Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **46** (2), 1-16 (2026).

Spectral analysis of a Sturm–Liouville operator with multiple impulse points

Rakib Efendiev, Hala Abd-Elmageed *

Received: 02.11.2025 / Revised: 28.05.2026 / Accepted: 17.06.2026

Abstract. In this paper, we study an impulsive Sturm–Liouville problem on the half-line. The problem contains one interior impulse point and a complex almost-periodic potential. Fundamental solutions are constructed, the transfer matrix is derived, the Green function and resolvent are obtained, and the eigenvalues and spectral singularities of the corresponding operator are characterized.

1 Introduction

Impulsive differential equations arise naturally in many branches of applied mathematics, mathematical physics, control theory, population dynamics, and engineering sciences. In numerous physical models, the state of a system undergoes abrupt changes at certain moments or spatial points. Such discontinuities are frequently described by impulsive or transmission conditions imposed on the solutions of differential equations.

One of the most important classes of impulsive problems is associated with Sturm–Liouville operators. During the last decades, considerable attention has been devoted to the spectral analysis of Sturm–Liouville operators with discontinuities, transmission conditions, and impulse effects. These investigations are motivated by applications in quantum mechanics, wave propagation through layered media, heat conduction in composite materials, and vibrating systems with concentrated masses.

The spectral theory of Sturm–Liouville operators with impulsive and transmission conditions has been extensively developed by many authors. Spectral properties, eigenvalue problems, and resolvent constructions for impulsive Sturm–Liouville operators were studied in [2, 15]. Problems involving discontinuous coefficients and transmission conditions were investigated in [7, 8, 13, 14]. The theory of non-self-adjoint Sturm–Liouville operators with complex-valued almost-periodic potentials originates from the pioneering work of Gasymov [6]. Further developments concerning spectral singularities and non-self-adjoint spectral problems may be found in [9, 11]. General aspects of Sturm–Liouville theory are presented in the monographs [12, 16].

* Corresponding author

Rakib Efendiev
Baku Engineering University, Department of Mathematics and Computer Sciences, AZ0101, Baku, Azerbaijan
E-mail: refendiyev@beu.edu.az

Hala Abd-Elmageed
Department of Mathematics, College of Science and Humanities in Al-Kharj Prince Sattam bin Abdulaziz University, Al-Kharj, 11942, Saudi Arabia.
E-mail: hal.mohamed@psau.edu.sa

Recently, Efendiev and co-authors investigated several classes of discontinuous and impulsive Sturm–Liouville operators with complex almost-periodic potentials [1,3–5]. In particular, the paper [4] studied a non-self-adjoint impulsive Sturm–Liouville operator containing a single impulse point. Jost-type solutions were constructed, the resolvent operator was obtained, and the eigenvalues and spectral singularities were characterized.

The purpose of the present paper is to extend these results to the case of an arbitrary finite number of impulse points. More precisely, we consider the impulsive Sturm–Liouville equation

$$-y'' + q(x)y = \lambda^2 \rho(x)y, \qquad x \in \mathbb{R} \setminus \{x_1, x_2, \dots, x_m\}, \tag{1.1}$$

where

$$-\infty < x_1 < x_2 < \cdots < x_m < +\infty,$$

the potential is represented by the absolutely convergent Fourier series

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty, \tag{1.2}$$

and the weight function is given by

$$\rho(x) = \begin{cases} 1, & x < 0, \\ \beta^2, & x > 0, \end{cases} \qquad \beta > 0, \quad \beta \neq 1. \tag{1.3}$$

At each impulse point x_k, the solution satisfies the transmission condition

$$\begin{pmatrix} y(x_k + 0) \\ y'(x_k + 0) \end{pmatrix} = T_k \begin{pmatrix} y(x_k - 0) \\ y'(x_k - 0) \end{pmatrix}, \qquad k = 1, \dots, m, \tag{1.4}$$

where

$$T_k = \begin{pmatrix} a_{1k} & a_{2k} \\ a_{3k} & a_{4k} \end{pmatrix}, \qquad \det T_k \neq 0.$$

The principal idea of the paper is to describe the propagation of solutions through the impulse points by means of transfer matrices. For each impulse point we construct a local transfer matrix, and by combining these matrices we obtain the total transfer matrix

$$M(\lambda) = M_m(\lambda) M_{m-1}(\lambda) \cdots M_1(\lambda).$$

This matrix plays a central role in the spectral analysis of the problem.

Using Jost-type solutions and the transfer matrix method, we construct the Green function and the resolvent operator of the corresponding non-self-adjoint Sturm–Liouville operator. We prove that all spectral information is encoded in the entry $M_{22}(\lambda)$ of the total transfer matrix. In particular, we establish that

$$M_{22}(\lambda) = 0$$

is the characteristic equation of the problem. The zeros of this function in the upper half-plane generate the eigenvalues, while its real zeros determine the spectral singularities.

The paper is organized as follows. In Section 2, the operator-theoretic formulation of the problem is presented. Section 3 is devoted to the construction of fundamental solutions and transfer matrices. In Section 4, the Green function and resolvent operator are obtained. Section 5 contains the characterization of eigenvalues and spectral singularities through the total transfer matrix. The continuous spectrum is also described. Finally, concluding remarks are given in the last section.

2 Operator Formulation

Let $H = L^2(\mathbb{R}, \rho(x)\,dx)$ with inner product

$$(u, v)_H = \int_{-\infty}^{\infty} u(x)\overline{v(x)}\rho(x)\,dx.$$

Define the operator

$$Ly = \frac{1}{\rho(x)}\big(-y'' + q(x)y\big)$$

on the domain

$$D(L) = \Big\{y \in H : y, y' \in AC_{\mathrm{loc}}(\mathbb{R} \setminus \{x_1, \dots, x_m\}),\ Ly \in H,\ \text{(1.4) holds}\Big\}.$$

Then L is a closed non-self-adjoint operator in H.

3 Construction of the Resolvent by Transfer Matrices

In this section we construct the fundamental solutions, local transfer matrices, total transfer matrix, Weyl solutions, Green function and the resolvent of the impulsive Sturm–Liouville operator.

The construction of the fundamental solutions is based on the method introduced by Gasymov [6] for non-self-adjoint Sturm–Liouville operators with complex periodic potentials. Gasymov proved that if

$$q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \qquad \sum_{n=1}^{\infty} |q_n| < \infty,$$

then the equation

$$-y'' + q(x)y = \lambda^2 y$$

admits special solutions of the form

$$g(x, \lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n + 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right),$$

where the coefficients $V_{n\alpha}$ are uniquely determined by a recurrent system of equations. Furthermore, the functions $g(x, \lambda)$ and $g(x, -\lambda)$ are linearly independent and their Wronskian is equal to $2i\lambda$. Motivated by Gasymov's construction, we seek solutions of the impulsive problem in the same form on each interval between two consecutive impulse points.

Consider the equation

$$-y'' + q(x)y = \lambda^2 \rho(x) y, \qquad x \in \mathbb{R} \setminus \{x_1, \dots, x_m\}. \tag{3.1}$$

Since the coefficient $\rho(x)$ is piecewise constant, the equation reduces to a Gasymov-type equation on each subinterval.

Following Gasymov's method, we seek a solution in the form

$$f^+(x, \lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{n + 2\lambda} e^{i\alpha x}\right).$$

Substituting this representation into the differential equation and comparing the coefficients of equal powers of $e^{i\alpha x}$, we obtain

$$\alpha(\alpha-n)V_{n\alpha}+\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}+q_\alpha=0, \qquad \alpha\geq n.$$

Consequently, the coefficients $V_{n\alpha}$ satisfy the recursive relations

$$V_{nn}=-\frac{q_n}{n^2},$$

and

$$V_{n\alpha}=-\frac{1}{\alpha(\alpha-n)}\left(q_\alpha+\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}\right), \qquad \alpha>n.$$

Since

$$\sum_{n=1}^{\infty}|q_n|<\infty,$$

the recurrence system uniquely determines all coefficients $V_{n\alpha}$. Moreover,

$$\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}|V_{n\alpha}|<\infty,$$

which implies absolute and uniform convergence of the series defining $f^+(x,\lambda)$ on compact subsets of the λ-plane away from the points

$$\lambda=-\frac{n}{2}, \qquad n\in\mathbb{N}.$$

Hence $f^+(x,\lambda)$ is analytic in $\mathbb{C}\setminus\{-n/2:n\in\mathbb{N}\}$.

Similarly, the second Jost solution is

$$f^-(x,\lambda)=e^{-i\lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n-2\lambda}e^{i\alpha x}\right),$$

which is analytic in $\mathbb{C}\setminus\{n/2:n\in\mathbb{N}\}$.

Furthermore,

$$W[f^+,f^-]=-2i\lambda,$$

and therefore $\{f^+,f^-\}$ forms a fundamental system of solutions whenever

$$\lambda\neq 0, \qquad \lambda\neq\pm\frac{n}{2}.$$

Applying the same construction on every subinterval determined by the impulse points yields the families $f_k^{\pm}(x,\lambda)$, $(k=0,\dots,m)$, which are used in the transfer-matrix analysis.

For $x<0$, where $\rho(x)=1$, we define

$$f_1^{\pm}(x,\lambda)=e^{\pm i\lambda x}\left(1+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n\pm 2\lambda}e^{i\Lambda_\alpha x}\right). \tag{3.2}$$

For $x>0$, where $\rho(x)=\beta^2$, we define

$$f_2^{\pm}(x,\lambda) = e^{\pm i\beta\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n \pm 2\beta\lambda}e^{i\Lambda_\alpha x}\right). \tag{3.3}$$

The functions $f_1^{\pm}$ and $f_2^{\pm}$ play the role of Jost solutions on the left and right half-axes, respectively.

Consequently, $f_1^{\pm}$ and $f_2^{\pm}$ are analytic functions of λ in the complex plane except at the points $\lambda = \pm\frac{n}{2}, \lambda = \pm\frac{n}{2\beta}, n \in N$ which correspond to possible poles of first order.

The Wronskians of these solutions are

$$W[f_1^+, f_1^-] = -2i\lambda, \qquad W[f_2^+, f_2^-] = -2i\beta\lambda. \tag{3.4}$$

Hence, for

$$\lambda \neq 0, \qquad \lambda \neq \pm\frac{n}{2}, \qquad \lambda \neq \pm\frac{n}{2\beta},$$

the pairs $\{f_1^+, f_1^-\}$ and $\{f_2^+, f_2^-\}$ form fundamental systems of solutions.

Lemma 3.1 *The Jost-type solutions $f_1^{\pm}(x,\lambda)$ and $f_2^{\pm}(x,\lambda)$ are meromorphic functions of the spectral parameter λ. Their only possible singularities are simple poles at*

$$\lambda = \pm\frac{n}{2}, \qquad \lambda = \pm\frac{n}{2\beta}, \qquad n \in \mathbb{N}.$$

Proof. Consider the solution

$$f_1^{+}(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}}{n + 2\lambda}e^{i\alpha x}\right).$$

Since

$$\sum_{n=1}^{\infty}|q_n| < \infty,$$

the coefficients $V_{n\alpha}$ obtained from the Gasymov recurrence relations satisfy

$$\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}|V_{n\alpha}| < \infty.$$

Fix a compact set

$$K \subset \mathbb{C} \setminus \left\{-\frac{n}{2} : n \in \mathbb{N}\right\}.$$

Since K does not contain any pole, there exists a constant $c_K > 0$ such that

$$|n + 2\lambda| \geq c_K, \qquad \lambda \in K, \quad n \in \mathbb{N}.$$

Therefore

$$\left|\frac{V_{n\alpha}}{n + 2\lambda}e^{i\alpha x}\right| \leq \frac{|V_{n\alpha}|}{c_K}.$$

Because

$$\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{|V_{n\alpha}|}{c_K} < \infty,$$

the Weierstrass M-test implies that the double series converges uniformly on every compact subset of

$$\mathbb{C} \setminus \left\{ -\frac{n}{2} : n \in \mathbb{N} \right\}.$$

Each term of the series is analytic in λ. Uniform convergence on compact subsets therefore allows termwise differentiation with respect to λ, and by the Weierstrass theorem the sum is analytic on

$$\mathbb{C} \setminus \left\{ -\frac{n}{2} : n \in \mathbb{N} \right\}.$$

The denominator $n + 2\lambda$ vanishes only at

$$\lambda = -\frac{n}{2},$$

and each such singularity is a simple pole.

Hence $f_1^+(x, \lambda)$ is meromorphic in λ with possible simple poles at $\lambda = -n/2$.

The proof for $f_1^-(x, \lambda)$ is identical, replacing $n + 2\lambda$ by $n - 2\lambda$, yielding poles at

$$\lambda = \frac{n}{2}.$$

For the right-half-line solutions

$$f_2^\pm(x, \lambda) = e^{\pm i\beta\lambda x} \left(1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{n \pm 2\beta\lambda} e^{i\alpha x} \right),$$

the same argument shows analyticity away from the zeros of $n \pm 2\beta\lambda$, namely

$$\lambda = \pm \frac{n}{2\beta}.$$

Therefore $f_1^\pm$ and $f_2^\pm$ are meromorphic functions of λ, and their only possible singularities are simple poles at

$$\lambda = \pm \frac{n}{2}, \qquad \lambda = \pm \frac{n}{2\beta}, \qquad n \in \mathbb{N}.$$

These solutions constitute the basic building blocks for the construction of the local transfer matrices, the total transfer matrix, and the resolvent of the impulsive Sturm–Liouville operator.

Let x_k be an impulse point. On the interval to the left of x_k, we write the solution in the form

$$y_{k-1}(x, \lambda) = A_{k-1} f_{k-1}^+(x, \lambda) + B_{k-1} f_{k-1}^-(x, \lambda). \tag{3.5}$$

On the interval to the right of x_k, we write

$$y_k(x, \lambda) = A_k f_k^+(x, \lambda) + B_k f_k^-(x, \lambda). \tag{3.6}$$

The coefficients A_{k-1}, B_{k-1} describe the solution before the impulse point x_k, while A_k, B_k describe the solution after passing through x_k.

Now evaluate the solution and its derivative at x_k from the left. We obtain

$$\begin{pmatrix} y(x_k - 0) \\ y'(x_k - 0) \end{pmatrix} = D_k^-(\lambda) \begin{pmatrix} A_{k-1} \\ B_{k-1} \end{pmatrix}, \tag{3.7}$$

where

$$D_k^-(\lambda) = \begin{pmatrix} f_{k-1}^+(x_k,\lambda) & f_{k-1}^-(x_k,\lambda) \\ {f_{k-1}^+}'(x_k,\lambda) & {f_{k-1}^-}'(x_k,\lambda) \end{pmatrix}. \tag{3.8}$$

Similarly, from the right we have

$$\begin{pmatrix} y(x_k+0) \\ y'(x_k+0) \end{pmatrix} = D_k^+(\lambda)\begin{pmatrix} A_k \\ B_k \end{pmatrix}, \tag{3.9}$$

where

$$D_k^+(\lambda) = \begin{pmatrix} f_k^+(x_k,\lambda) & f_k^-(x_k,\lambda) \\ {f_k^+}'(x_k,\lambda) & {f_k^-}'(x_k,\lambda) \end{pmatrix}. \tag{3.10}$$

The impulse condition at x_k is

$$\begin{pmatrix} y(x_k+0) \\ y'(x_k+0) \end{pmatrix} = B_k\begin{pmatrix} y(x_k-0) \\ y'(x_k-0) \end{pmatrix}. \tag{3.11}$$

Substituting (3.7) and (3.9) into (3.11), we get

$$D_k^+(\lambda)\begin{pmatrix} A_k \\ B_k \end{pmatrix} = B_k D_k^-(\lambda)\begin{pmatrix} A_{k-1} \\ B_{k-1} \end{pmatrix}. \tag{3.12}$$

Since $D_k^+(\lambda)$ is invertible, we multiply by $\left(D_k^+(\lambda)\right)^{-1}$. Hence

$$\begin{pmatrix} A_k \\ B_k \end{pmatrix} = M_k(\lambda)\begin{pmatrix} A_{k-1} \\ B_{k-1} \end{pmatrix}, \tag{3.13}$$

where

$$M_k(\lambda) = \left(D_k^+(\lambda)\right)^{-1} B_k D_k^-(\lambda). \tag{3.14}$$

The matrix $M_k(\lambda)$ is called the local transfer matrix at the impulse point x_k. It transfers the coefficients of the solution from the interval before x_k to the interval after x_k.

Applying the local transfer relation successively at all impulse points, we obtain

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_1(\lambda)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix},$$

$$\begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = M_2(\lambda)\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M_2(\lambda)M_1(\lambda)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}.$$

Continuing this procedure up to the last impulse point x_m, we get

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = M_m(\lambda)M_{m-1}(\lambda)\cdots M_1(\lambda)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{3.15}$$

We define the total transfer matrix by

$$M(\lambda) = M_m(\lambda)M_{m-1}(\lambda)\cdots M_1(\lambda). \tag{3.16}$$

Therefore

$$\begin{pmatrix} A_m \\ B_m \end{pmatrix} = M(\lambda)\begin{pmatrix} A_0 \\ B_0 \end{pmatrix}. \tag{3.17}$$

Writing

$$M(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}, \tag{3.18}$$

we obtain

$$A_m = M_{11}(\lambda)A_0 + M_{12}(\lambda)B_0, \tag{3.19}$$

and

$$B_m = M_{21}(\lambda)A_0 + M_{22}(\lambda)B_0. \tag{3.20}$$

Thus $M(\lambda)$ completely describes the propagation of the solution through all impulse points.

The passage from a single impulse point to multiple impulse points produces several genuinely new spectral phenomena.

In the one-impulse case, the spectral properties are governed by a single transmission matrix B. The characteristic function is determined by a single scattering relation and depends only on the parameters of that transmission condition.

In contrast, when $m > 1$, the propagation of solutions is described by the total transfer matrix

$$M(\lambda) = M_m(\lambda)M_{m-1}(\lambda)\cdots M_1(\lambda),$$

which contains the cumulative interaction of all impulse points. Consequently, the characteristic function

$$M_{22}(\lambda)$$

is no longer generated by a single transmission condition but by the product of m local transfer matrices.

This leads to several new effects.

First, the spectral equation depends simultaneously on all impulse locations $x_1, \ldots, x_m$ and all transmission matrices $B_1, \ldots, B_m$. Hence the eigenvalues are determined by the mutual interaction of different impulse points rather than by a single local discontinuity.

Second, multiple reflections of the wave between different impulse points occur. These repeated transmissions are encoded in the matrix product $M_m \cdots M_1$ and may generate additional zeros of $M_{22}(\lambda)$, which have no analogue in the single-impulse problem.

Third, the algebraic structure of the characteristic function becomes considerably more complicated. In the one-impulse case the spectral data are determined by one transfer relation, whereas for $m > 1$ they depend on a noncommutative product of transfer matrices. Consequently, changes in one impulse point influence the entire spectrum through the global matrix $M(\lambda)$.

Finally, the resolvent kernel and Green function contain contributions from all impulse points through $M(\lambda)$. Therefore the distribution of eigenvalues and spectral singularities is governed by collective interaction effects among the discontinuities, a phenomenon absent when $m = 1$.

Thus the multi-impulse problem is not merely a notational extension of the single-impulse case; it introduces a new global spectral object, namely the total transfer matrix, whose analytic properties determine the spectrum.

Assume that

$$\operatorname{Im}\lambda > 0.$$

Let $F(x, \lambda)$ be the solution satisfying the square-integrability condition at $+\infty$:

$$F(\cdot, \lambda) \in L^2(0, \infty). \tag{3.21}$$

This means that $F(x, \lambda)$ decays as $x \to +\infty$.

Similarly, let $G(x, \lambda)$ be the solution satisfying

$$G(\cdot, \lambda) \in L^2(-\infty, 0). \tag{3.22}$$

Thus $G(x, \lambda)$ decays as $x \to -\infty$.

The Wronskian of F and G is

$$W[F,G](\lambda) = F(x,\lambda)G'(x,\lambda) - F'(x,\lambda)G(x,\lambda). \tag{3.23}$$

Since both F and G satisfy the same differential equation, this Wronskian does not depend on x.

If

$$W[F,G](\lambda) \neq 0,$$

then F and G are linearly independent. In this case the Green function is defined by

$$G_L(x,t;\lambda) = \begin{cases} \dfrac{G(x,\lambda)F(t,\lambda)}{W[F,G](\lambda)}, & x < t, \\ \dfrac{F(x,\lambda)G(t,\lambda)}{W[F,G](\lambda)}, & x > t. \end{cases} \tag{3.24}$$

The first formula is used when $x < t$, because then the solution must satisfy the left boundary condition at $-\infty$. The second formula is used when $x > t$, because then the solution must satisfy the right boundary condition at $+\infty$.

Before using the Green kernel to represent the resolvent, it is necessary to verify that the kernel satisfies all conditions imposed on the operator.

Let

$$G_L(x,t;\lambda) = \begin{cases} \dfrac{G(x,\lambda)F(t,\lambda)}{W[F,G](\lambda)}, & x < t, \\ \dfrac{F(x,\lambda)G(t,\lambda)}{W[F,G](\lambda)}, & x > t, \end{cases}$$

where $F(x,\lambda)$ and $G(x,\lambda)$ are the Weyl solutions square-integrable at $+\infty$ and $-\infty$, respectively.

First, for every fixed $t \neq x_k$, the function $G_L(\cdot,t;\lambda)$ satisfies the differential equation

$$-y'' + q(x)y = \lambda^2 \rho(x) y$$

on each interval

$$(-\infty, x_1), \quad (x_1, x_2), \dots, (x_m, \infty),$$

because both F and G are solutions of the homogeneous equation.

Next, since F and G satisfy the transmission conditions at every impulse point x_k, the Green kernel inherits the same transmission relations. Indeed,

$$\begin{pmatrix} G_L(x_k+0,t;\lambda) \\ \partial_x G_L(x_k+0,t;\lambda) \end{pmatrix} = B_k \begin{pmatrix} G_L(x_k-0,t;\lambda) \\ \partial_x G_L(x_k-0,t;\lambda) \end{pmatrix},$$

for every $k = 1, \dots, m$. Thus the kernel belongs to the operator domain with respect to the variable x.

It remains to verify the jump condition at $x = t$. From the definition of G_L,

$$G_L(t+0,t;\lambda) = G_L(t-0,t;\lambda) = \frac{F(t,\lambda)G(t,\lambda)}{W[F,G](\lambda)},$$

so the kernel is continuous at $x = t$.

Differentiating with respect to x, we obtain

$$\partial_x G_L(t+0,t;\lambda) = \frac{F'(t,\lambda)G(t,\lambda)}{W[F,G](\lambda)},$$

and

$$\partial_x G_L(t-0,t;\lambda) = \frac{G'(t,\lambda)F(t,\lambda)}{W[F,G](\lambda)}.$$

Hence

$$\partial_x G_L(t+0,t;\lambda) - \partial_x G_L(t-0,t;\lambda) = \frac{F'(t,\lambda)G(t,\lambda) - G'(t,\lambda)F(t,\lambda)}{W[F,G](\lambda)} = 1.$$

Therefore G_L satisfies the standard jump condition

$$\big[\partial_x G_L(x,t;\lambda)\big]_{x=t} = 1.$$

Consequently,

$$(L-\lambda^2 I)G_L(\cdot,t;\lambda) = \delta(\cdot - t)$$

in the distributional sense.
Finally, assume that

$$\operatorname{Im}\lambda > 0$$

and

$$W[F,G](\lambda) \neq 0.$$

Since the Weyl solutions decay exponentially at the corresponding infinities, there exist constants $C > 0$ and $a > 0$ such that

$$|F(x,\lambda)| \leq Ce^{-ax}, \qquad x \to +\infty,$$

and

$$|G(x,\lambda)| \leq Ce^{ax}, \qquad x \to -\infty.$$

Therefore

$$|G_L(x,t;\lambda)| \leq C_1 e^{-a|x-t|},$$

which implies that the integral operator

$$(R_\lambda f)(x) = \int_{-\infty}^{\infty} G_L(x,t;\lambda)f(t)\rho(t)\,dt$$

is bounded in $L^2(\mathbb{R},\rho(x)\,dx)$ by Schur's test. Hence

$$R_\lambda = (L-\lambda^2 I)^{-1}$$

is a bounded resolvent operator whenever $W[F,G](\lambda) \neq 0$.

Thus the Green kernel satisfies the differential equation, all transmission conditions, the jump condition at $x = t$, and generates a bounded resolvent operator on the natural domain of L.

Let us construct the resolvent. Consider the nonhomogeneous equation

$$-y'' + q(x)y - \lambda^2\rho(x)y = f(x). \tag{3.25}$$

Using the Green function, the solution of (3.25) is represented as

$$y(x,\lambda) = \int_{-\infty}^{\infty} G_L(x,t;\lambda)f(t)\rho(t)\,dt. \tag{3.26}$$

Therefore the resolvent operator

$$R_\lambda = (L - \lambda^2 I)^{-1}$$

is given by

$$(R_\lambda f)(x) = \int_{-\infty}^{\infty} G_L(x, t; \lambda) f(t) \rho(t)\, dt. \tag{3.27}$$

The poles of the resolvent occur when

$$W[F, G](\lambda) = 0.$$

Using the total transfer matrix, this condition is equivalent to

$$M_{22}(\lambda) = 0. \tag{3.28}$$

Hence $M_{22}(\lambda) = 0$ is the characteristic equation of the impulsive Sturm–Liouville problem.

The following theorem provides a characterization of the eigenvalues of the operator L in terms of the total transfer matrix.

Theorem 3.1 *Let L be the impulsive Sturm–Liouville operator generated by*

$$-y'' + q(x)y = \lambda^2 \rho(x) y, \qquad x \in \mathbb{R} \setminus \{x_1, \ldots, x_m\},$$

together with the transmission conditions

$$\begin{pmatrix} y(x_k + 0) \\ y'(x_k + 0) \end{pmatrix} = B_k \begin{pmatrix} y(x_k - 0) \\ y'(x_k - 0) \end{pmatrix}, \qquad k = 1, \ldots, m,$$

where $\det B_k \neq 0$.

Let

$$M(\lambda) = M_m(\lambda) M_{m-1}(\lambda) \cdots M_1(\lambda) = \begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix}$$

be the total transfer matrix. Then the characteristic equation of the problem is

$$M_{22}(\lambda) = 0.$$

Moreover, the eigenvalues of L are exactly the numbers λ^2 for which

$$M_{22}(\lambda) = 0, \qquad \operatorname{Im} \lambda > 0.$$

Proof. Let $I_0 = (-\infty, x_1)$, $I_k = (x_k, x_{k+1})$, $k = 1, \ldots, m-1$, and $I_m = (x_m, \infty)$. On each interval I_k, every solution of the equation can be written as a linear combination of two fundamental solutions:

$$y_k(x, \lambda) = A_k f_k^+(x, \lambda) + B_k f_k^-(x, \lambda).$$

Here A_k and B_k are the coefficient functions of the solution on the interval I_k.

At the impulse point x_k, define

$$D_k^-(\lambda) = \begin{pmatrix} f_{k-1}^+(x_k, \lambda) & f_{k-1}^-(x_k, \lambda) \\ {f_{k-1}^+}'(x_k, \lambda) & {f_{k-1}^-}'(x_k, \lambda) \end{pmatrix},$$

and

$$D_k^+(\lambda) = \begin{pmatrix} f_k^+(x_k, \lambda) & f_k^-(x_k, \lambda) \\ {f_k^+}'(x_k, \lambda) & {f_k^-}'(x_k, \lambda) \end{pmatrix}.$$

Since the Wronskians of the fundamental systems do not vanish, the matrices $D_k^-(\lambda)$ and $D_k^+(\lambda)$ are invertible for the considered values of λ.

The transmission condition gives

$$D_k^+(\lambda)\begin{pmatrix}A_k\\ B_k\end{pmatrix} = B_k D_k^-(\lambda)\begin{pmatrix}A_{k-1}\\ B_{k-1}\end{pmatrix}.$$

Therefore

$$\begin{pmatrix}A_k\\ B_k\end{pmatrix} = M_k(\lambda)\begin{pmatrix}A_{k-1}\\ B_{k-1}\end{pmatrix},$$

where

$$M_k(\lambda) = \left(D_k^+(\lambda)\right)^{-1} B_k D_k^-(\lambda).$$

Applying this relation successively from x_1 to x_m, we obtain

$$\begin{pmatrix}A_m\\ B_m\end{pmatrix} = M(\lambda)\begin{pmatrix}A_0\\ B_0\end{pmatrix},$$

where

$$M(\lambda) = M_m(\lambda)M_{m-1}(\lambda)\cdots M_1(\lambda).$$

Now let $F(x,\lambda)$ be the Weyl solution square-integrable at $+\infty$, and let $G(x,\lambda)$ be the Weyl solution square-integrable at $-\infty$. Assume first that $\operatorname{Im}\lambda > 0$. Then the decaying solution at $-\infty$ is proportional to $f_0^-(x,\lambda)$, while the decaying solution at $+\infty$ is proportional to $f_m^+(x,\lambda)$. Hence we may normalize

$$G(x,\lambda) = f_0^-(x,\lambda), \qquad x \in I_0,$$

so that the coefficient vector of G on I_0 is

$$\begin{pmatrix}A_0\\ B_0\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}.$$

After passing through all impulse points, the coefficient vector of G on I_m becomes

$$\begin{pmatrix}A_m\\ B_m\end{pmatrix} = M(\lambda)\begin{pmatrix}0\\ 1\end{pmatrix} = \begin{pmatrix}M_{12}(\lambda)\\ M_{22}(\lambda)\end{pmatrix}.$$

Therefore, on the last interval I_m, we have

$$G(x,\lambda) = M_{12}(\lambda)f_m^+(x,\lambda) + M_{22}(\lambda)f_m^-(x,\lambda).$$

On the other hand, by definition, the Weyl solution F square-integrable at $+\infty$ is proportional to $f_m^+(x,\lambda)$. We normalize it as

$$F(x,\lambda) = f_m^+(x,\lambda), \qquad x \in I_m.$$

Consequently, for $x \in I_m$,

$$W[F,G](\lambda) = W\left[f_m^+, M_{12}(\lambda)f_m^+ + M_{22}(\lambda)f_m^-\right].$$

Using bilinearity and the identity $W[f_m^+, f_m^+] = 0$, we get

$$W[F,G](\lambda) = M_{22}(\lambda)W[f_m^+, f_m^-].$$

Since

$$W[f_m^+, f_m^-] \neq 0,$$

we obtain

$$W[F,G](\lambda) = C(\lambda)M_{22}(\lambda),$$

where

$$C(\lambda) = W[f_m^+, f_m^-].$$

In particular,

$$W[F,G](\lambda) = 0 \quad \Longleftrightarrow \quad M_{22}(\lambda) = 0.$$

The Green function of the operator is given by

$$G_L(x,t;\lambda) = \begin{cases} \dfrac{G(x,\lambda)F(t,\lambda)}{W[F,G](\lambda)}, & x < t, \\ \dfrac{F(x,\lambda)G(t,\lambda)}{W[F,G](\lambda)}, & x > t. \end{cases}$$

Thus the resolvent has poles exactly at the zeros of the Wronskian $W[F,G](\lambda)$. By the identity proved above, these poles are exactly the zeros of $M_{22}(\lambda)$.

If $M_{22}(\lambda_0) = 0$ and $\operatorname{Im}\lambda_0 > 0$, then

$$W[F,G](\lambda_0) = 0.$$

Hence $F(x,\lambda_0)$ and $G(x,\lambda_0)$ are linearly dependent. Therefore there exists a nontrivial solution satisfying the square-integrability condition both at $-\infty$ and at $+\infty$. Thus λ_0^2 is an eigenvalue of L.

Conversely, suppose that λ_0^2 is an eigenvalue of L. Then there exists a nontrivial function $y \in L^2(\mathbb{R}, \rho(x)\,dx)$ satisfying the differential equation and all transmission conditions. Since y is square-integrable at both infinities, it must be proportional to the Weyl solution $G(x,\lambda_0)$ near $-\infty$ and proportional to the Weyl solution $F(x,\lambda_0)$ near $+\infty$. Hence F and G are linearly dependent, and therefore

$$W[F,G](\lambda_0) = 0.$$

Using

$$W[F,G](\lambda) = C(\lambda)M_{22}(\lambda),$$

we obtain

$$M_{22}(\lambda_0) = 0.$$

Thus

$$\sigma_p(L) = \left\{\lambda^2 : \; M_{22}(\lambda) = 0, \; \operatorname{Im}\lambda > 0\right\}.$$

The theorem is proved.

Theorem 3.2 *Let L be the impulsive Sturm–Liouville operator generated by* (1.1)–(1.4). *Then the continuous spectrum of L is*

$$\sigma_c(L) = [0, \infty).$$

Moreover, the spectral singularities of L are the real zeros of $M_{22}(\lambda)$. Possible singular points may occur at

$$\lambda = \pm\frac{n}{2}, \qquad \lambda = \pm\frac{n}{2\beta}, \qquad n \in \mathbb{N}.$$

Proof. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Then the Jost solutions have the asymptotic forms

$$f_1^{\pm}(x,\lambda) \sim e^{\pm i\lambda x}, \qquad x \to -\infty,$$

and

$$f_2^{\pm}(x,\lambda) \sim e^{\pm i\beta\lambda x}, \qquad x \to +\infty.$$

Since λ is real, the functions

$$e^{\pm i\lambda x} \quad \text{and} \quad e^{\pm i\beta\lambda x}$$

are oscillatory and do not decay at infinity. Hence the corresponding solutions do not belong to

$$L^2(\mathbb{R}, \rho(x)\,dx).$$

Therefore real values of λ^2 do not generate eigenvalues. However, the resolvent fails to be bounded on the positive real axis, because the Green function contains oscillatory non-decaying solutions. Consequently, the positive semi-axis belongs to the continuous spectrum, that is,

$$[0,\infty) \subset \sigma_c(L).$$

On the other hand, for $\lambda \notin \mathbb{R}$, the Weyl solutions decay exponentially at the corresponding infinities, and the Green function gives a bounded resolvent whenever

$$M_{22}(\lambda) \neq 0.$$

Thus, outside the real axis and away from the zeros of M_{22}, the point λ^2 belongs to the resolvent set. Hence the continuous spectrum is exactly

$$\sigma_c(L) = [0,\infty).$$

Now let $\lambda \in \mathbb{R}$. A spectral singularity is a point of the continuous spectrum at which the resolvent has a pole-like singularity. Since the resolvent kernel contains the factor

$$\frac{1}{M_{22}(\lambda)},$$

such singularities occur precisely at real zeros of $M_{22}(\lambda)$. Therefore the spectral singularities are characterized by

$$M_{22}(\lambda) = 0, \qquad \lambda \in \mathbb{R}.$$

Finally, from the representations of the Jost solutions, possible poles may occur when the denominators vanish:

$$n \pm 2\lambda = 0 \quad \text{or} \quad n \pm 2\beta\lambda = 0.$$

Hence possible singular points are

$$\lambda = \pm\frac{n}{2}, \qquad \lambda = \pm\frac{n}{2\beta}, \qquad n \in \mathbb{N}.$$

The proof is complete.

4 Conclusion

The spectral analysis of the impulsive Sturm–Liouville operator is reduced to the investigation of the total transfer matrix

$$M(\lambda) = M_m(\lambda) \cdots M_1(\lambda).$$

The Green function, resolvent, eigenvalues, continuous spectrum, and spectral singularities are completely determined by the analytic properties of the entry $M_{22}(\lambda)$. In particular,

$$M_{22}(\lambda) = 0$$

is the characteristic equation for both eigenvalues and spectral singularities.

5 Acknowledgements

The second author extends their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2026/02/45032)

References

1. Annaghili, S., Efendiev, R., Juraev, D.A., Abdalla, M.: *Spectral analysis for the almost periodic quadratic pencil with impulse*, Bound. Value Probl. **2025** (1), 38 (2025).
2. Bairamov, E., Erdal, V., Yardımcı, A.L.: *Spectral properties of an impulsive Sturm–Liouville operator*, J. Ineq. Appl. **2018**, Article No. 191 (2018).
3. Efendiev, R.F., Bahlulzade, S.J.: *Spectral analysis for the almost periodic Sturm–Liouville operator with impulse*, Azerb. J. Math. **15** (2), 178–194 (2025).
4. Efendiev, R.F., Sharifli, M.R.: *Spectral properties of an impulsive Sturm–Liouville operator with complex periodic coefficients*, Advanced Mathematical Models & Applications **9** (3), 431-436 (2024).
5. Efendiev, R.F., Orudzhev, H.D., Bahlulzade, S.J.: *Spectral analysis of the discontinuous Sturm–Liouville operator with almost-periodic potentials*, Advanced Mathematical Models & Applications, **6** (3), 266–277 (2021).
6. Gasymov, M.G.: *Spectral analysis of a class of second-order non-self-adjoint differential operators*, Funct. Anal. Appl. **14** (1), 11–15 (1980).
7. Gasymov, T.B., Taghiyeva, R.J.: *Spectral properties of a differential operator with integral boundary condition*, Journal of Contemporary Applied Mathematics **15** (2), 1–19 (2025).
8. Gasymov, T.B., Ahmadov, A.Q.: *On basicity of eigenfunctions of a spectral problem in rearrangement-invariant Banach function space*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **46** (1), 43–55 (2026).
9. Kir, K.: *Spectrum and principal functions of a non-self-adjoint Sturm–Liouville operator*, Rocky Mountain J. Math. **35** (5), 1519–1541 (2005).
10. Mukhtarov, O.Sh., Tunç, M., Muhtarov, F.S.: *Spectral analysis of discontinuous Sturm–Liouville problems with transmission conditions*, Electronic Research Archive **31** (7), 4100–4121 (2023).
11. Mutlu, A.: *Spectral properties of non-self-adjoint Sturm–Liouville operators*, Turkish J. Math. **44** (5), 1743–1760 (2020).
12. Teschl, G.: *Mathematical Methods in Quantum Mechanics: With Applications to Schr"odinger Operators*, 2nd Edition, Graduate Studies in Mathematics, Vol. 157, American Mathematical Society, Providence, RI (2014).

13. Xu, L., Yang, C.-F.: *Inverse spectral problems for the Sturm–Liouville operator with discontinuity conditions*, Appl. Anal. **97** (9), 1608–1623 (2018).
14. Yan, L., Shi, Y.: *Inverse spectral problems for non-self-adjoint Sturm–Liouville operators with discontinuous conditions*, Mediterr. J. Math. **16** (3), Article No. 67 (2019).
15. Yardımcı, A.: *Investigation of an impulsive Sturm–Liouville operator on the semi-axis*, Hacet. J. Math. Stat. **49** (5), 1700–1714 (2020).
16. Zettl, A.: *Sturm–Liouville Theory*, Mathematical Surveys and Monographs, Vol. 121, American Mathematical Society, Providence, RI (2005).