned} i.&\ \ \beta = 0,\\ ii.&\ \ L_V R_{ij} = \alpha R_{ij},\\ iii.&\ \ v^{\overline{h}} = y^a A_a^h + B^h,\\ iv.&\ \ v^c R_{cja}^{\ \ \ m} + \nabla_j A_a^m = 0,\\ v.&\ \ \nabla_j B_i = 0,\\ vi.&\ \ L_V \Gamma_{km}^m = n\nabla_k \alpha. \end{aligned}$$

Let $\widetilde{X} = (v^h E_h + v^{\overline{h}} E_{\overline{h}})$ be a vector field on TM relative to the adapted frame $\{E_\beta\}$. Then, $\widetilde{X}$ is a vertical vector field on TM if and only if $v_h = 0$. In this case, the vector field $\widetilde{X}$ in Theorem 3.1 simplifies to $\widetilde{X} = y^a A_a^h + B^h E_h$. Therefore, from Theorem 3.1, we can draw the following conclusion.

Corollary 3.2 *Let $\widetilde{X} = y^a A_a^h + B^h E_h$ be a vertical vector field on $(TM, \overline{g})$ with respect to the adapted frame $\{E_\beta\}$. The scalar function α on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). Then $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field with respect to the synectic lift metric if and only if the following conditions are satisfied:*

$$\begin{aligned} i.&\ \ \beta = 0,\\ ii.&\ \ R_{ij} = 0, \textit{i.e., the base manifold is Ricci flat.}\\ iii.&\ \ \nabla_j A_a^j = 0,\\ iv.&\ \ 2\left(\nabla_j B_i\right) = \alpha a_{ij},\\ v.&\ \ \nabla_k \alpha = 0. \end{aligned}$$

In particular, if U is a local vector field that remains constant on each fiber T_xM such that $U = y$, then its vertical lift $U^V = y^h E_h$ is referred to as the canonical vertical vector field, also known as the Liouville vector field on TM. In this case, the vector field $\widetilde{X}$ in Theorem 3.1 reduces to $\widetilde{X} = y^h E_h$. Therefore, Theorem 3.1 gives the following conclusion.

Corollary 3.3 *Let $\widetilde{X} = y^h E_h$ be the Liouville vector field on $(TM, \overline{g})$ with respect to the adapted frame $\{E_\beta\}$. The scalar function α on TM depends only on the variables (x^h) with respect to the induced coordinates (x^h, y^h). Then $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field if and only if the following conditions are satisfied:*

$i.$ $\beta = 0$,
$ii.$ $R_{ij} = 0$, *i.e., the base manifold is Ricci flat.*
$iii.$ $a_{ij} = 0$, *i.e., the synectic lift metric reduces to the complete lift metric.*
$iv.$ $\nabla_k \alpha = 0$.

3.2 Ricci bi-conformal vector fields with respect to Sasaki metric on $\left(TM,\ {}^S g\right)$

We will now investigate the necessary and sufficient conditions for a vector field on the tangent bundle TM to be a fibre-preserving Ricci bi-conformal vector field with respect to the Sasaki metric ${}^S g$.

Theorem 3.2 *Let $\widetilde{X} = v^h E_h + v^{\overline{h}} E_{\overline{h}}$ be a vector field on $(TM,\ {}^S g)$ with respect to the adapted frame $\{E_\beta\}$ and α,β depends only on the variables (x^h). Then $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field with respect to the Sasaki metric ${}^S g$ if and only if the following conditions are satisfied:*

$i.$ $L_V R_{ij} = \alpha R_{ij}$,
$ii.$ $v^{\overline{h}} = y^a A_a^h + B^h$,
$iii.$ $\nabla_i A_s^m = v^c R_{ics}{}^m$,
$iv.$ $\nabla_i B^m = 0$,
$v.$ $R_{ish}{}^m R_{jpm}{}^h = 0$,
$vi.$ $R_{mis}{}^h R_{hpj}{}^m = 0$,
$vii.$ $L_V \Gamma_{km}^m = -\dfrac{n}{2} \nabla_k \alpha + \nabla_k \left(\nabla_m v^m\right)$,
$viii.$ $B^h \left(\nabla_h R_{ij} - \nabla_i R_{hj}\right) = 0$,
$ix.$ $L_V g_{ij} = \alpha g_{ij}$,
$x.$ $\alpha \left(\nabla_t R_{ij} - \nabla_i R_{tj}\right) = v^h \left[\nabla_h \left(\nabla_t R_{ij} - \nabla_i R_{tj}\right)\right] + \left(\nabla_j v^h\right) \left(\nabla_t R_{ih} - \nabla_i R_{th}\right)$
$+A_t^h \left(\nabla_h R_{ij} - \nabla_i R_{hj}\right) + A_i^h \left(\nabla_t R_{hj} - \nabla_h R_{tj}\right).$

Proof. We begin by considering the relation (1.1), represented as follows:

$$\left(L_{\widetilde{X}}\ {}^S g\right)(\xi_1, \xi_2) = \alpha\ {}^S g(\xi_1, \xi_2) + \beta\ {}^S R(\xi_1, \xi_2). \tag{3.9}$$

Using the expressions (2.5), (2.6) and (2.7) in (3.9) we write

$$\begin{aligned}
&\{L_V g_{ij}\} dx^i dx^j - 2g_{mj}\left\{y^s v^c R_{ics}^{\ \ \ m} - v^{\bar{b}} \Gamma_{bi}^m - E_i\left(v^{\overline{m}}\right)\right\} dx^i \delta y^j \\
&+\left\{L_V g_{ij} - 2g_{mj}\left(\nabla_i v^m - E_{\bar{i}}\left(v^{\overline{m}}\right)\right)\right\} \delta y^i \delta y^j \\
&= \alpha\left\{g_{ij} dx^i dx^j + g_{ij} \delta y^i \delta y^j\right\} + \beta\{\left(R_{ij} + \frac{1}{4} y^s y^p (R_{mis}^{\ \ \ \ h} R_{phj}^{\ \ \ \ m} + R_{msi}^{\ \ \ \ h} R_{jhp}^{\ \ \ \ m})\right) dx^i dx^j \\
&+ y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right) dx^i \delta y^j - \left(\frac{1}{4} y^s y^p R_{ish}^{\ \ \ \ m} R_{jpm}^{\ \ \ \ h}\right) \delta y^i \delta y^j\}.
\end{aligned}$$

From this equation we obtain the followings:

$$L_V g_{ij} = \alpha g_{ij} + \beta\left[R_{ij} + \frac{1}{4} y^s y^p (R_{mis}^{\ \ \ \ h} R_{phj}^{\ \ \ \ m} + R_{msi}^{\ \ \ \ h} R_{jhp}^{\ \ \ \ m})\right], \tag{3.10}$$

$$L_V g_{ij} - 2g_{mj}\left(\nabla_i v^m\right) + 2g_{mj}\left(E_{\bar{i}}\left(v^{\overline{m}}\right)\right) = \alpha g_{ij} - \frac{1}{4}\beta\left(y^s y^p R_{ish}^{\ \ \ \ m} R_{jpm}^{\ \ \ \ h}\right), \tag{3.11}$$

and

$$-2g_{mj}\left\{y^s v^c R_{ics}^{\ \ \ m} - v^{\bar{b}} \Gamma_{bi}^m - E_i\left(v^{\overline{m}}\right)\right\} = \beta y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right). \tag{3.12}$$

In equation(3.10), since $L_V g_{ij}$, αg_{ij} and βR_{ij} depend only on the variable $\left(x^h\right)$, we obtain

$$R_{mis}^{\ \ \ \ h} R_{hpj}^{\ \ \ \ m} = 0. \tag{3.13}$$

Considering equations (3.10) and (3.11) together, we get

$$\begin{aligned}
&\alpha g_{ij} + \beta\left[R_{ij} + \frac{1}{4} y^s y^p \underbrace{(R_{mis}^{\ \ \ \ h} R_{phj}^{\ \ \ \ m} + R_{msi}^{\ \ \ \ h} R_{jhp}^{\ \ \ \ m})}_{0}\right] - 2\left(\nabla_i v_j\right) + 2g_{mj}\left(E_{\bar{i}}\left(v^{\overline{m}}\right)\right) \\
&= \alpha g_{ij} - \frac{1}{4}\beta\left(y^s y^p R_{ish}^{\ \ \ \ m} R_{jpm}^{\ \ \ \ h}\right),
\end{aligned}$$

from which we write

$$\beta R_{ij} = 2\left(\nabla_i v_j\right) - 2g_{mj} E_{\bar{i}}\left(v^{\overline{m}}\right), \tag{3.14}$$

and

$$R_{ish}^{\ \ \ \ m} R_{jpm}^{\ \ \ \ h} = 0. \tag{3.15}$$

Applying $E_{\bar{k}}$ to the both sides of Equation (3.14), we obtain

$$v^{\overline{m}} = y^a A_a^m + B^m, \tag{3.16}$$

where A_a^m and B^m are certain functions which depend only on the variable $\left(x^h\right)$. Furthermore one can show that A_a^h and B^h are the components of a $(1,1)$ tensor field and a contravariant vector field on M, respectively. Substituting (3.16) into the Equation (3.12) we have

$$\begin{aligned}
&-2g_{mj}\left\{y^s v^c R_{ics}^{\ \ \ m} - \left(y^a A_a^b + B^b\right)\Gamma_{bi}^m - \left(\partial_i - y^s \Gamma_{si}^h \partial_{\bar{h}}\right)\left(y^a A_a^m + B^m\right)\right\} \\
&= \beta y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right).
\end{aligned}$$

After making the necessary adjustments to the last equation, we write

$$-2g_{mj}\left\{-y^s v^c R_{cis}^{\ \ \ m} - y^s\left(\nabla_i A_s^m\right) - \nabla_i B^m\right\} = \beta y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right),$$

from which we conclude:

$$2g_{mj}\left(v^c R_{cis}^{\ \ \ m} + \nabla_i A_s^m\right) = \beta\left(\nabla_s R_{ji} - \nabla_j R_{si}\right), \tag{3.17}$$

and

$$\nabla_i B^m = 0.$$

Substituting (3.15) and (3.16) into the Equation (3.11), we have

$$L_V g_{ij} - 2g_{mj}\left(\nabla_i v^m\right) + 2g_{mj}\left(\partial_{\bar{i}}\left(y^a A_a^m + B^m\right)\right) = \alpha g_{ij},$$

from which we get

$$L_V g_{ij} - 2g_{mj}\left(\nabla_i v^m\right) + 2g_{mj} A_i^m = \alpha g_{ij}. \tag{3.18}$$

Applying the covariant derivative ∇_k to both sides of (3.18)

$$\begin{aligned} 2g_{mj}\left(\nabla_k A_i^m\right) &= \nabla_k\ \left(\alpha g_{ij} + 2\left(\nabla_i v_j\right) - L_V g_{ij}\right) \\ &= (\nabla_k \alpha) g_{ij} + 2\left(\nabla_k \nabla_i v_j\right) - \nabla_k\ \left(L_V g_{ij}\right) \\ &= (\nabla_k \alpha) g_{ij} + 2\left(\nabla_k \nabla_i v_j\right) - \nabla_k\ \left[v^m\left(\nabla_m g_{ij}\right) + \left(\nabla_i v_j\right) + \left(\nabla_j v_i\right)\right] \\ &= (\nabla_k \alpha) g_{ij} + 2\left(\nabla_k \nabla_i v_j\right) - \nabla_k\ \left(\nabla_i v_j\right) - \nabla_k\ \left(\nabla_j v_i\right) \\ &= (\nabla_k \alpha) g_{ij} + \nabla_k\ \left(\nabla_i v_j\right) - \nabla_k\ \left(\nabla_j v_i\right). \end{aligned} \tag{3.19}$$

Also from the equation(3.17), (if $i \to k$ and $s \to i$ index is changed) we write

$$2g_{mj}\left(\nabla_k A_i^m\right) = \beta\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - 2g_{mj}\left(v^c R_{cki}^{\ \ \ m}\right). \tag{3.20}$$

Using expression (3.20) in (3.19), we obtain

$$\beta\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - 2g_{mj}\left(v^c R_{cki}^{\ \ \ m}\right) = (\nabla_k \alpha) g_{ij} + \nabla_k\left(\nabla_i v_j\right) - \nabla_k\left(\nabla_j v_i\right).$$

From this formula: $L_V \mathit{\Gamma}_{ki}^m = \nabla_k \nabla_i v^m + R_{cki}^{\ \ \ m} v^c$, we can write

$$\begin{aligned} \beta\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - 2g_{mj}\left(L_V \mathit{\Gamma}_{ki}^m - \nabla_k \nabla_i v^m\right) &= (\nabla_k \alpha) g_{ij} + \nabla_k\left(\nabla_i v_j - \nabla_j v_i\right) \\ \beta\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - 2g_{mj}\left(L_V \mathit{\Gamma}_{ki}^m\right) + 2\left(\nabla_k \nabla_i v_j\right) &= (\nabla_k \alpha) g_{ij} + \nabla_k\left(\nabla_i v_j - \nabla_j v_i\right) \\ \beta\left(\nabla_i R_{jk} - \nabla_j R_{ik}\right) - 2g_{mj}\left(L_V \mathit{\Gamma}_{ki}^m\right) &= (\nabla_k \alpha) g_{ij} - \nabla_k\left(\nabla_i v_j - \nabla_j v_i\right). \end{aligned}$$

Contracting both sides of the last equation with g^{ij}, we have

$$\begin{aligned} \beta\left(\overbrace{\nabla_i R_k^i - \nabla_j R_k^j}^{0}\right) - 2\delta_m^i\left(L_V \mathit{\Gamma}_{ki}^m\right) &= n(\nabla_k \alpha) - \nabla_k\ \left(\nabla_i v^m\right)\delta_m^i - \nabla_k\ \left(\nabla_j v^m\right)\delta_m^j \\ -2L_V \mathit{\Gamma}_{km}^m &= n(\nabla_k \alpha) - \nabla_k\ \left(\nabla_m v^m\right) - \nabla_k\ \left(\nabla_m v^m\right) \\ -2L_V \mathit{\Gamma}_{km}^m &= n(\nabla_k \alpha) - 2\nabla_k\ \left(\nabla_m v^m\right) \\ L_V \mathit{\Gamma}_{km}^m &= -\frac{n}{2}(\nabla_k \alpha) + \nabla_k\ \left(\nabla_m v^m\right). \end{aligned}$$

Furthermore using the expressions (3.13) and (3.15) in (2.7) , the components ${}^S R_{\alpha\beta}$ of the Ricci tensor of ${}^S g$ become simpler as follows:

$$\begin{aligned} &i)\ \ {}^s R_{ij} = R_{ij}, \\ &ii)\ \ {}^s R_{\bar{i}j} = \tfrac{1}{2} y^s\left(\nabla_s R_{ij} - \nabla_i R_{sj}\right), \\ &iii)\ {}^s R_{i\bar{j}} = \tfrac{1}{2} y^s\left(\nabla_s R_{ji} - \nabla_j R_{si}\right), \\ &iv)\ \ {}^s R_{\bar{i}\bar{j}} = 0. \end{aligned} \tag{3.21}$$

In line with relation (1.2), we derive the equation:

$$\left(L_{\widetilde{X}}\ {}^{S}R\right)(\xi_1,\xi_2)=\alpha\ {}^{S}R(\xi_1,\xi_2)+\beta\ {}^{S}g(\xi_1,\xi_2). \tag{3.22}$$

Using the expressions (2.5) , (2.8) and (3.21) in (3.22) we write

$$\begin{aligned}
&\{L_V R_{ij}+\frac{1}{2}y^a y^t\left[\left(\nabla_i A_a^h\right)(\nabla_t R_{hj}-\nabla_h R_{tj})+\left(\nabla_j A_a^h\right)(\nabla_t R_{hi}-\nabla_h R_{ti})\right]\\
&\frac{1}{2}v^h y^a y^t\left[R_{hia}{}^{m}(\nabla_t R_{mj}-\nabla_m R_{tj})+R_{hja}{}^{m}(\nabla_t R_{mi}-\nabla_m R_{ti})\right]\}dx^i dx^j\\
&\frac{1}{2}\{v^h y^t\left[\nabla_h(\nabla_t R_{ij}-\nabla_i R_{tj})\right]+y^t\left(\nabla_j v^h\right)(\nabla_t R_{ih}-\nabla_i R_{th})\\
&+y^t A_t^h(\nabla_h R_{ij}-\nabla_i R_{hj})+y^t A_i^h(\nabla_t R_{hj}-\nabla_h R_{tj})+B^h(\nabla_h R_{ij}-\nabla_i R_{hj})\}dx^j\delta y^i\\
&=\alpha\left\{R_{ij}dx^i dx^j+\frac{1}{2}y^t(\nabla_t R_{ij}-\nabla_i R_{tj})dx^j\delta y^i\right\}+\beta\left\{g_{ij}dx^i dx^j+g_{ij}\delta y^i\delta y^j\right\},
\end{aligned}$$

from which we get

$$\beta g_{ij}\delta y^i\delta y^j=0\Rightarrow\beta=0, \tag{3.23}$$

$$\begin{aligned}
\alpha(\nabla_t R_{ij}-\nabla_i R_{tj})&=v^h\left[\nabla_h(\nabla_t R_{ij}-\nabla_i R_{tj})\right]+\left(\nabla_j v^h\right)(\nabla_t R_{ih}-\nabla_i R_{th})\\
&+A_t^h(\nabla_h R_{ij}-\nabla_i R_{hj})+A_i^h(\nabla_t R_{hj}-\nabla_h R_{tj}),\\
&B^h(\nabla_h R_{ij}-\nabla_i R_{hj})=0,
\end{aligned}$$

and

$$\begin{aligned}
&2L_V R_{ij}+y^a y^t\left[\left(\nabla_i A_a^h\right)(\nabla_t R_{hj}-\nabla_h R_{tj})+\left(\nabla_j A_a^h\right)(\nabla_t R_{hi}-\nabla_h R_{ti})\right]\\
&+v^h y^a y^t\left[R_{hia}{}^{m}(\nabla_t R_{mj}-\nabla_m R_{tj})+R_{hja}{}^{m}(\nabla_t R_{mi}-\nabla_m R_{ti})\right]\\
&=2\alpha R_{ij}+2\beta g_{ij}.
\end{aligned} \tag{3.24}$$

Substituting (3.23) into the Equation (3.17) we get

$$v^c R_{cis}^{\ \ \ m}+\nabla_i A_s^m=0\Rightarrow\nabla_i A_s^m=v^c R_{ics}^{\ \ \ m}. \tag{3.25}$$

Using the expression (3.25) and (3.23) into the Equation (3.24) we conclude

$$\begin{aligned}
&2L_V R_{ij}+y^a y^t\left[\left(v^c R_{ica}^{\ \ \ h}\right)(\nabla_t R_{hj}-\nabla_h R_{tj})+\left(v^c R_{jca}^{\ \ \ h}\right)(\nabla_t R_{hi}-\nabla_h R_{ti})\right]\\
&+y^a y^t\left[v^h R_{hia}{}^{m}(\nabla_t R_{mj}-\nabla_m R_{tj})+v^h R_{hja}{}^{m}(\nabla_t R_{mi}-\nabla_m R_{ti})\right]\\
&=2\alpha R_{ij}+\underbrace{\beta g_{ij}}_{0}.
\end{aligned}$$

It follows that

$$L_V R_{ij}=\alpha R_{ij}.$$

Also using the expression (3.23) in (3.10) we get

$$L_V g_{ij}=\alpha g_{ij}.$$

This shows that V is a conformal vector field on M.

Conversely, if B^h, A_j^h,α and β are taken so that they satisfy $(i)-(x)$, reserving the above steps, we see that $\widetilde{X}$ is a fibre-preserving Ricci bi-conformal vector field on TM with respect to the Sasaki metric ${}^{S}g$. This completes the proof.

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