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On strong solvability in L_p of one nonlocal boundary value problem for Laplace equation on half disc

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Abstract. *The spectral problem for an ordinary differential equation with non-local boundary conditions on a finite interval* $[0,\pi]$ *is investigated. In this case, the corresponding spectral problem has two series of asymptotically close eigenfunctions that do not form a basis in the space* $L_p\,[0,\pi]$*, and a direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed that already forms a basis in* $L_p\,[0,\pi]$*. Using the new system, we investigate the unique solvability (in the strong sense) of the nonlocal boundary value problem for the Laplace equation in the Sobolev space* $W_p^2\,(D)$*, where* $D=(0,\pi)\times(0,1)$ *is the half-disk of the complex plane.*

Keywords. Laplace equation, Sobolev space, non-local BVP, strong solvability, eigenfunction, basis.

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1 Introduction

Solving some elliptic equations with nonlocal boundary conditions using the Fourier method leads to spectral problems with boundary conditions that are regular, but not strongly regular. For this reason, the root functions of these problems generally do not form a basis in the corresponding function space. In such cases, direct application of the Fourier method is impossible. One such problem is the following

$$\left.\begin{array}{c} \varDelta U\,(\theta;\varphi)=0,\quad (\theta,\varphi)\in D, \\ u\,\left(1;\theta\right)=\varphi\left(\theta\right),\ \ 0\leq\theta\leq\pi, \\ U\,\left(r;0\right)=0,\ \ r\in\,\left[0,1\right], \\ \frac{\partial U(r;0)}{\partial\theta}=\,\frac{\partial U(r;\pi)}{\partial\theta}+\alpha\,U\left(r;\pi\right),r\in\,\left[0,1\right], \end{array}\right\} \tag{1.1}$$

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where $D = \{(r,\theta) : 0 < r < 1; 0 < \theta < \pi\}$ is half-disc in polar coordinates, $\alpha \epsilon R$ – some parameter. This problem leads to the following spectral problem

$$\left.\begin{array}{c} -u''(x) = \lambda\, u(x)\,,\ 0 < x < \pi, \\ u(0) = 0,\ \ u'(0) = u'(\pi) + \alpha\, u\ (\pi)\,. \end{array}\right\} \tag{1.2}$$

The boundary conditions of this spectral problem are regular, but not strongly regular [1]. The studies of such problems in the Lebesgue space $L_2\,[0,\pi]$are devoted to the works [2–6]. It should be noted that questions related to this topic, in the case of $\alpha = 0$, were also considered in the works [7,8]. All these spectral problems are not self-adjoint. The cases $\alpha = 0$ and $\alpha \neq 0$ differ significantly from each other. The fact is that in the first case, all the eigenvalues are double and they correspond to one eigenfunction and one associated function, and together they form a basis in $L_2\,[0,\pi]$. In the second case, all the eigenvalues are simple, but the eigenfunctions corresponding to them are not a basis in $L_2\,[0,\pi]$. From linear combinations of these eigenfunctions, a new system of functions is constructed, which already forms a basis in $L_2\,[0,\pi]$. Although the resulting system is not a system of eigenfunctions of the spectral problem, it nevertheless allows one to solve the equation under consideration using the Fourier method. This fact is applied to solving the nonlocal boundary value problem (1.1) for the Laplace equation. This approach is used in [4] to solve the initial boundary value problem for the heat equation, in [5] to solve the nonlocal boundary value problem for the Helmholtz operator in a half-disk, and in [6] to solve the inverse nonlocal boundary value problem for the heat equation in the classical formulation. We also note the works [9–20], where nonlocal boundary value problems for the elliptic equations are studied in various domains, in various functional spaces, using spectral methods. In the present paper, we apply this method to study the strong solvability of problem (1.1) in the Sobolev class $W_p^2\,(D)$.

The spectral problem for an ordinary differential equation with non-local boundary conditions on a finite interval $[0,\pi]$ is investigated. Such problems arise when solving nonlocal boundary value problems for partial differential equations using the Fourier method. They arise, for example, when solving nonstationary diffusion problems with Samarskii-Ionkin-type boundary conditions, or when solving a stationary diffusion problem with counter current flows over a portion of an interval. The boundary conditions of this problem are regular, but not strengthened regular in the Birkhoff sense. In this case, the corresponding spectral problem has two series of asymptotically close eigenfunctions that do not form a basis in the space $L_p\,[0,\pi]$, and a direct application of the Fourier method is impossible. Based on these eigenfunctions, a new system of functions is constructed that already forms a basis in $L_p\,[0,\pi]$. Using the new system, we investigate the unique solvability (in the strong sense) of the nonlocal boundary value problem for the Laplace equation in the Sobolev space $W_p^2\,(D)$, where $D = (0,\pi) \times (0,1)$ is the half-disk of the complex plane.

2 Preliminaries

21 Notations.

Accept the following notations (in general, standard notation) which we use in this work. $\mathbb{N}-$positive integers; $\mathbb{Z}-$all integers; $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$; $\mathbb{R}-$real axis; $\mathbb{C}-$complex numbers; $L\,[\cdot]-$linear span; $\overline{M}-$ closure of M; B - space – Banach space; $\|\cdot\|_X-$norm in X; X^*-dual space of X; $[X;Y] - B$ -space of linear bounded operators, acting from B-space X to B-space Y; $[X] = [X;X]$, p' is a conjugate to p number: $\frac{1}{p} + \frac{1}{p'} = 1$, for $x \in X$ and $x^* \in X^* : \langle x, x^* \rangle = x^*\,(x)\,;$ for $x \in X$ and $y \in Y$ the relation $\|x\|_X \lesssim \|y\|_Y$ means $\|x\|_X \leq const\ \|y\|_Y$.

22 Some concepts and facts from the theory of bases.

Let us present briefly the main definitions and facts which will be used in what follows. Let X be a B-space. A system $\{x_n\}_{n\in\mathbb{N}}$ of elements X is said to be complete in X if $\overline{L\left(\{x_n\}_{n\in\mathbb{N}}\right)} = X$; that is, any element of the space X can be approximated by a linear combination of elements of this system with any accuracy in the norm of the space X.

A system $\{x_n\}_{n\in\mathbb{N}}$ of elements X is said to be minimal in X if $x_n \notin \overline{L\left(\{x_k\}_{k\neq n}\right)}$. It is well known that a system $\{x_n\}_{n\in\mathbb{N}}$ is minimal if and only if there exists a biorthogonal system which is dual to it, that is, a system of linear functionals $\{x_n^*\}_{n\in\mathbb{N}}$ from X^* such that $\langle x_n, x_k^*\rangle = \delta_{nk}$ for all $n,\ k \in \mathbb{N}$. Moreover, if the initial system is complete and minimal in X, then the biorthogonal system is uniquely defined.

We say that a system $\{x_n\}_{n\in\mathbb{N}}$ is uniformly minimal in X, if there exists $\gamma > 0$ such that for all $n \in \mathbb{N}$,

$$\operatorname{dist}(x_n, X_n) \geq \gamma \|x_n\|_X \ ,$$

where $X_n = \overline{L\left[\{x_k\}_{k\neq n}\right]}$. It is also well known that a complete and minimal system $\{x_n\}_{n\in\mathbb{N}}$ is uniformly minimal in X if and only if:

$$\sup_{n\in\mathbb{N}} \|x_n\|_X \|x_n^*\|_{X^*} < \infty \ .$$

A system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis of the space X if, for any element $x \in X$, there exists a unique expansion into a series

$$x = \sum_{n=1}^{\infty} c_n x_n$$

converging in the norm of the space X.

Two systems $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ of a B-space X are called equivalent if there exists an automorphism T that maps one of these systems to the other: $\exists T \in [X] : Tx_n = y_n,\ \forall n \in \mathbb{N}$. A system equivalent to a basis is itself a basis in the same space.

A system $\{x_n\}_{n\in\mathbb{N}}$ is called a basis with brackets in a B-space X if there exists a sequence $\{n_k\}_{k\in\mathbb{N}}$ of positive integers such that $n_1 < n_2 < \ldots < n_k < n_{k+1} < \ldots$, and for any $x \in X$ there is a unique expansion into a series

$$x = \sum_{k=0}^{\infty} \sum_{i=n_k+1}^{n_{k+1}} c_i x_i, \ \ (n_0 = 0)$$

converging in the norm of the space X.

We say that a system $\{x_n\}_{n\in\mathbb{N}}$ is almost normalized in X, if

$$0 < \inf_{n\in\mathbb{N}} \|x_n\| \leq \sup_{n\in\mathbb{N}} \|x_n\| < +\infty \ .$$

A uniformly minimal system is almost normalized if and only if its bioorthogonal system is almost normalized.

Any basis is a complete and minimal system in X, and, therefore, we can uniquely find its biorthogonal dual system $\{x_n^*\}_{n\in\mathbb{N}}$ and hence the expansion of any element $x \in X$ with respect to the basis $\{x\}_{n\in\mathbb{N}}$ coincides with its biorthogonal expansion, that is, $c_n = \langle x, x_n^*\rangle$ for all $n \in \mathbb{N}$.

We will use also some facts about p - closure bases. Concerning these facts more details one can see the works [20–24].

Systems $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}\subset X$ in B -space X are called p-closure if

$$\sum_{n=1}^{\infty}\|x_n-y_n\|_X^p<\infty.$$

The minimal system $\{x_n\}_{n\in\mathbb{N}}\subset X$ with biorthogonal system $\{x_n^*\}_{n\in\mathbb{N}}\subset X^*$ is called p-besselian, if for any $x\in X$

$$\left(\sum_{n=1}^{\infty}|\langle x,x_n^*\rangle|^p\right)^{\frac{1}{p}}\leq M\|x\|_X.$$

If the basis $\{x_n\}_{n\in\mathbb{N}}$ for X is p- besselian, then we call it as p-basis.
It is valid the following

Theorem 2.1 *[23,24] Let the system $\{x_n\}_{n\in\mathbb{N}}$ is p-basis for B-space X and the system $\{y_n\}_{n\in\mathbb{N}}\subset X$ is p'– clouser to it, $1<p<\infty$. Then the following assertions are equivalent:*
$\{y_n\}_{n\in\mathbb{N}}$ is complete in X;
$\{y_n\}_{n\in\mathbb{N}}$ is minimal in X;
$\{y_n\}_{n\in\mathbb{N}}$ is isomorphic to $\{x_n\}_{n\in\mathbb{N}}$ basis for X.

It is valid the following

Proposition 2.1 *[25,26] Let system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis with parentheses for Banach space X. If the system $\{x_n\}_{n\in\mathbb{N}}$ is uniformly minimal and condition*

$$\sup_{k\in\mathbb{N}}(n_{k+1}-n_k)\ <\infty$$

hold, then the system $\{x_n\}_{n\in\mathbb{N}}$ forms a basis for X.

In obtaining the main results we essentially will use the basicity in $L_p\ (0,\pi)$ the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ where

$$w_0(x)=x,\ \ w_{2k-1}(x)=\sin 2kx,\quad w_{2k}(x)=x\cos 2kx\ ,\ \ k\in\mathbb{N},\tag{2.1}$$

which is a collection of root functions, corresponding to the spectral problem

$$\left.\begin{array}{c}-w''(x)=\lambda w(x)\,,\ x\in(0,\pi)\,,\\ w(0)=w'(0)-w'(\pi)=0.\end{array}\right\}\tag{2.2}$$

It is valid

Theorem 2.2 *The system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ forms a s-basis for $L_p\ (0,\pi)\,,\ \ 1<p<+\infty$, where $s=\max\{p,\ p'\}$.*

Proof. The spectral problem (2.2) is regular, but not strongly regular in the sense of Birkhoff (see [1]). From the results of [27], in particular, it follows that the eigenfunctions and associated functions of this problem, i.e., the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ forms a basis with brackets in $L_p\ (0,\pi)\,,\ \ 1<p<+\infty$, and in the biorthogonal expansion, pairs of terms corresponding to w_{2k-1} and w_{2k} must be combined in brackets, i.e., $n_{k+1}-n_k=2$. The problem adjoint to (2.2) has the form

$$\left.\begin{array}{c}-z''(x)=\lambda\,z(x)\,,\ \ x\in(0,\pi)\,,\\ z'(0)=z(0)-z(\pi)=0.\end{array}\right\}\tag{2.3}$$

The system of eigen and associated functions of the spectral problem (2.3) is the system $\{z_n(x)\}_{n\in\mathbb{Z}_+}$, where

$$z_0(x)=\frac{2}{\pi^2},\quad z_{2k-1}(x)=\frac{4}{\pi^2}(\pi-x)\sin 2kx,\quad z_{2k}(x)=\frac{4}{\pi^2}\cos 2kx,\quad k\in\mathbb{N}. \tag{2.4}$$

The systems $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ and $\{z_n(x)\}_{n\in\mathbb{Z}_+}$ are biorthonormal, i.e.

$$\langle w_n,z_m\rangle=\delta_{nm},\quad n,m\in\mathbb{Z}_+.$$

From the formulas (2.1) and (2.4) it is obvious that

$$\sup_{n\in\mathbb{Z}_+}\|w_n\|_{L_p(0,\pi)}\|z_n\|_{L_{p'}(0,\pi)}<+\infty.$$

Thus, all the conditions of Statement 2.1 are satisfied, according to which the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ forms a basis in the space $L_p(0,\pi)$, $1<p<+\infty$. We show that the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is also an s-basis in this space, where $s=\max\{p,\ p'\}$. Let $1<p\le 2$, then $s=p'$ and, as follows from the Hausdorff-Young inequality (see [28]), for any function $f(x)$ from $L_p(0,\pi)$

$$\left(\sum_{k=0}^{\infty}|\langle f,z_{2k}\rangle|^{p'}\right)^{\frac{1}{p'}}\lesssim\|f\|_{L_p},$$

$$\left(\sum_{k=1}^{\infty}|\langle f,z_{2k-1}\rangle|^{p'}\right)^{\frac{1}{p'}}=\left(\sum_{k=1}^{\infty}\left|\int_0^{\pi}f(x)\frac{4}{\pi^2}(\pi-x)\sin 2kx\ dx\right|^{p'}\right)^{\frac{1}{p'}}$$

$$\lesssim\left(\sum_{k=1}^{\infty}\left|\int_0^{\pi}\tilde f(x)\sin 2kx\ dx\right|^{p'}\right)^{\frac{1}{p'}}\lesssim\left\|\tilde f\right\|_{L_p}\lesssim\|f\|_{L_p},$$

where $\tilde f(x)=(\pi-x)f(x)$ is denoted, hence the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is a p'-basis in $L_p(0,\pi)$.

If $p\in(2;+\infty)$, then $p'\in(1;2)$ and $s=p$, and again applying the Hausdorff-Young inequality and taking into account the embedding $L_p(0,\pi)\subset L_{p'}(0,\pi)$, we obtain

$$\left(\sum_{n=0}^{\infty}|\langle f,z_n\rangle|^{p}\right)^{\frac{1}{p}}\lesssim\|f\|_{L_{p'}}\lesssim\|f\|_{L_p},$$

i.e., the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is a p-basis in $L_p(0,\pi)$. Thus, the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is an s-basis in $L_p(0,\pi)$, where $s=\max\{p,\ p'\}$.

Theorem is proved.

Carrying out similar reasoning, we obtain that for the system $\{z_n(x)\}_{n\in\mathbb{Z}_+}$ the following corollary is true.

Corollary 2.1 *The system $\{z_n(x)\}_{n\in\mathbb{Z}_+}$ of eigenfunctions and associated functions of the adjoint problem* (2.3) *forms an s-basis in the space $L_p(0,\pi)$, $1<p<+\infty$, where $s=\max\{p,p'\}$.*

From Theorem 2.2, as well as from Corollary 2.1, in particular for $p=2$, we obtain the following corollary.

Corollary 2.2 *The systems* $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ *and* $\{z_n(x)\}_{n\in\mathbb{Z}_+}$ *form a Riesz basis in the space* $L_2(0,\pi)$.

We will also need some concepts and facts from the theory of positive operators. Recall that an operator L with a dense domain $D(L)$ is called positive if $[0,+\infty)$ belongs to the resolvent set of the operator L and there exists a constant $C>0$ such that the estimate

$$\left\|(L+tI)^{-1}\right\| \le \frac{C}{t+1},\ t\in[0,+\infty).$$

For positive operators, fractional powers $L^\nu,\ 0<\nu<1$, are defined (see, for example, [29]), and in the study of the convergence of spectral decompositions, the question of describing the domains of definition of $D(L^\nu)$ is very important. For ordinary differential operators, such results were obtained in [30–32]. In particular, from the results of [32] it follows that the following theorem is valid.

Theorem 2.3 *[32] Let the operator L be generated by the differential expression* $l(y)=-y''$ *and the boundary conditions from* (1.2). *Then for the domain of definition* $D(L^\nu)$ *of fractional powers of the operator L the following holds:*

$$D(L^\nu)=W^{2\nu}_{p,U}(0,\pi),\ 0\le\nu\le1,$$

where $W^{2\nu}_{p,U}(0,\pi)=\left[L_p(0,\pi),\ W^2_{p,U}(0,\pi)\right]_\nu$*;* $[\ ,\ ]_\nu$ *denotes the interpolation spaces corresponding to the complex method, and* $W^2_{p,U}(0,\pi)$ *denotes the space of functions belonging to* $W^2_p(0,\pi)$ *and satisfying the boundary conditions from* (1.2).

From the results of [31] it follows that the following theorem is true.

Theorem 2.4 *[31] For* $2\nu\ne\frac{1}{p};\ 1+\frac{1}{p}$, $W^{2\nu}_{p,U}(0,\pi)$ *consists of functions belonging to the Sobolev-Slobodetskii space* $W^{2\nu}_p(0,\pi)$ *and satisfying the boundary conditions from* (1.2)*, the order of which does not exceed* $2\nu-\frac{1}{p}$.

3 Basicity questions of eigenfunctions for $L_p(0,\pi)$

In this section we will consider some basis issues for the eigenfunctions of the spectral problem (1.2) in the general case $\alpha\in\mathbb{C}$, which we will use to obtain the main results. Similar to the work [4], the system of eigenfunctions of the spectral problem (1.2) for any $\alpha\ne0,\ \alpha\in\mathbb{C}$, has the form $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ where

$$u_{2k-1}(x)=\sin 2kx\ ,\ \ k\in\mathbb{N},\ u_{2k}(x)=\sin\rho_{2k}x\ ,\ \ k\in\mathbb{Z}_+,$$

and ρ_{2k} are roots of the equation

$$\tan\frac{\pi\rho}{2}=\frac{\alpha}{\rho},$$

which satisfy the asyimptotics

$$\rho_{2k}=2k+2\delta_k,\ \ k\in\mathbb{Z}_+,\ \ \delta_k=O\left(\frac{1}{k}\right). \tag{3.1}$$

Let's show that the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is not uniformly minimal in $L_p(0,\pi)$.

Lemma 3.1 *For any* $\alpha \neq 0,\ \alpha \in \mathbb{C}$, *the system* $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ *of eigenfunctions of problem* (1.2) *is complete, minimal, and almost normalized, but is not uniformly minimal in* $L_p(0,\pi), 1 < p < +\infty$.

Proof. Let $1 < p < +\infty$. As in the case of the spectral problem (2.2), (2.3), problem (1.2) is also not strongly regular, and it follows from [27] that the eigenfunctions and associated functions of problem (1.2) form a basis with brackets in $L_p(0,\pi)$. This, in particular, implies the completeness and minimality of the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ in the space $L_p(0,\pi)$. Moreover, the spectral problem adjoint to (1.2) has the form

$$\left.\begin{array}{c} -\vartheta''(x) = \lambda\,\vartheta(x),\ \ 0 < x < \pi, \\ \vartheta(\pi) - \vartheta(0) = \vartheta'(\pi) + \alpha\,\vartheta(0) = 0. \end{array}\right\}. \tag{3.2}$$

The eigenfunctions of the problem (3.2) are

$$\left.\begin{array}{c} \vartheta_{2k-1}(x) = \frac{2}{\pi}\left(\sin 2kx - \frac{2k}{\alpha}\cos 2kx\right), k \in \mathbb{N}, \\ \vartheta_{2k}(x) = c_{2k}\left(\sin\rho_{2k}x + \frac{\rho_{2k}}{\alpha}\cos\rho_{2k}x\right),\ k \in \mathbb{Z}_+, \end{array}\right\} \tag{3.3}$$

where the constants c_{2k} such that $\langle u_{2k}, \vartheta_{2k}\rangle = 1$. By direct calculation we find that

$$c_{2k} = \frac{2}{\pi} + O\left(\frac{1}{k}\right). \tag{3.4}$$

The system $\{\vartheta_n(x)\}_{n\in\mathbb{Z}_+}$ is a biorthogonal to $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ system regarding the space $L_p(0,\pi)$, and therefore the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is minimal in $L_p(0,\pi)$.

Let us show that the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is almost normalized. For u_{2k-1}, we have an obvious relation

$$\|u_{2k-1}\|_{L_p} = \left(\int_0^\pi |\sin 2k\,x|^p dx\right)^{\frac{1}{p}} \leq \pi^{\frac{1}{p}}.$$

And for u_{2k}, taking into account the asymptotic relation $\sin(2k+2\delta_k)\,x = \sin 2k\,x + O\left(\frac{1}{k}\right),\ k \to \infty$, we obtain

$$\|u_{2k}\|_{L_p} = \left(\int_0^\pi |\sin(2k+2\delta_k)\,x|^p dx\right)^{\frac{1}{p}} \leq \pi^{\frac{1}{p}} + O\left(\frac{1}{k}\right).$$

From this we obtain

$$\overline{\lim_{k\to\infty}}\,\|u_{2k}\|_{L_p} \leq \pi^{\frac{1}{p}}. \tag{3.5}$$

For the lower bound, we first consider the case $1 < p \leq 2$. Then we have

$$\|u_{2k-1}\|_{L_p}^p = \int_0^\pi |\sin 2kx\,|^p dx \geq \int_0^\pi \sin^2 2kx\,dx = \frac{\pi}{2}.$$

This implies

$$\|u_{2k-1}\|_{L_p} \geq \left(\frac{\pi}{2}\right)^{\frac{1}{p}}.$$

Similarly, for larger values of k we have

$$\|u_{2k}\|_{L_p} = \left(\int_0^\pi |\sin(2k+2\delta_k)\,x\,|^p dx\right)^{\frac{1}{p}}$$

$$\geq \left(\int_0^{\pi} |\sin 2kx\,|^p dx\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right) \geq \left(\frac{\pi}{2}\right)^{\frac{1}{p}} - O\left(\frac{1}{k}\right) \to \left(\frac{\pi}{2}\right)^{\frac{1}{p}}, \qquad k \to \infty.$$

Therefore,

$$\underline{\lim}_{k\to\infty} \|u_{2k}\|_{L_p} \geq \left(\frac{\pi}{2}\right)^{\frac{1}{p}}.$$

Hence, taking into account (3.5), we obtain that for $1 < p \leq 2$, the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is almost normalized.

Now let $p > 2$. Then there is a continuous embedding $L_p(0,\pi) \subset L_2(0,\pi)$, and therefore there exists a number $C > 0$ such that

$$\|u_{2k-1}\|_p \geq C\|u_{2k-1}\|_2 = C\left(\frac{\pi}{2}\right)^{\frac{1}{2}};$$

$$\|u_{2k}\|_p \geq C\|u_{2k}\|_2 \geq C\left(\|u_{2k-1}\|_2 + O\left(\frac{1}{k}\right)\right) \geq C\left(\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + O\left(\frac{1}{k}\right)\right).$$

From this we have

$$\underline{\lim}_{n\to\infty} \|u_{2k}\|_{L_p} \geq C\left(\frac{\pi}{2}\right)^{\frac{1}{p}}.$$

Thus, for all $p \in (1, +\infty)$

$$0 < \inf_{n\in\mathbb{Z}_+} \|u_n\|_{L_p} \leq \sup_{n\in\mathbb{Z}_+} \|u_n\|_{L_p} < +\infty, \tag{3.6}$$

i.e., the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is almost normalized in $L_p(0,\pi)$.

We now proceed to the proof of the last statement of the lemma. To do this, it is sufficient to show that

$$\overline{\lim}_{k\to\infty} \|u_{2k-1}\|_{L_p} \|\vartheta_{2k-1}\|_{L_{p'}} = +\infty.$$

From the biorthonormality relations we have

$$\langle u_{2k-1}, \vartheta_{2k-1}\rangle = 1, \ \ \langle u_{2k}, \vartheta_{2k-1}\rangle = 0, \ \ k \in \mathbb{N}.$$

Hence, $\langle u_{2k} - u_{2k-1}, \vartheta_{2k-1}\rangle = 1$. Using Hölder's inequality, we obtain

$$\|\vartheta_{2k-1}\|_{L_{p'}} \geq \|u_{2k} - u_{2k-1}\|_{L_p}^{-1}. \tag{3.7}$$

On the other hand, from the asymptotics $\rho_{2k} = 2k + O\left(\frac{1}{k}\right)$ it follows

$$u_{2k}(x) - u_{2k-1}(x) = O\left(\frac{1}{k}\right).$$

Consequently

$$\|u_{2k} - u_{2k-1}\|_{L_p} = O\left(\frac{1}{k}\right).$$

Taking into account the last relation, from (3.6) and (3.7) we obtain

$$\lim_{k\to\infty} \|u_{2k-1}\|_{L_p} \|\vartheta_{2k-1}\|_{L_{p'}} = +\infty.$$

Thus, the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ is not uniformly minimal. The lemma is proved.

From this lemma follows

Corollary 3.1 *For any* $\alpha \neq 0,\ \alpha \in \mathbb{C}$, *the system* $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ *of eigenfunctions of the spectral problem* (1.2) *does not form a basis for* $L_p(0,\pi)\,,\ 1<p<+\infty$.

Introduce to the consideration the following system

$$\varphi_0(x)=u_0(x)\,,\ \varphi_{2k-1}(x)=u_{2k-1}(x)\,,k\in\mathbb{N},$$
$$\varphi_{2k}(x)=\left(u_{2k}(x)-u_{2k-1}(x)\right)(2\delta_k)^{-1},k\in\mathbb{N}, \tag{3.8}$$

which is a linear combination of the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$. It is valid the following theorem.

Theorem 3.1 *The system* $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ *forms an equivalent to the system* $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ *basis for* $L_p(I)\,,1<p<+\infty$, *with biorthogonal system* $\{\psi_n(x)\}_{n\in\mathbb{Z}_+}$ *where*

$$\psi_0(x)=\vartheta_0(x)\,,\ \psi_{2k-1}(x)=\vartheta_{2k}(x)+\vartheta_{2k-1}(x)\,,\ \psi_{2k}(x)=2\delta_k\vartheta_{2k}(x)\,. \tag{3.9}$$

Proof. Let us show that the system of functions $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ forms a basis in $L_p(0,\pi)\,,\ 1<p<+\infty$. Obviously, it is complete and minimal in this space. Completeness follows from the completeness of the system $\{u_n(x)\}_{n\in\mathbb{Z}_+}$ in $L_p(0,\pi)$. On the other hand, the systems $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ and $\{\psi_n(x)\}_{n\in\mathbb{Z}_+}$ are biorthogonal, which is verified directly, from here we obtain the minimality of this system.

According to Theorem 2.2, the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ forms a basis in $L_p(0,\pi)$. From formulas (3.8) we have

$$\varphi_{2k-1}(x)-w_{2k-1}(x)=0,\ \ k\in\mathbb{N};$$
$$\varphi_{2k}(x)=\frac{1}{2\delta_k}\left(\sin\left((2k+2\delta_k)x\right)\ -\sin 2k\ x\right)=$$
$$=\frac{\sin(\delta_k x)}{\delta_k x}\cdot x\cos\left((2k+2\delta_k)x\right)\ =$$
$$=\left(1+O(\delta_k)\right)x\cos(2kx)\ \left(1+O\left(\delta_k^2\right)\right)=x\cos(2kx)\ +O(\delta_k)=$$
$$=w_{2k}(x)+O\left(\frac{1}{k}\right),$$

or

$$\varphi_{2k}(x)-w_{2k}(x)=O\left(\frac{1}{k}\right).$$

Then, for any $p>1$, we have

$$\|\varphi_n-w_n\|_{L_p}=O\left(\frac{1}{k}\right).$$

As a result, we find that for any $s,\ p\in(1,+\infty)$, we have

$$\sum_{n=0}^{\infty}\|\varphi_n-w_n\|_{L_p}^s<+\infty, \tag{3.10}$$

i.e., the systems $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ and $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ are s-close in the space $L_p(0,\pi)$.

In addition, as follows from the Hausdorff-Young inequality (see [28]), for $1<p\le 2$ the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is a p'-basis in $L_p(0,\pi)$, i.e.

$$\left(\sum_{i=1}^{2}\sum_{k=0}^{\infty}\left|\left\langle f,w_k^{(i)}\right\rangle\right|^{p'}\right)^{\frac{1}{p'}}\le C\|f\|_{L_p},\forall f\in L_p(0,\pi)\,,$$

and choosing $s = p$ in (3.6) we obtain that the systems $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ and $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ are p-close.

If $p \in (2;+\infty)$, then $p' \in (1;2)$ and from the embedding $L_p(0,\pi) \subset L_{p'}(0,\pi)$ we have:

$$\left(\sum_{i=1}^{2}\sum_{k=0}^{\infty}\left|\left\langle f, w_k^{(i)}\right\rangle\right|^p\right)^{\frac{1}{p}} \le C\|f\|_{L_{p'}} \le C_1\|f\|_{L_p}, \forall f \in L_p(0,\pi),$$

i.e. the system $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ is a p-basis in $L_p(0,\pi)$, and choosing) $s = p'$ in (3.10), we obtain that the systems $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ and $\{w_n(x)\}_{n\in\mathbb{Z}_+}$ are p'-close. Thus, all the conditions of Theorem 2.1 are satisfied, and therefore the system $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ forms a basis in $L_p(0,\pi)$, equivalent to the basis $\{w_n(x)\}_{n\in\mathbb{Z}_+}$.

The theorem is proved.

Corollary 3.2 *The systems* $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ *and* $\{\psi_n(x)\}_{n\in\mathbb{Z}_+}$ *defined by formulas* (3.8) *and* (3.9) *form an s-basis in* $L_p(0,\pi)$ *for any* $p \in (1, +\infty)$, *where* $s = \max\{p, p'\}$.

From Theorem 3.1, as well as from Corollary 3.2, we obtain, in particular, that for $p = 2$ the following corollary holds.

Corollary 3.3 *Both systems* $\{\varphi_n(x)\}_{n\in\mathbb{Z}_+}$ *and* $\{\psi_n(x)\}_{n\in\mathbb{Z}_+}$ *defined by formulas* (3.8) *and* (3.9) *form a Riesz basis in the space* $L_2(0,\pi)$.

4 Strong solution of BVP

Before formulating the statement of the BVP, accept the following notations: $L_p(D)$ is the Lebesgue space with the norm

$$\|u\|_{L_p(D)} = \left(\iint_D |u(x,y)|^p dxdy\right)^{\frac{1}{p}} = \left(\int_0^1\int_0^\pi |U(r,\theta)|^p r dr d\theta\right)^{\frac{1}{p}}, \tag{4.1}$$

where $U(r,\theta) = u(r\cos\theta\ ,\ r\sin\theta\ )$. $W_p^2(D)$ is the Sobolev space generated by the norm (4.1), i.e., the space of functions with generalized derivatives up to second order with finite norm

$$\|u\|_{W_p^2(D)} = \sum_{|\alpha|\le 2}\left\|\frac{\partial^{|\beta|}u}{\partial x^{\beta_1}\partial y^{\beta_2}}\right\|_{L_p(D)},$$

where $\beta = (\beta_1,\beta_2)$; $\beta_1,\beta_2 \in \mathbb{Z}_+$, $|\beta| = \beta_1+\beta_2$;

$$\Gamma_0 = \{(r,\theta) : r \in (0,1),\ \theta = 0\},$$
$$\Gamma_1 = \{(r,\theta) : r = 1,\ \theta \in (0,\pi)\},$$
$$\Gamma_2 = \{(r,\theta) : r \in (0,1),\ \theta = \pi\}.$$

Consider the following BVP

$$\left.\begin{array}{c} \Delta U = 0 \text{ in } D, \\ B_{\Gamma_1}U = f, \\ B_{\Gamma_0}U = 0, \\ B_{\Gamma_0}\left(\frac{\partial U}{\partial\theta}\right) = B_{\Gamma_2}\left(\frac{\partial U}{\partial\theta}\right) + \alpha B_{\Gamma_2}U, \end{array}\right\} \tag{4.2}$$

where B_{Γ_i}, $i = 0,1,2$, denotes corresponding trace operator, $\alpha \in \mathbb{R}$; $f \in W_p^2(0,\pi)$, $1 < p < +\infty$, is a given function and $U \in W_p^2(D)$.

Firstly let's prove that the problem (4.2) may have only a unique solution in $W_p^2(D)$.

Theorem 4.1 *For any* $f \in L_p(0,\pi)$, $1 < p < +\infty$, *the boundary value problem* (4.2) *can have only one solution in the Sobolev class* $W_p^2(D)$.

Proof. It is sufficient to prove that the corresponding homogeneous problem (i.e. the case $f = 0$) has only zero solution. Let $V(r,\theta)$ any solution of the homogeneous problem in Sobolev space $W_p^2(D)$:

$$\left.\begin{array}{c} \Delta V(r,\theta) = 0 \text{ in } D, \\ V(1,\theta) = 0, \\ V(r,0) = 0, \\ \frac{\partial V(r,0)}{\partial\theta} = \frac{\partial V(r,\pi)}{\partial\theta} + \alpha V(r,\pi). \end{array}\right\} \tag{4.3}$$

Since the system $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ forms basis for $L_p(I)$, then any solution of the problem (4.3) representable in the form

$$V(r,\theta) = \sum_{n=0}^{\infty} R_n^0(r)\,\varphi_n(\theta), \tag{4.4}$$

where

$$R_n^0(r) = \int_0^\pi V(r,\theta)\,\psi_n(\theta)\,d\theta, \quad n \in \mathbb{Z}_+.$$

Let's take into account that the functions $\{\psi_n\}_{n\in\mathbb{N}}$ satisfy the boundary conditions of the spectral problem (3.2). It is known that for a.e. $\theta \in (0,\pi)$ it holds $V(\cdot,\theta) \in W_p^2(0,1)$ (see, f.e. the monograph [33, p.41, Theorem 2.7.1]). Then it is evident that for every fixed $n \in \mathbb{Z}_+ :\ R_n^0(r) \in W_p^2(0,1)$, and we have

$$\frac{d^2R_n^0(r)}{dr^2} = \int_0^\pi \frac{\partial^2 V(r,\theta)}{\partial r^2}\psi_n(\theta)\,d\theta.$$

Paying attention to the relation

$$\frac{\partial^2 V(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial V(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V(r,\theta)}{\partial\theta^2} = 0, \quad (r,\theta) \in D,$$

we obtain

$$\frac{d^2R_n^0(r)}{dr^2} = -\frac{1}{r}\frac{dR_n^0(r)}{dr} - \frac{1}{r^2}\int_0^\pi \frac{\partial^2 V(r,\theta)}{\partial\theta^2}\psi_n(\theta)\,d\theta.$$

Further considering the boundary conditions for $V(r,\theta)$ and $\psi_n(\theta)$ we establish that $R_n^0(r)$ satisfies the equation

$$\left.\begin{array}{c} r^2\frac{d^2R_{2k}^0(r)}{dr^2} + r\frac{dR_{2k}^0(r)}{dr} - \lambda_{2k}R_{2k}^0(r) = 0, \\ r^2\frac{d^2R_{2k-1}^0(r)}{dr^2} + r\frac{dR_{2k-1}^0(r)}{dr} - \lambda_{2k-1}R_{2k-1}^0(r) - \frac{\lambda_{2k}-\lambda_{2k-1}}{2\delta_k}R_{2k}^0(r) = 0, \end{array}\right\} \tag{4.5}$$

where $\lambda_{2k-1} = (2k)^2,\ k \in \mathbb{N},\ \lambda_{2k} = (\rho_{2k})^2,\ k \in \mathbb{Z}_+$. Since $R_n^0(r) \in W_p^2(0,1),\ n \in \mathbb{Z}_+$, the solution of the system (4.5) has the form

$$\left.\begin{array}{c} R_{2k}^0(r) = c_{2k}r^{\sqrt{\lambda_{2k}}},\ \ r \in (0,1], \\ R_{2k-1}^0(r) = c_{2k-1}r^{\sqrt{\lambda_{2k-1}}} + c_{2k}\frac{1}{2\delta_k}\left(r^{\sqrt{\lambda_{2k}}} - r^{\sqrt{\lambda_{2k-1}}}\right),\ \ r \in (0,1], \end{array}\right\} \tag{4.6}$$

where $c_n,\ n \in \mathbb{Z}_+$, are constants, non-depending of r. Moreover, completely analogously to the work [16] we establish

$$R_n^0(1) = \int_0^\pi V(1,\theta)\,\psi_n(\theta)\,d\theta,\ n \in \mathbb{Z}_+,$$

where $V(1,\theta) = B_{\Gamma_1}V$. Taking into account the boundary conditions (4.2) for $V(r,\theta)$ we obtain $R_n^0(1) = 0$, consequently $c_n = 0,\ \forall n \in \mathbb{N}$. Finally, we have $R_n^0(r) \equiv 0, \forall n \in \mathbb{Z}_+$. From basicity of the system $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ for $L_p(0,\pi)$ follows the basicity of the biorthogonal to it system $\{\psi_n\}_{n\in\mathbb{Z}_+}$ for $L_{p'}(0,\pi)$. Then from expression (4.4) we obtain that $V(r,\theta) = 0$, a.e. in D. From Sobolev's embedding theorems (see, for example, [29,34,35]) follows that $W_p^2(D) \subset C\left(\hat{D}\right), p > 1$. Therefore, we can state that $V(r,\theta) \equiv 0,\ \forall (r,\theta) \in \overline{D}$.

And now consider the solvability question of the BVP (4.2) in $W_p^2(D)$. By $W_{p,U}^{2+\gamma}(0,\pi)$ we denote the space of functions f from $W_{p,U}^2(0,\pi)$ for which $f'' \in W_{p,U}^{\gamma}(0,\pi)$.

Theorem 4.2 *Let* $\alpha \in \mathbb{R},\ \alpha \neq 0,\ f \in W_{p,U}^{2+\gamma}(0,\pi),\ 1 < p < +\infty,$ *and* $\frac{1}{s'} - \frac{1}{s} < \gamma < 1,$ *where* $s' = \min\{p,p'\}$ *. Then for* $1 < p < \frac{1}{1-\delta_0}$ *the BVP* (4.2) *is solvable in* $W_p^2(D)$ *and the solution has the following representation:*

$$U(r,\theta) = \sum_{k=1}^{\infty}\left(f_{2k-1}r^{2k} + \frac{1}{2\delta_k}f_{2k}\left(r^{\rho_{2k}} - r^{2k}\right)\right)\varphi_{2k-1}(\theta) + \sum_{k=0}^{\infty} f_{2k}r^{\rho_{2k}}\varphi_{2k}(\theta),$$

where $\{f_n\}_{n\in\mathbb{Z}_+}$ *are the biorthogonal coefficients of* f *on the basis* $\{\varphi_n(\theta)\}_{n\in\mathbb{Z}_+}$ *in* $L_p(0,\pi)$ *. For* $\frac{1}{1-\delta_0} \le p < \infty$*, BVP* (4.2) *is solvable if and only if the following condition is additionally satisfied:*

$$f_0 = \int_0^{\pi} f(\theta)\psi_0(\theta)\,d\theta = 0.$$

Under this condition, the solution can be represented in the form

$$U(r,\theta) = \sum_{k=1}^{\infty}\left(f_{2k-1}r^{2k} + \frac{1}{2\delta_k}f_{2k}\left(r^{\rho_{2k}} - r^{2k}\right)\right)\varphi_{2k-1}(\theta) + \sum_{k=1}^{\infty} f_{2k}r^{\rho_{2k}}\varphi_{2k}(\theta).$$

Moreover, for the solution it is valid the following estimation:

$$\|U\|_{W_p^2(D)} \le c\|f\|_{W_p^{2+\gamma}(0,\pi)}.$$

Proof. According to Theorem 3.1 the system $\{\varphi_n\}_{n\in\mathbb{Z}_+}$ forms basis for $L_p(0,\pi),\ 1 < p < +\infty$, and in result the solution $U(r,\theta)$ for a.e. $r \in (0,1)$ in $L_p(0,\pi)$ is representable in the form (because by Fubini's theorem for a.e. $r \in (0,1)$ it holds $U(r,\cdot) \in L_p(0,\pi)$)

$$U(r,\theta) = \sum_{n=0}^{\infty} R_n(r)\varphi_n(\theta), \tag{4.7}$$

where the coefficients $R_n(r),\ n \in \mathbb{N},$ are defined by the expressions

$$R_n(r) = \int_0^{\pi} U(r,\theta)\,\psi_n(\theta)\,d\theta,\ n \in \mathbb{Z}_+.$$

Using the fact that $B_{\Gamma_i} \in \left[W_p^2(D); L_p(\mathit{\Gamma}_i)\right]$, $i = 0,1,2$, [29,34,35] similarly to the previous case, we obtain that the coefficients $R_n(r),\ n \in \mathbb{Z}_+$, satisfy the system of equations (4.5) and it have the expressions (4.6). From here direct follows

$$\left.\begin{array}{c} R_{2k}(r) = R_{2k}(1)\,r^{\rho_{2k}},\ \forall r \in (0,1], \\ R_{2k-1}(r) = R_{2k-1}(1)\,r^{2k} + R_{2k}(1)\,\frac{1}{2\delta_k}\left(r^{\rho_{2k}} - r^{2k}\right), \forall r \in (0,1]. \end{array}\right\} \tag{4.8}$$

Assume

$$U_m(r,\theta)=\sum_{n=0}^{m}R_n(r)\varphi_n(\theta)\,,\ m\in\mathbb{Z}_+.$$

It is evident that

$$B_{\Gamma_1}U_m=\sum_{n=0}^{m}R_n(1)\varphi_n(\theta)\,,\ \ \forall m\in\mathbb{Z}_+.$$

Then from $B_{\Gamma_1}\in\left[W_p^2(D)\,,L_p(\Gamma_1)\right]$ follows

$$B_{\Gamma_1}U=\sum_{n=0}^{\infty}R_n(1)\varphi_n(\theta)\,.$$

Let's expand the function $f(\theta)$ on basis in $L_p(0,\pi)$:

$$f(\theta)=\sum_{n=0}^{\infty}f_n\varphi_n(\theta)\,,$$

where

$$f_n=\int_0^{\pi}f(\theta)\,\psi_n(\theta)\,d\theta,\ n\in\mathbb{Z}_+.$$

Note that the system $\{\psi_n(\theta)\}_{n\in\mathbb{Z}_+}$ has the expressions (3.9). Let $f\in W_{p,U}^{2+\gamma}(0,\pi)\,.$ Then for biorthogonal coefficients $f_n,\ n\in\mathbb{Z}_+$, we have

$$\begin{aligned}f_{2k}=\langle f,\psi_{2k}\rangle=\left\langle L^{-\frac{2+\gamma}{2}}g,\psi_{2k}\right\rangle=\left\langle g,(L^*)^{-\frac{2+\gamma}{2}}\psi_{2k}\right\rangle\\=\frac{1}{(\lambda_{2k})^{\frac{2+\gamma}{2}}}\langle g,\psi_{2k}\rangle=\frac{g_{2k}}{(\lambda_{2k})^{\frac{2+\gamma}{2}}},\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}f_{2k-1}=\langle f,\psi_{2k}\rangle=\langle f,\vartheta_{2k}\rangle+\langle f,\vartheta_{2k-1}\rangle=\left\langle L^{-\frac{2+\gamma}{2}}g,\vartheta_{2k}\right\rangle\\+\left\langle L^{-\frac{2+\gamma}{2}}g,\vartheta_{2k-1}\right\rangle=\frac{g_{2k}}{(\lambda_{2k})^{\frac{2+\gamma}{2}}}+\frac{g_{2k-1}}{(\lambda_{2k-1})^{\frac{2+\gamma}{2}}}.\end{aligned}\tag{4.10}$$

Let us show that $U(r,\theta)$ is a solution of the BVP (4.2) in $W_p^2(D)\,.$ To do this we show that series (4.7) can be differentiated twice and the resulting series converge according to the norm $L_p(D)$. From the expressions (4.9), (4.10) we directly obtain

$$|f_{2k}|\leq\frac{|g_{2k}|}{|\lambda_{2k}|^{\frac{2+\gamma}{2}}}\leq c\frac{|g_{2k}|}{k^{2+\gamma}};\tag{4.11}$$

$$|f_{2k-1}|\leq\frac{|g_{2k}|}{|\lambda_{2k}|^{\frac{2+\gamma}{2}}}+\frac{|g_{2k-1}|}{|\lambda_{2k-1}|^{\frac{2+\gamma}{2}}}\leq c\frac{|g_{2k}|+|g_{2k-1}|}{k^{2+\gamma}}.\tag{4.12}$$

Let's estimate the series (4.7)

$$\|U\|_{L_p(D)}=\left(\int_0^1\int_0^{\pi}\left|\sum_{n=0}^{\infty}R_n(r)\,\varphi_n(\theta)\right|^p rdr\,d\theta\right)^{\frac{1}{p}}$$

$$\le \sum_{n=0}^{\infty}\left(\int_0^{\pi} |\varphi_n(\theta)|^p d\theta \int_0^1 r|R_n(r)|^p dr\right)^{\frac{1}{p}} \lesssim \sum_{n=0}^{\infty}\left(\int_0^1 r|R_n(r)|^p dr\right)^{\frac{1}{p}}$$

$$= \sum_{k=1}^{\infty}\left(\int_0^1 r|R_{2k-1}(r)|^p dr\right)^{\frac{1}{p}} + \sum_{k=0}^{\infty}\left(\int_0^1 r|R_{2k}(r)|^p dr\right)^{\frac{1}{p}}. \tag{4.13}$$

Then, taking into account that $R_n(1) = f_n$, from (4.8) we obtain

$$\left(\int_0^1 r|R_{2k}(r)|^p dr\right)^{\frac{1}{p}} = \left(\int_0^1 r|R_{2k}(1)\, r^{\rho_{2k}}|^p dr\right)^{\frac{1}{p}}$$

$$= |f_{2k}|\left(\int_0^1 r^{1+p(2k+2\delta_k)} dr\right)^{\frac{1}{p}} = \frac{|f_{2k}|}{(2+p(2k+2\delta_k))^{\frac{1}{p}}} \lesssim \frac{|f_{2k}|}{k^{\frac{1}{p}}};$$

$$\left(\int_0^1 r|R_{2k-1}(r)|^p dr\right)^{\frac{1}{p}} \le \left(\int_0^1 r\left|R_{2k-1}(1)\, r^{2k}\right|^p dr\right)^{\frac{1}{p}}$$

$$+\left(\int_0^1 r\left|\frac{R_{2k}(1)\, r^{2k}\left(r^{2\delta_k}-1\right)}{2\delta_k}\right|^p dr\right)^{\frac{1}{p}}$$

$$\lesssim |R_{2k-1}(1)|\left(\frac{1}{2kp+1}\right)^{\frac{1}{p}} + |R_{2k}(1)|\left(\frac{1}{2kp+1}\right)^{\frac{1}{p}} \lesssim \frac{|f_{2k-1}|+|f_{2k}|}{k^{\frac{1}{p}}}. \tag{4.14}$$

taking into account inequalities (4.12) and (4.14) in (4.13), for $U(r,\theta)$ we obtain

$$\|u\|_{L_p(D)} \lesssim \|g\|_{L_p} \lesssim \|f\|_{W_p^{2+\gamma}}.$$

To obtain the estimate $\|U\|_{W_p^2(D)} \lesssim \|f\|_{W_p^{2+\gamma}(0,\pi)}$, we need to separately estimate the norms of all derivatives $D^{\beta}u = \frac{\partial^{|\beta|}u}{\partial x^{\beta_1}\partial y^{\beta_2}}$, $|\beta| \le 2$; $\beta = (\beta_1, \beta_2)$, $|\beta| = \beta_1 + \beta_2$. Using the formulas for the derivatives $D^{\beta}u$ in polar coordinates, we have

$$max\left\{\left|\frac{\partial u}{\partial x}\right|; \left|\frac{\partial u}{\partial y}\right|\right\} \le \left|\frac{\partial U}{\partial r}\right| + \frac{1}{r}\left|\frac{\partial U}{\partial \theta}\right|,$$

$$max\left\{\left|\frac{\partial^2 u}{\partial x^2}\right|; \left|\frac{\partial^2 u}{\partial y^2}\right|; \left|\frac{\partial^2 u}{\partial x \partial y}\right|\right\} \le \left|\frac{\partial^2 U}{\partial r^2}\right| + \frac{1}{r^2}\left|\frac{\partial^2 U}{\partial \theta^2}\right| + \frac{1}{r^2}\left|\frac{\partial U}{\partial \theta}\right| + \frac{1}{r}\left|\frac{\partial^2 U}{\partial r \partial \theta}\right| + \frac{1}{r}\left|\frac{\partial U}{\partial r}\right|.$$

Let us estimate the norm of $\frac{1}{r^2}\frac{\partial^2 U(r,\theta)}{\partial \theta^2}$. The norm of the remaining terms is estimated similarly. From this we obtain

$$\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}\frac{\partial^2 U(r,\theta)}{\partial \theta^2}\right|^p r dr d\theta\right)^{\frac{1}{p}} = \left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}\sum_{n=0}^{\infty} R_n(r)\, \varphi_n''(\theta)\right|^p r dr d\theta\right)^{\frac{1}{p}}$$

$$\le \sum_{n=0}^{\infty}\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}R_n(r)\, \varphi_n''(\theta)\right|^p r dr d\theta\right)^{\frac{1}{p}} \le \rho_0^2\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}R_0(r)\, \varphi_0(\theta)\right|^p r dr d\theta\right)^{\frac{1}{p}}$$

$$+\sum_{k=1}^{\infty}\rho_{2k-1}^2\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}R_{2k-1}(r)\, \varphi_{2k-1}(\theta)\right|^p r dr d\theta\right)^{\frac{1}{p}}$$

$$+\sum_{k=1}^{\infty}\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}R_{2k}\left(r\right)\left(\rho_{2k}^2\varphi_{2k}\left(\theta\right)-\rho_{2k-1}^2\varphi_{2k-1}\left(\theta\right)\right)\right|^p rdrd\theta\right)^{\frac{1}{p}} := I_0+\ I_1+I_2.$$

Now, taking into account (4.8) and substituting there $R_n\left(1\right)=f_n,\ n\in\mathbb{Z}_+$, we obtain

$$I_1\le\sum_{k=1}^{\infty}k^2\left(\int_0^1\left|\frac{1}{r^2}R_{2k-1}\left(r\right)\right|^p rdr\right)^{\frac{1}{p}}$$

$$\le\sum_{k=1}^{\infty}k^2\left(\int_0^1\left|\frac{1}{r^2}R_{2k-1}\left(1\right)r^{2k}\right|^p r\,dr\right)^{\frac{1}{p}}+\sum_{k=1}^{\infty}k^2\left(\int_0^1\left|\frac{1}{r^2}\frac{R_{2k}\left(1\right)r^{2k}\left(r^{2\delta_k}-1\right)}{2\delta_k}\right|^p r\,dr\right)^{\frac{1}{p}}$$

$$=\sum_{k=1}^{\infty}k^2\left|f_{2k-1}\right|\left(\int_0^1 r^{(2k-2)p+1}\,dr\right)^{\frac{1}{p}}+\sum_{k=1}^{\infty}k^2\left|f_{2k}\right|\left(\int_0^1 r^{(2k-2)p+1}\left|\frac{\left(r^{2\delta_k}-1\right)}{2\delta_k}\right|^p dr\right)^{\frac{1}{p}}$$

(taking into account the boundedness of $\frac{\left(r^{2\delta_k}-1\right)}{2\delta_k}$ for $\delta_k=O\left(\frac{1}{k}\right)$, as well as estimates (4.11) and (4.12))

$$\lesssim\sum_{k=1}^{\infty}k^2\frac{\left|f_{2k-1}\right|+\left|f_{2k}\right|}{\left(\left(2k-2\right)p+2\right)^{\frac{1}{p}}}\lesssim\sum_{k=1}^{\infty}k^2\frac{\left|f_{2k-1}\right|+\left|f_{2k}\right|}{k^{\frac{1}{p}}}$$

$$\lesssim\sum_{n=1}^{\infty}\frac{\left|g_n\right|}{n^{\gamma+\frac{1}{p}}}\lesssim\left(\sum_{n=1}^{\infty}\frac{1}{n^{\left(\gamma+\frac{1}{p}\right)s'}}\right)^{\frac{1}{s'}}\left(\sum_{n=1}^{\infty}\left|g_n\right|^s\right)^{\frac{1}{s}}\lesssim\left\|g\right\|_{L_{s'}}\lesssim\left\|g\right\|_{L_p}\lesssim\left\|f\right\|_{W_p^{2+\gamma}}.$$

I_2 is estimated similarly:

$$I_2\le\sum_{k=1}^{\infty}k^2\left(\int_0^1\int_0^{\pi}\left|\frac{1}{r^2}R_{2k}\left(r\right)\right|^p rdrd\theta\right)^{\frac{1}{p}}\lesssim\sum_{k=1}^{\infty}k^2\left|f_{2k}\right|\left(\int_0^1 r^{(\rho_{2k}-2)p+1}\,dr\right)^{\frac{1}{p}}$$

$$\lesssim\sum_{k=1}^{\infty}k^2\frac{\left|f_{2k}\right|}{\left(\left(\rho_{2k}-2\right)p+2\right)^{\frac{1}{p}}}\lesssim\sum_{k=1}^{\infty}k^2\frac{\left|f_{2k}\right|}{k^{\frac{1}{p}}}\lesssim\sum_{k=1}^{\infty}\frac{\left|g_{2k}\right|}{k^{\gamma+\frac{1}{p}}}$$

$$\le\left(\sum_{k=1}^{\infty}\frac{1}{k^{\left(\gamma+\frac{1}{p}\right)s'}}\right)^{\frac{1}{s'}}\left(\sum_{k=1}^{\infty}\left|g_{2k}\right|^s\right)^{\frac{1}{s}}\lesssim\left\|g\right\|_{L_{s'}}\lesssim\left\|g\right\|_{L_p}\lesssim\left\|f\right\|_{W_p^{2+\gamma}}.$$

The convergence of the integral in I_0 depends on the location of $\rho_0=2\delta_0$, as well as on the value of p. Note that (see [2,4]) under the condition $\alpha\in\mathbb{R},\ \alpha\neq 0$, depending on the sign of α estimate (3.1) can be refined as follows:

$$2k<\rho_{k,2}<2k+1,\ k\in\mathbb{Z}_+,\quad\text{if}\quad\alpha>0,$$

$$2k+1<\rho_{k,2}<2k+2,\ k\in\mathbb{Z}_+,\quad\text{if}\quad\alpha<0.$$

From here, in particular, we have $0<\rho_0<1,$ if $\alpha>0$ and $1<\rho_0<2$, if $\alpha<0$. This means that $0<\delta_0<\frac{1}{2},$ if $\alpha>0$ and $\frac{1}{2}<\delta_0<1$, if $\alpha<0$. Therefore, in both cases, for

$1 < p < \frac{1}{1-\delta_0}$, the integral in I_0 converges, and for $p \geq \frac{1}{1-\delta_0}$, it diverges, and a solution to the problem exists under the additional condition $f_0 = 0$. Therefore

$$\left(\int_0^1 \int_0^\pi \left|\frac{1}{r^2}\frac{\partial^2 U\left(r,\theta\right)}{\partial\theta^2}\right|^p rdrd\theta\right)^{\frac{1}{p}} \lesssim \|f\|_{W_p^{2+\gamma}}.$$

Completely analogously to above we establish

$$\|u\|_{W_p^2(D)} \lesssim \|f\|_{W_p^{2+\gamma}}$$

From here direct follows that the function $U\left(r,\theta\right)$, defined by series (4.7), is a solution of the equation $\mathit{\Delta} U = 0$ in $W_p^2\left(D\right)$. Moreover, from $B_{\Gamma_1} \in \left[W_p^2\left(D\right),\ L_p\left(\Gamma_1\right)\right]$ follows

$$B_{\Gamma_1}U = \sum_{n=0}^{\infty} R_n\left(1\right)\varphi_n\left(\theta\right) = \sum_{n=0}^{\infty} f_n\varphi_n = f.$$

At the same time, since the functions $\left\{\varphi_n\left(\theta\right)\right\}_{n\in\mathbb{Z}_+}$ satisfy the boundary conditions (1.2), also from $B_{\Gamma_i} \in \left[W_{p,1}^2\left(D\right),\ L_p\left(\Gamma_i\right)\right]$, $i = 0,1,2$, follows that the function $U\left(r,\theta\right)$ satisfies

$$B_{\Gamma_0}U = 0,\ \ B_{\Gamma_0}\left(\frac{\partial U}{\partial\theta}\right) - B_{\Gamma_2}\left(\frac{\partial U}{\partial\theta}\right) - \alpha B_{\Gamma_2}U = 0.$$

To complete the proof, we will show that under the conditions of the theorem for $p \geq \frac{1}{1-\delta_0}$, the condition $f_0 = 0$ is also necessary for the existence of a solution to *BVP* (4.2). Suppose that the function $V\left(r,\theta\right)$ is a solution to BVP (4.2). We put $U\left(r,\theta\right) = V\left(r,\theta\right) - R_0\left(r\right)\varphi_0\left(\theta\right)$. It is directly verified that $U\left(r,\theta\right)$ is also a solution to problem (4.2). Then, from the uniqueness of the solution (Theorem 4.1), it follows that $U\left(r,\theta\right) = V\left(r,\theta\right)$ a.e. in D. Considering that $R_0\left(r\right)\varphi_0\left(\theta\right) = f_0 r^{\rho_0}\sin\rho_0\theta$, $\rho_0 > 0$, this is possible only in the case $f_0 = 0$.

Theorem is proved.

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