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On (p,q)-truncated exponential polynomials

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Abstract. *In this paper, we introduce the integral transform of (p,q)-truncated exponential polynomials $E_{n,p,q}(x)$ and obtain their series definition and other properties. Also, we introduce the integral transforms of higher-order (p,q)-truncated exponential polynomials $E^{\alpha}_{n,p,q}(x)$, ${}_{[2]}E_{n,p,q}(x)$ and ${}_{[m]}E^{\alpha}_{n,p,q}(x)$. Finally, certain higher-order properties of (p,q)-truncated exponential polynomials are established.*

Keywords. (p,q)-calculus, (p,q)-Hermite polynomials, (p,q)-Gamma function, (p,q)-Gould Hopper polynomials.

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1 Introduction

In the late 20^{th} century, the mathematical framework for post-quantum calculus, often labeled (p,q)-calculus was constructed. The (p,q)-calculus corresponds to the q-calculus when $p=1$. Researchers in several physics and mathematics disciplines are currently pursuing an interest in (p,q)-calculus. The (p,q)-analogues of various special polynomials were introduced and explored [2–4, 9–11, 13, 15, 16, 18, 19, 22–28, 32]

The (p,q)-analogue of a non-negative integer ν also called (p,q)-number and denoted by $[\nu]_{p,q}$ is defined as (see [18, 19, 25]):

$$[\nu]_{p,q} = \frac{p^{\nu}-q^{\nu}}{p-q}, \qquad \nu \in \mathbb{N}. \tag{1.1}$$

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The (p,q)-analogue of the factorial or for short the (p,q)-factorial is defined as (see [19, 25]):

$$[\nu]_{p,q}! = \prod_{k=1}^{\nu}[k]_{p,q}, \qquad \nu \in \mathbb{N},$$

where $[0]_{p,q}! = 1$ and $[\nu+1]_{p,q}! = [\nu+1]_{p,q}[\nu]_{p,q}!$.

The ones that follow are the (p,q)-binomial coefficients (see [25]):

$$\begin{bmatrix}\nu\\k\end{bmatrix}_{p,q} = \frac{[\nu]_{p,q}!}{[\nu-k]_{p,q}![k]_{p,q}!}. \tag{1.2}$$

The raising (p,q)-power is expressed through (see [25]):

$$(x\oplus\zeta)^n_{p,q} = \begin{cases}(x+\zeta)(px+\zeta q)...(p^{n-2}x+\zeta q^{n-2})(p^{n-1}x+\zeta q^{n-1}), & \text{if } n\geq 1,\\ 1, & \text{if } n=0.\end{cases} \tag{1.3}$$

Equation (1.3) can equivalently be expressed as (see [6])

$$(x\oplus\zeta)^n_{p,q} = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q} q^{\binom{k}{2}}p^{\binom{n-k}{2}}\zeta^k x^{n-k}. \tag{1.4}$$

For $x=0$ in equation (1.3), we can write

$$(\zeta)^n_{p,q} = \zeta^n p^{\binom{n}{2}}. \tag{1.5}$$

Also, from equation (1.3), we can write

$$(x\ominus\zeta)^n_{p,q} = \big(x\oplus(-\zeta)\big)^n_{p,q}.$$

Thus, for $x, n=1, \zeta=t$ the above equation gives:

$$(1\ominus t)^1_{p,q} = (1-t).$$

According to [18,4], the two (p,q)-exponential functions, $e_{p,q}(x)$ and $E_{p,q}(x)$, are defined by:

$$e_{p,q}(x) = \sum_{n=0}^{\infty} p^{\binom{n}{2}}\frac{x^n}{[n]_{p,q}!} \tag{1.6}$$

and

$$E_{p,q}(x) = \sum_{n=0}^{\infty} q^{\binom{n}{2}}\frac{x^n}{[n]_{p,q}!}, \tag{1.7}$$

respectively.

The Taylor's expansion of the function $f(x) = \dfrac{1}{(1-x)^n}$ has a (p,q)-analogue (see [18, 25]):

$$\frac{1}{(1\ominus t)^m_{p,q}} = \sum_{n=0}^{\infty}\begin{bmatrix}m+n-1\\n\end{bmatrix}_{p,q} t^n, \qquad |\, t\,| <1,\ m\in\mathbb{R}. \tag{1.8}$$

According to [3,18,25], the (p,q)-derivative of a function f with regard to x, denoted by $D_{p,q,x}\ f(x)$, is defined by:

$$D_{p,q,x}f(x) = \frac{f(px) - f(qx)}{(p-q)x} \qquad (x \neq 0)$$

and satisfies the rule (see [25]):

$$D_{p,q,x}(f(x)g(x)) = g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x).$$

In particular, we have [3,26]:

$$D_{p,q,x}e_{p,q}(\alpha x) = \alpha e_{p,q}(\alpha px) \tag{1.9}$$

and

$$D_{p,q,x}E_{p,q}(\alpha x) = \alpha E_{p,q}(\alpha qx). \tag{1.10}$$

Likewise, the (p,q)-definite integral of a function f is defined as [3,25]:

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{n=0}^{\infty}\frac{p^n}{q^{n+1}}\ f\left(\frac{p^n}{q^{n+1}}a\right), \qquad \mid \frac{p}{q} \mid <1.$$

The 2-variable (p,q)-Hermite polynomials $H_{n,p,q}(x,y)$ were introduced and investigated by Raza, Fadel, Nisar, and Zakarya in 2021 [27] by their generating function below:

$$e_{p,q}(xt)\ e_{p,q}(yt^2) = \sum_{n=0}^{\infty} H_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}\ !}$$

and series definition

$$H_{n,p,q}(x,y) = [n]_{p,q}!\sum_{k=0}^{[n/2]}\frac{p^{\binom{n-2k}{2}}p^{\binom{k}{2}}x^{n-2k}y^k}{[n-2k]_{p,q}![k]_{p,q}!}. \tag{1.11}$$

The q-Gould Hopper polynomials $H_{n,q}^{(m)}(x,y)$ are defined in 2024 by Raza, Fadel and Cesarano in [29] employing a particular generating function:

$$e_q(xt)e_q(yt^m) = \sum_{k=0}^{\infty} H_{n,q}^{(m)}(x,y)\frac{t^n}{[n]_q!}. \tag{1.12}$$

In the physical sciences, truncated exponential polynomials are important for evaluating integrals involving products of special functions. They can be described using a variety of techniques, including generating functions, orthogonality criteria, operational formulae. Researchers in mathematics and physical science appreciate the usefulness of generalizations and extensions in their applications; for more information, see [8,14,17,20,21,29–31, 34].

The q-truncated exponential polynomials $E_{n,q}(x)$ is provided by the next integral form [29]:

$$E_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)(\zeta + x)_q^n d_q\zeta. \tag{1.13}$$

The subsequent integral form is described the m^{th} order q-truncated exponential polynomials ${}_{[m]}E_{n,q}(x)$ [29]:

$$ {}_{[m]}E_{n,q}(x) = \frac{1}{[n]_q!}\int_0^{\frac{1}{1-q}} E_q(-q\zeta)H_{n,q}^{(m)}(x,\zeta)d_q\zeta. \tag{1.14}$$

The work of Raza *et al.* [27] on the characteristics of the q-truncated exponential polynomials $E_{n,q}(x)$ and the realization that ordinary truncated exponential polynomials $e_n(x)$ have applications in many areas of mathematics and science served as inspiration. Likewise, (p,q)-calculus's applicability in numerous mathematical and scientific fields serves as inspiration. The rest of the document is structured as follows: The integral transform for (p,q)-truncated exponential polynomials $E_{n,p,q}(x)$ is established and their properties are provided in Section 2. Also, the definition of the series, the generating function, and other characteristics of these polynomials are derived. In section 3, the integral transforms of higher-order (p,q)-truncated exponential polynomials, such as $E^{\alpha}_{n,p,q}(x)$, ${}_{[2]}E_{n,p,q}(x)$, and ${}_{[m]}E_{n,p,q}(x)$ and their related features are offered and investigated.

2 The (p,q)-truncated exponential polynomials

The (p,q)-truncated exponential polynomials $E_{n,p,q}(x)$ are presented in this section employing integral form, and their series definition, generating function, and other characteristics are obtained.

In the beginning, we employ the subsequent integral form to define the (p,q)-truncated exponential polynomials $E_{n,p,q}(x)$:

$$E_{n,p,q}(x) = \frac{1}{[n]_{p,q}!}\int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta)(\zeta\oplus x)^n_{p,q}d_{p,q}\zeta. \tag{2.1}$$

Combining equation (1.4) with the following Euler integral representation of (p,q)-Gamma function [26,1]:

$$p^{\frac{\alpha(\alpha-1)}{2}}\int_0^{\frac{1}{p-q}} \zeta^{\alpha}E_{p,q}(-q\zeta)\,d_{p,q}\zeta = \mathit{\Gamma}_{p,q}(\alpha+1) = [\alpha]_{p,q}!, Re(\alpha)>0, \alpha\neq 0,-1,-2,\dots, \tag{2.2}$$

yields

$$E_{n,p,q}(x) = \frac{1}{[n]_{p,q}!}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} q^{\binom{k}{2}}x^k p^{\binom{n-k}{2}}\int_0^{\frac{1}{p-q}}\zeta^{n-k}E_{p,q}(-q\zeta)d_{p,q}\zeta, \tag{2.3}$$

gives

$$E_{n,p,q}(x) = \frac{1}{[n]_{p,q}!}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} q^{\binom{k}{2}}x^k[n-k]_{p,q}!, \tag{2.4}$$

and simplifications using equation (1.2), we gain the following series definition of $E_{n,p,q}(x)$:

$$E_{n,p,q}(x) = \sum_{k=0}^{n} q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}. \tag{2.5}$$

The following theorem is used to define the generating function of (p,q)-truncated exponential polynomials $E_{n,p,q}(x)$:

Theorem 2.1 *The (p,q)-truncated exponential polynomials have a specific generating function:*

$$\frac{1}{(1-t)}E_{p,q}(xt)=\sum_{n=0}^{\infty}E_{n,p,q}(x)t^n, \qquad |\,t\,|<1. \tag{2.6}$$

where $E_{n,p,q}(x)$ is the (p,q)-exponential finite series which defined through in formula (2.5).

Proof. Considering of equation (2.5), there is

$$\sum_{n=0}^{\infty}E_{n,p,q}(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}t^n.$$

Applying the subsequent series rearrangement procedure ([5] page 782, Eq. 1.4):

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}A(m,n)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}A(m,n-m), \tag{2.7}$$

gives

$$\sum_{n=0}^{\infty}E_{n,p,q}(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}t^{n+k}, \tag{2.8}$$

or, equivalently

$$\sum_{n=0}^{\infty}E_{n,p,q}(x)t^n=\sum_{n=0}^{\infty}t^n\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}t^{k}. \tag{2.9}$$

Applying equations (1.8) and (1.7) yields assertion (2.6).

After assuming that $x=0$ in equation (2.6), then via equation (1.8), we acquire the subsequent initial condition through comparing the same powers of t:

$$E_{n,p,q}(0)=1. \tag{2.10}$$

From equation (2.6), we have

$$E_{n,p,q}(-x)=-E_{n,p,q}(x). \tag{2.11}$$

Considering the (p,q)-derivative of each part for equation (2.6) alongside a reference to x and then employing equation (2.6) in the consequent equation, we receive

$$t\frac{1}{(1-t)}E_{p,q}(qxt)=\sum_{n=0}^{\infty}D_{p,q,x}E_{n,p,q}(x)t^n, \qquad n=0,1.2.\dots. \tag{2.12}$$

Applying equation (2.6) to the left part of the equation (2.12), we receive

$$\sum_{n=0}^{\infty}E_{n,p,q}(qx)t^{n+1}=\sum_{n=0}^{\infty}D_{p,q,x}E_{n,p,q}(x)t^n. \tag{2.13}$$

When we evaluate the identical powers for t on every part of the formula previously, we derive

$$D_{p,q,x}E_{n,p,q}(x)=E_{n-1,p,q}(qx), \qquad n\geq 1. \tag{2.14}$$

Once more, applying the (p,q)-derivative through equation (2.6) about x, then utilizing equation (2.6) on the left part of the resulting equation, we obtain the 2^{nd} and k^{th}-(p,q)-partial derivatives of $E_{n,p,q}(x)$:

$$D^2_{p,q,x}E_{n,p,q}(x) = qE_{n-2,p,q}(q^2x), \qquad x\in\mathbb{C},\ n\geq 2,$$

$$D^k_{p,q,x}E_{n,p,q}(x) = q^{\binom{k}{2}}E_{n-k,p,q}(q^kx), \qquad x\in\mathbb{C},\ n\geq k.$$

3 Higher-order for (p,q)-truncated exponential polynomials

This section provides the corresponding (p,q)-truncated exponential polynomials $E^{(\alpha)}_{n,p,q}(x)$, higher order of (p,q)-truncated exponential polynomials ${}_{[2]}E_{n,p,q}(x)$, ${}_{[m]}E_{n,p,q}(x)$ and ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$. This is demonstrated by their integral forms, series definitions and generating functions.

The associated (p,q)-truncated exponential polynomials (A(p,q)TEP) $E^{(\alpha)}_{n,p,q}(x)$ were recently introduced employing the integral representation shown below:

$$E^{(\alpha)}_{n,p,q}(x) = \frac{1}{[n]_{p,q}!}\int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta)(\zeta)^{\alpha}_{p,q}(\zeta\oplus x)^n_{p,q}\, d_{p,q}\zeta, \tag{3.1}$$

which on using equation (1.4), gives

$$E^{(\alpha)}_{n,p,q}(x) = \frac{1}{[n]_{p,q}!}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} q^{\binom{k}{2}}x^kp^{\binom{n-k+\alpha}{2}}\int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta)\zeta^{n-k+\alpha}d_{p,q}\zeta.$$

Applying the Euler integral representation of (p,q)-gamma function (2.2) in the previous equation, we receive

$$E^{(\alpha)}_{n,p,q}(x) = \sum_{k=0}^{n} q^{\binom{k}{2}}\frac{x^k\Gamma_{p,q}(n-k+\alpha+1)}{[k]_{p,q}![n-k]_{p,q}!}, \qquad x\in\mathbb{C},\ 0\leq k\leq n. \tag{3.2}$$

Presently yields the subsequent result for A(p,q)TEP $E^{(\alpha)}_{n,p,q}(x)$:

Theorem 3.1 *The associated (p,q)-truncated exponential polynomials $E^{(\alpha)}_{n,p,q}(x)$ has the following generating function:*

$$\frac{\Gamma_{p,q}(\alpha+1)}{(1-t)^{\alpha+1}_{p,q}}E_{p,q}(xt) = \sum_{n=0}^{\infty}E^{(\alpha)}_{n,p,q}(x)t^n, \qquad x\in\mathbb{C},\alpha\in\mathbb{R},\ |t|<\frac{1}{p-q}, \tag{3.3}$$

where $E_{n,p,q}(x)$ is the (p,q)-exponential finite series which defined through equation (2.5).

Proof. As per equation (3.2), there is

$$\sum_{n=0}^{\infty}E^{(\alpha)}_{n,p,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}![n-k]_{p,q}!}\Gamma_{p,q}(n-k+\alpha+1)t^n.$$

When applied to equation (2.7), we posses

$$\sum_{n=0}^{\infty}E^{(\alpha)}_{n,p,q}(x)t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}q^{\binom{k}{2}}\frac{x^k}{[k]_{p,q}!}\frac{\Gamma_{p,q}(\alpha+k+1)}{[n]_{p,q}!}t^{n+k}.$$

The right part of the above formula is multiplied by $\frac{\Gamma_{p,q}(\alpha+1)}{\Gamma_{p,q}(\alpha+1)}$, that we have

$$\sum_{n=0}^{\infty} E_{n,p,q}^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty} q^{\binom{k}{2}} \frac{x^k}{[k]_{p,q}!} t^k \Gamma_{p,q}(\alpha+1) \sum_{n=0}^{\infty} \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_{p,q} t^n$$

when the right part of the equation as mentioned above is resolved employing equations (1.7) and (1.8), yields a claim (3.3). Theorem 3.1 has been fully proved.

Taking into account the 2^{nd} order (p,q)-truncated exponential polynomials via the integral form listed below:

$$_{[2]}E_{n,p,q}(x) = \frac{1}{[n]_{p,q}!} \int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta) H_{n,p,q}(x,\zeta) d_{p,q}\zeta. \tag{3.4}$$

Through employing equation (1.11) to the formula (3.4), employing equation (2.2) to the right part of the resulting equation then identical the same powers of t on each part of this equation, the next set of definition for $_{[2]}E_{n,p,q}(x)$ can be gained:

$$_{[2]}E_{n,p,q}(x) = \sum_{k=0}^{[n/2]} \frac{p^{\binom{n-2k}{2}} x^{n-2k}}{[n-2k]_{p,q}!}, \qquad |x| < \frac{1}{p-q}, 0 \le k \le \frac{n}{2}. \tag{3.5}$$

Presently, we derive the subsequent outcome for generating function of $_{[2]}E_{n,p,q}(x)$:

Theorem 3.2 *The 2^{nd} order (p,q)-truncated exponential polynomials $_{[2]}E_{n,p,q}(x)$ have the following generating functions:*

$$\sum_{n=0}^{\infty} {}_{[2]}E_{n,p,q}(x)t^n = \frac{e_{p,q}(xt)}{1-t^2}, \qquad |x|, |t| < \frac{1}{p-q}, \tag{3.6}$$

where $n = 0, 1.2. \ldots.$

Proof. With reference to equation (3.5), we possess

$$\sum_{n=0}^{\infty} {}_{[2]}E_{n,p,q}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} \frac{p^{\binom{n-2k}{2}} x^{n-2k}}{[n-2k]_{p,q}!} t^n,$$

that employs the subsequent series restructuring technique ([5] page 782, Eq. 1.6):

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k), \tag{3.7}$$

gives

$$\sum_{n=0}^{\infty} {}_{[2]}E_{n,p,q}(x)t^n = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} x^n}{[n]_{p,q}!} t^n \sum_{k=0}^{\infty} t^{2k},$$

which yields the claim (3.6) when the equations (1.7) and (1.8) are applied to the right part of the equation described previously.

Theorem 3.2 has been fully proved.

Introducing the (p,q)-Gould Hopper polynomials to illustrate the higher-order (p,q)-truncated exponential polynomials is worthwhile. The generalized heat equation [7] is satisfied by the Gould-Hopper polynomials [12], which are a generalization of 2-variable Hermite polynomials.

The (p,q)-Gould Hopper polynomials (p,q)GHP $H^{(m)}_{n,p,q}(x,y)$ are introduced via a particular generating function in consideration of equation (1.12):

$$e_{p,q}(xt)e_{p,q}(yt^m)=\sum_{k=0}^{\infty}H^{(m)}_{n,p,q}(x,y)\frac{t^n}{[n]_{p,q}!},$$

which yields the subsequent series description of (p,q)GHP $H^{(m)}_{n,p,q}(x,y)$ after being simplified, applying equation (1.6) on the left part, then identifying the same powers of t from each parts of the resulting equation:

$$H^{(m)}_{n,p.q}(x,y)=[n]_{p,q}!\sum_{k=0}^{[n/m]}\frac{p^{\binom{n-mk}{2}}x^{n-mk}p^{\binom{k}{2}}y^k}{[n-mk]_{p,q}![k]_{p,q}!},\qquad 0\leq k\leq\frac{n}{m}. \tag{3.8}$$

Presently, considering the subsequent integral form, we provide the m^{th} order (p,q)-truncated exponential polynomials ${}_{[m]}E_{n,p,q}(x)$ within the context of equations (1.14) and (2.1):

$${}_{[m]}E_{n,p,q}(x)=\frac{1}{[n]_{p,q}!}\int_0^{\frac{1}{p-q}}E_{p,q}(-q\zeta)H^{(m)}_{n,p,q}(x,\zeta)d_{p,q}\zeta. \tag{3.9}$$

The subsequent series definition of m^{th}-order (p,q)-truncated exponential polynomials ${}_{[m]}E_{n,p,q}(x)$ are obtained via employing equation (3.8) in equation (3.9), then followed by equation (2.2) in the right part for the resulting formula, then identical the same powers of t from each part of this equation:

$${}_{[m]}E_{n,p,q}(x)=\sum_{k=0}^{[n/m]}\frac{p^{\binom{n-mk}{2}}x^{n-mk}}{[n-mk]_{p,q}!},\qquad |x|<\frac{1}{p-q},0\leq k\leq\frac{n}{m}. \tag{3.10}$$

The subsequent outcome is achieved for generating functions of ${}_{[m]}E_{n,p,q}(x)$:

Theorem 3.3 *The generating function for the m^{th} order (p,q)-truncated exponential polynomials ${}_{[m]}E_{n,p,q}(x)$ is outlined below:*

$$\sum_{n=0}^{\infty}{}_{[m]}E_{n,p,q}(x)t^n=\frac{e_{p,q}(xt)}{1-t^m},\qquad |x|,|t|<\frac{1}{p-q}. \tag{3.11}$$

Proof. Since equation (3.10) is present, we gain

$$\sum_{n=0}^{\infty}{}_{[m]}E_{n,p,q}(x)t^n=\sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{p^{\binom{n-mk}{2}}x^{n-mk}}{[n-mk]_{p,q}!}t^n,$$

that employs the series rearranging method listed below ([5] page 782, Eq. 1.1):

$$\sum_{\nu=0}^{\infty}\sum_{k=0}^{\infty}A(k,\nu)=\sum_{\nu=0}^{\infty}\sum_{k=0}^{[\nu/m]}A(k,\nu-mk), \tag{3.12}$$

gives

$$\sum_{n=0}^{\infty} {}_{[m]}E_{n,p,q}(x)t^n = \sum_{k=0}^{\infty} t^{mk} \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}} x^n}{[n]_{p,q}!} t^n,$$

which yields the claim (3.11) whenever the right part of the aforementioned equation is determined via equations (1.6) and (1.8). Theorem 3.3 has been fully proved.

The subsequent integral form is employed to gain the m^{th}-order associated q-truncated exponential polynomials ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$ within context of equations (3.1) and (1.14):

$$ {}_{[m]}E^{(\alpha)}_{n,p,q}(x) = \frac{1}{[n]_q!} \int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta)(\zeta)^{\alpha}_{p,q} H^{(m)}_{n,p,q}(x,\zeta) d_{p,q}\zeta. \tag{3.13}$$

Considering equation (3.8), gives

$$ {}_{[m]}E^{(\alpha)}_{n,p,q}(x) = \sum_{k=0}^{[n/m]} \frac{p^{\binom{n-mk}{2}} x^{n-mk}}{[k]_{p,q}![n-mk]_{p,q}!} p^{\binom{k+\alpha}{2}} \int_0^{\frac{1}{p-q}} E_{p,q}(-q\zeta)\zeta^{k+\alpha} d_{p,q}\zeta. \tag{3.14}$$

The subsequent series definitions for ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$ can be gained through applying equation (2.2) to the right part of equation (3.14):

$$ {}_{[m]}E^{(\alpha)}_{n,p,q}(x) = \sum_{k=0}^{[n/m]} \frac{\Gamma_{p,q}(\alpha+k+1)}{[k]_{p,q}![n-mk]_{p,q}!} p^{\binom{n-mk}{2}} x^{n-mk}, \alpha \in \mathbb{R}, |x| < \frac{1}{p-q}, 0 \le k \le \frac{n}{m}. \tag{3.15}$$

The subsequent outcome can be gained by generating function of ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$:

Theorem 3.4 *The m^{th} order associated (p,q)-truncated exponential polynomials ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$ have the following generating function:*

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,p,q}(x)t^n = \frac{\Gamma_{p,q}(\alpha+1)}{(1-t^m)^{\alpha+1}_{p,q}} e_{p,q}(xt), \qquad |x|,|t| < \frac{1}{p-q}. \tag{3.16}$$

Proof. Considering the equation (3.15), there is

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,p,q}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{p^{\binom{n-mk}{2}} x^{n-mk} \Gamma_{p,q}(k+\alpha+1)}{[k]_{p,q}![n-mk]_{p,q}!} t^n,$$

which on using equation (3.12), we gain

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,p,q}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{p^{\binom{n-mk}{2}} x^{n-mk} \Gamma_{p,q}(\alpha+k+1)}{[n]_{p,q}![k]_{p,q}!} t^{n+mk}.$$

The right part of the above formula is multiplied by $\frac{\Gamma_{p,q}(\alpha+1)}{\Gamma_{p,q}(\alpha+1)}$, we achieve

$$\sum_{n=0}^{\infty} {}_{[m]}E^{(\alpha)}_{n,p,q}(x)t^n = \Gamma_{p,q}(\alpha+1) \sum_{k=0}^{\infty} \begin{bmatrix} \alpha+k \\ k \end{bmatrix}_{p,q} (t^m)^k \sum_{n=0}^{\infty} \frac{p^{\binom{n-mk}{2}} x^{n-mk}}{[n]_{p,q}!} t^n, \tag{3.17}$$

which yields the claim (3.16) when $m = \alpha + 1$ is calculated via equations (1.6) and (1.8). We have finished a claim for Theorem 3.4.

Remark 3.1 Since, for $m = 2$ the m^{th}-order associated (p, q)-truncated exponential polynomials ${}_{[m]}E^{(\alpha)}_{n,p,q}(x)$ reduces to the 2^{nd} order associated (p, q)-truncated exponential polynomials ${}_{[2]}E^{(\alpha)}_{n,p,q}(x)$, which we introduce in view of equation (3.13) as:

$$ {}_{[2]}E^{(\alpha)}_{n,p,q}(x) = \frac{1}{[n]_{p,q}!} \int_0^{\frac{1}{p-q}} E_q(-q\zeta)(\zeta)^{\alpha}_{p,q} H_{n,p,q}(x, \zeta) d_{p,q}\zeta. \tag{3.18} $$

The subsequent table lists the series definition and generating function of ${}_{[2]}E^{(\alpha)}_{n,p,q}(x)$, which is derived by inserting $m = 2$ in equations (3.15) and (3.16):

Table 3.1 Series definition and generating function of ${}_{[2]}E^{(\alpha)}_{n,p,q}(x)$

S. No.	Polynomials	Series definition	Generating function
I.	${}_{[2]}E^{(\alpha)}_{n,p,q}(x)$	${}_{[2]}E^{(\alpha)}_{n,p,q}(x) = \sum_{k=0}^{[n/2]} \frac{p^{\binom{n-2k}{2}} x^{n-2k} \Gamma_{p,q}(\alpha+k+1)}{[k]_{p,q}![n-2k]_{p,q}!}$	$\sum_{n=0}^{\infty} {}_{[2]}E^{(\alpha)}_{n,p,q}(x)t^n = \frac{\Gamma_{p,q}(\alpha+1)}{(1-t^2)^{\alpha+1}_{p,q}} e_{p,q}(xt)$

Remark 3.2 Finding the k times (p, q)-partial derivatives with regard with x of each part of equation (3.3) through equation (1.10) and the k times (p, q)-partial derivative of each part of equation (3.16) employing equation (1.9), and then once again by employing equations (3.3) and (3.16) in the resulting equations, we receive

$$ D^k_{p,q,x} E^{(\alpha)}_{n,p,q}(x) = q^{\binom{k}{2}} E^{(\alpha)}_{n-k,p,q}(q^k x), \qquad 0 \le k \le n \tag{3.19} $$

and

$$ D^k_{p,q,x}\ {}_{[m]}E^{(\alpha)}_{n,p,q}(x) = p^{\binom{k}{2}}\ {}_{[m]}E^{(\alpha)}_{n-mk,p,q}(p^k x), \qquad 0 \le k \le \frac{n}{m}. \tag{3.20} $$

For $\alpha = 0$, equations (3.19) and (3.20), give the next k^{th}-derivatives of $E_{n,p,q}(x)$ and ${}_{[m]}E_{n,p,q}(x)$:

$$ D^k_{p,q,x} E_{n,p,q}(x) = q^{\binom{k}{2}} E_{n-k,p,q}(q^k x), \qquad 0 \le k \le n \tag{3.21} $$

and

$$ D^k_{p,q,x}\ {}_{[m]}E_{n,p,q}(x) = p^{\binom{k}{2}}\ {}_{[m]}E_{n-mk,p,q}(p^k x), \qquad 0 \le k \le \frac{n}{m}, \tag{3.22} $$

respectively.

Further, for $m = 2$, equation (3.22) gives the following k^{th} derivative of ${}_{[2]}E_{n,p,q}(x)$:

$$ D^k_{p,q,x}\ {}_{[2]}E_{n,p,q}(x) = p^{\binom{k}{2}}\ {}_{[2]}E_{n-2k,p,q}(p^k x), \qquad 0 \le k \le \frac{n}{2}. \tag{3.23} $$

4 Conclusions

Special function experts frequently study post-quantum calculus, sometimes called (p,q)-calculus. In applied science, this tool is frequently employed. The characteristics of q-truncated and classical truncated polynomials and their relatives have generated attention in many quantum mechanical and optical problems. In assessing integrals involving products of special functions, they were also crucial. As a result, (p,q)-truncated polynomials arrived, and their characteristics were shown through the integral formulation of the (p,q)-Gamma function and several (p,q)-calculus identities. In previous sections, various types of (p,q)-truncated polynomials were employed to improve the situation. The most important of these polynomials was the development of the integral representation, generating function and series formulation of the (p,q)-truncated polynomials $E_{n,p,q}(x)$. Some of these polynomials' features, including their integral representation, generating function and series definition are established in Section 2. The connected (p,q)-truncated exponential polynomials $E^{(\alpha)}_{n,p,q}(x)$, higher order (p,q)-truncated exponential polynomials ${}_{[2]}E_{n,p,q}(x)$ and ${}_{[m]}E_{n,p,q}(x)$, and higher order related (p,q)-truncated exponential polynomials are introduced in section 3. Additionally, we obtain their series definitions, generating functions, and integral forms.

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