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The existence of strong solutions of quasi-linear elliptic equations and second-order systems

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Abstract. *Boundary value problems for quasi-linear elliptic equations and systems with a principal quasi-linear elliptic operator of the second order in Sobolev space are considered. An interpolation method is described for obtaining a priori estimates of strong solutions of quasi-linear elliptic equations and systems with unbounded singularities on the right-hand side, provided that there is a first a priori estimate in the space of summable functions.*

Keywords. interpolation method, upper and lower solutions, Sobolev space.

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1 Introduction

Let $R_+ = \left\{ t_0 \in R \,\middle|\, t_0 \geq 0 \right\}$. All functions entered are considered to be real-valued. The following functional spaces are used [1], [10], [11], [14], [21], [22], [9, p.123] : the space of summable functions $L_p(\Omega)$, $p \geq 1$, with norm

$$\|u\|_{p;\Omega} = \left(\int\limits_{\Omega} |u(x)|^p dx \right)^{1/p} ;$$

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anisotropic Sobolev space $W_p^2(\Omega)$ with norm

$$\|u\|_{W_p^2(\Omega)} = \|u\|_{p;\Omega} + \sum_{i=1}^{N} \left\| \frac{\partial^2 u}{\partial x_i^2} \right\|_{p;\Omega};$$

space $W_p^{2-1/p}(\partial\Omega)$ with norm $\|u\|_{W_p^{2-1/p}(\partial\Omega)}$, to which the traces of functions from $W_p^2(\Omega)$ belong. By $D^i u$ we denote a vector from partial derivatives $D^\alpha u$ of the function $u(x)$, $\alpha = (\alpha_1, \ldots, \alpha_N)$, $|\alpha| = i$, that has N components for $i = 1$ and N^2 for $i = 2$, by $\vec{u}(x) = (u_1(x), \ldots, u_n(x))$ a vector of functions $u_j(x)$, by $D^i\vec{u}$ a vector $\left(D^i u_1, \ldots, D^i u_n\right)$. We will write that $\vec{u} \in \vec{L}_p(\Omega)$ if $u_j \in L_p(\Omega)$ for each $j = \overline{1,n}$ and $\|\vec{u}\|_{p;\Omega} \equiv \sum\limits_{j=1}^{n} \|u_j\|_{p;\Omega}$. We will adhere to similar agreements in the case of spaces $\vec{C}\left(\overline{\Omega}\right)$, $\vec{W}_p^{2-1/p}(\partial\Omega)$ and $\vec{W}_p^2(\Omega)$ as well.

Consider the following boundary value problem for a single equation

$$\begin{cases} Lu \equiv \sum\limits_{|\alpha|=2} a_\alpha(x, u, Du)\, D^\alpha u = f(x, u, Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $u = u(x)$ and $Du = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right)$.

The problem (1.1) is studied in the class of real functions from the Sobolev space $W_p^2(\Omega)$ for $p > 1$, so that the boundary function $\varphi(x)$ is the trace $\varphi(x) = \tilde{\varphi}(x)\Big|_{\partial\Omega}$ of the function $\tilde{\varphi}(x)$ from $W_p^2(\Omega)$, i.e., belongs to the space $W_p^{2-1/p}(\partial\Omega)$ with norm $\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \le c_0\|\tilde{\varphi}\|_{W_p^2(\Omega)}$, where c_0 does not depend on $\tilde{\varphi}(x)$.It is assumed that for any strong solution of problem (1.1) $u \in W_p^2(\Omega)$ there is the first a priori estimate $\|u\|_{W_2^1(\Omega)} \le M$ with $M > 0$ holds. Thus, the boundary value problem (1.1) is considered at the intersection of spaces $W_p^2(\Omega) \bigcap W_2^1(\Omega)$. Note that for $p \ge \frac{2N}{N+2}$ takes place in the inclusion $W_p^2(\Omega) \subset W_2^1(\Omega)$ (see [22]).

The following conditions are assumed to be fulfilled

A.1) Let the function $f(x, \xi_0, \xi_1)$ be determined on $\Omega \times R \times R^N$ with the values in R and be a caratheodorian function, i.e. measurable with respect to x for all $(\xi_0, \xi_1) \in R \times R^N$ and continuous with respect to $(\xi_0, \xi_1) \in R \times R^N$ almost for all $x \in \Omega$.

A.2) Let

$$|f(x, \xi_0, \xi_1)| \le b(x, \xi_0) + b_1(x, \xi_0) \cdot |\xi_1|^\mu$$

almost for everyone $x \in \Omega$ and for all $\xi_0 \in R$, $\xi_1 \in R^N$ with non-negative carateodory functions $b(x, \xi_0)$ and $b_1(x, \xi_0)$ such that for any $r \ge 0$

$$b_r(x) = \sup\left\{ b(x, \xi_0) \,\Big|\, |\xi_0| \le r \right\}$$

belongs to the space $L_p(\Omega)$ with $p > N$ and $p > \frac{2N}{N+2}$, the function

$$b_{1,r}(x) = \sup\left\{ b_1(x, \xi_0) \,\Big|\, |\xi_0| \le r \right\}$$

belongs to the space $L_q(\Omega)$ with $q > p$ and $\mu > 1$.

A.3) Let $\mu = \frac{N+2}{N} - \frac{2}{q}$.

A.4) Let $L(u)$ be a second order quasilinear elliptic operator of the form

$$Lu \equiv \sum_{|\alpha|=2} a_\alpha(x,u,Du)\,D^\alpha u,$$

the coefficients $a_\alpha\,(|\alpha|=2)$ be real and continuous function on $\overline{\Omega}\times R\times R^N$.

Let the operator $L_v u$, which is linear with respect to $u(x)$, be equal to

$$L_v u \equiv \sum_{|\alpha|=2} a_\alpha(x,v(x),Dv(x))\,D^\alpha u$$

for any function $v\in C^1\left(\overline{\Omega}\right)$, there exists a linear elliptic operator with real and continuous coefficients such that the linear (with respect to $u(x)$) boundary value problem

$$\begin{cases} L_v u = g(x), & x\in\Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), & x\in\partial\Omega \end{cases} \tag{1.2}$$

is coercive in the space $W_p^2(\Omega)$, i.e., the a priori estimate is satisfied.

$$\|u\|_{W_p^2(\Omega)} \le C\left(\|g\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} + \|u\|_{p;\Omega}\right) \tag{1.3}$$

with a positive constant C, independent of $g\in L_p(\Omega)$, of $\varphi\in W_p^{2-1/p}(\partial\Omega)$ and of the solution $u\in W_p^2(\Omega)$ of the linear problem (1.2).

In this case, the constant C may depend on the continuity modules of the coefficients $a_\alpha\,(|\alpha|=2)$, and the continuity modules of the functions $v(x)$ and $Dv(x)$.

Note that sufficient conditions for the coefficients of the operator L ensuring the coercivity of the linear (with respect to $u(x)$) boundary value problem (1.2) are given in [1].

The first part of the article presents a diagram of the method using the example of a second-order equation. Limitations on the summability of the right-hand side have been found, in which the compactness of the corresponding nonlinear superposition operator holds. It is shown how these limitations are related to the proposed method for obtaining a second a priori estimate. The second part proves the theorem on a priori estimates for a second-order quasi-linear elliptic system. Quasi-linear and semi-linear systems with a bounded right-hand side were considered in [7], [14], [25] and many other works. However, they assume the existence of an a priori estimate of the maximum modulus of the solution and allow any growth of the right-hand side relative to the desired function, but require that all equations of the system have the same leading terms. At the same time, the following from the condition of coercivity, the latter is not necessary. It is also important to note that a nonlinear term can have unbounded features.Such equations were studied in [4], [5], [6], [15], [16], [18], [19], [20], systems in work [16].

The main focus is on the question of the attainability of the limiting degrees of growth of the nonlinear component of the system and the restrictions on its summability index. As is known, for elliptic equations [19], [5] and parabolic [16], [4], [6] of the second order, the limiting degrees are reached. However, in high-order elliptic equations, this is not true, as shown by the examples constructed in [20], [23] . In the parabolic case, Wahl [23], [24] proved that when the condition of coerativity is satisfied for high-order equations and systems, a limit growth rate is permissible.

The condition for growth of the right-hand side obtained in this work, which provides a strong a priori estimate of the solution of a quasi-linear system through the maximum modulus of this solution, differs somewhat from the conditions in [19], [20], [25], [3], [13]. A new exact condition has been obtained for the growth of the considered nonlinear functions

$f(x, \xi_0, \xi_1)$ with respect to $\xi_1 \in R^N$ under which an a priori estimate $\|u\|_{W_2^1(\Omega)}$ of the solution to problem (1.1) yields an a priori estimate $\|Du\|_{S,\Omega}$ for $S = \frac{\mu pq}{q-p}$. One particular example shows non-improvement the corresponding growth indicator. The theory of solvability of problems of the form (1.1) is considered under the conditions of the existence of upper and lower solutions to these problems. Here, along with the theorem on the a priori estimate $\|u\|_{W_p^2(\Omega)}$, the maximum principle established by A.D. Aleksandrov [2] is used. Based on the general solvability theorem, specific theorems on the solvability of problems of the form (1.1) have been obtained. Theorems on a priori estimates for quasi-linear elliptic systems of the second order are proven.

In investigating the problem (1.1), the embedding theorem for anisotropic spaces is used extensively $\|u\|_{W_p^2(\Omega)}$, which yields the following multiplicative inequality. Let us formulate a general theorem, investments [16] in the specific form we need.

Lemma 1.1 *Let $u(x) \in W_p^2(\Omega)$, $p > \frac{2N}{N+2}$ and conditions fulfilled A.1) -A.3). Then*

$$\|Du\|_{S;\Omega} \leq c_1 \|u\|_{W_p^2(\Omega)}^{1/\mu} \cdot \|u\|_{W_2^1(\Omega)}^{1/1-\mu} + c_2 \|u\|_{W_2^1(\Omega)} \tag{1.4}$$

with positive constants c_1 and c_2, independent of function $u(x)$ of $W_p^2(\Omega)$, where

$$\begin{cases} \frac{1}{S} = \frac{1}{2} + \frac{1}{\mu}\left(\frac{1}{p} - \frac{N+2}{2N}\right) \\ 1 < \mu \leq \frac{N+2}{N} \end{cases} \tag{1.5}$$

Lemma 1.2 *Let conditions A.1)-A.3) be fulfilled. Then the operator $F(u)(x) \equiv f(x, u(x), Du(x))$ acts completely continuously from $W_p^2(\Omega)$ to $L_p(\Omega)$ with $p > 1$ and $p > \frac{2N}{N+2}$.*

Proof. Estimate $\|F(u)\|_{p;\Omega}$ by means of conditions A.1) -A.3) and Hölder's inequality.

$$\|\vec{F}(u)\|_{p;\Omega} \leq \|b_r\|_{p;\Omega} + \|b_{1,r}\|_{q;\Omega} \cdot \|Du\|_{S;\Omega}^{\mu} \tag{1.6}$$

with $r = \text{const} \cdot \|u\|_{W_2^1(\Omega)}$ and $S = \frac{\mu pq}{q-p}$. Then, based on equalities A.3) and interpolation inequalities (1.4) with the specified S and μ corresponding to them according to formula (1.5), we obtain

$$\begin{aligned} \|F(u)\|_{p;\Omega} &\leq \|b_r\|_{p;\Omega} + \|b_{1,r}\|_{q;\Omega} \\ &\times \left[c_1 \cdot \|u\|_{W_p^2(\Omega)}^{1/\mu} \cdot \|u\|_{W_2^1(\Omega)}^{1-1/\mu} + c_2 \cdot \|u\|_{W_2^1(\Omega)}\right]^{\mu} \leq \|b_r\|_{p;\Omega} \\ &+ \|b_{1,r}\|_{q;\Omega} \cdot 2^{\mu-1} \cdot c_1^{\mu} \cdot \|u\|_{W_2^1(\Omega)}^{\mu-1} \cdot \|u\|_{W_p^2(\Omega)} + \|b_{1,r}\|_{q;\Omega} \cdot 2^{\mu-1} \cdot c_2^{\mu} \cdot \|u\|_{W_2^1(\Omega)}^{\mu} \\ &\leq \|b_r\|_{p;\Omega} + \Phi_1\left(\|u\|_{W_2^1(\Omega)}\right) \cdot \|u\|_{W_p^2(\Omega)} + \Phi_2\left(\|u\|_{W_2^1(\Omega)}\right), \end{aligned} \tag{1.7}$$

where $\Phi_1, \Phi_2 : R_+ \to R_+$ are the increasing functions determined by the known data.

This proves the boundedness of the operator $F(u)$. Since $p(N+2) > 2N$. Under the constraints of this lemma, the embedding operator $F_1 : W_p^2(\Omega) \to L_S(\Omega)$ is completely continuous. By the estimate (1.7), the operator $F : L_S(\Omega) \to L_p(\Omega)$ is bounded and by the common properties of the superposition operator it is continuous. Then the operator $F : W_p^2(\Omega) \to L_p(\Omega)$ is completely continuous as a composition of completely continuous and continuous operators.

Lemma 1.2 is proved.

Let $u(x) \in W_p^2(\Omega)$ be the solution of problem (1.1) and let conditions A.1) - A.4) be fulfilled. Then we can consider the problem (1.1) as a linear problem with a fixed right part $F(u) \equiv f(x, u(x), Du(x)) \in L_p(\Omega)$ with $p > 2N/(N+2)$.

Theorem 1.1 *Let conditions A.1) - A.4) be fulfilled with some $p > \frac{2N}{(N+2)}$. Then there exists such a function $\Phi : R_+^2 \to R_+$, bounded on each compact that for any solution $u(x)$ from $W_p^2(\Omega)$ with $p > \frac{2N}{N+2}$ of the problem* (1.1) *the following inequality*

$$\|u\|_{W_p^2(\Omega)} \leq \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right),$$

where $\|u\|_{W_2^1(\Omega)} \leq M$, $R_+^2 = R_+ \times R_+$.

Proof. We have
$$Lu = \frac{f(x, u, Du)}{1 + |Du|^\mu} \cdot (1 + |Du|^\mu).$$

Hence
$$Lu - c(x) u(x) = \frac{f(x, u(x), Du(x))}{1 + |Du|^\mu} \cdot (1 + |Du|^\mu) - c(x) u(x),$$

where $c(x) = b_r(x) \geq 0, c(x) \in L_p(\Omega)$.

Then, for the function $u(x)$, we have

$$\begin{cases} Lu - c(x) u(x) = f_1(x) \cdot |Du|^\mu + f_0(x), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \quad x \in \partial\Omega. \end{cases} \tag{1.8}$$

Here $f_1(x) = \frac{f(x,u,Du)}{1+|Du|^\mu}$, $f_0(x) = f_1(x) - c(x) u(x)$.

Our goal obtain the estimate $\|u\|_{W_p^2(\Omega)}$ that bounds from above $\|u\|_{W_2^1(\Omega)}$, $\|f_0\|_{p;\Omega}$, $\|f_1\|_{q;\Omega}$ and $\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}$. For that let's consider in the space $W_p^2(\Omega)$ with $p > \frac{2N}{N+2}$ we consider the boundary value problem

$$\begin{cases} LV - c(x) V = f_1(x) |DV|^\mu + \lambda \cdot f_0(x), \quad x \in \Omega, \\ V\Big|_{\partial\Omega} = \lambda \cdot \varphi(x), \quad x \in \partial\Omega, \end{cases} \tag{1.9}$$

with the parameter $\lambda \in [0,1]$ and the functions $c(x)$, $f(x)$, $f_1(x)$ and $\varphi(x)$ determined above.

In connection with the introduction of the parameter, we note that the method of introducing parameters in elliptic and parabolic problems to obtain a priori estimates has been used by many authors (see [3], [4], [13], [19] .)

For problem (1.9), the following lemma about the uniqueness of the solution holds.

Lemma 1.3 *At any fixed $\lambda \in [0,1]$ the problem* (1.9) *has at most one solution from $W_p^2(\Omega)$ with $p > 2N/(N+2)$.*

Proof. Let us assume the opposite. Then for the difference $\mathcal{W} = V - Z$ of two possible solutions $V(x)$ and $Z(x)$ we have:

$$\begin{cases} L\mathcal{W} - c(x) \mathcal{W} = f_1(x) \sum\limits_{i=1}^{N} h_i(x) D_i \mathcal{W}, \quad x \in \Omega, \\ \mathcal{W}\Big|_{\partial\Omega} = 0, \quad x \in \partial\Omega. \end{cases}$$

Here $h_i(x) = \int\limits_0^1 H_i(x,\tau)\,d\tau$, where

$$H_i(x,\tau) = \mu \left[\sum_{k=1}^{N} (\tau \cdot D_k \mathcal{W} + D_k Z)^2\right]^{\frac{\mu}{2}-1} \cdot (\tau \cdot D_i \mathcal{W} + D_i Z)(x)$$

for $\sum\limits_{k=1}^{N} (\tau \cdot D_k \mathcal{W} + D_k Z)^2(x) \neq 0$ and $H_i(x,\tau) = 0$ for $\sum\limits_{k=1}^{N} (\tau \cdot D_k \mathcal{W} + D_k Z)^2(x) = 0$.

Since $c(x) \geq 0,\ c(x) \in L_p(x),\ f_1(x) \in L_q(\Omega),\ \ h_i \in L_p(\Omega)\ (i = 1, \ldots, N)$ and $p > 2N/(N+2)$, from the results of A.D.Alexandrov's [2] paper we obtain $\mathcal{W}(x) \equiv 0, x \in \overline{\Omega}$.

Let now V_1 and V_2 be the solutions of problem (1.9) corresponding to the values λ_1 and $\lambda_2 > \lambda_1$ of the parameter λ. Then for the difference $\tilde{V} = V_2 - V_1$ we have

$$\begin{cases} L\tilde{V} - c(x)\tilde{V} = f_1(x) \sum\limits_{i=1}^{N} \tilde{h}_i(x) D_i \tilde{V} + (\lambda_2 - \lambda_1) f_0(x), \ \ x \in \Omega, \\ \tilde{V}\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\varphi(x), \ \ x \in \partial\Omega. \end{cases}$$

Here $\tilde{h}_i(x) = \int\limits_0^1 \tilde{H}_i(x,\tau)\,d\tau$, where

$$\tilde{H}_i(x,\tau) = \mu \left[\sum_{k=1}^{N} \left(\tau \cdot D_k \tilde{V} + D_k V_1\right)^2\right]^{\frac{\mu}{2}-1} \cdot \left(\tau \cdot D_i \tilde{V} + D_i V_1\right)(x)$$

for $\sum\limits_{k=1}^{N} \left(\tau \cdot D_k \tilde{V} + D_k V_1\right)^2(x) \neq 0$ and $\tilde{H}_i(x,\tau) = 0$ for $\sum\limits_{k=1}^{N} \left(\tau \cdot D_k \tilde{V} + D_k V_1\right)^2(x) = 0$.

Assume $K = (\lambda_2 - \lambda_1)\left(1 + \|u\|_{W_2^1(\Omega)}\right),\ \lambda_2 > \lambda_1$.

Lemma 1.4 $\|V_2 - V_1\|_{W_2^1(\Omega)} \leq (\lambda_2 - \lambda_1) \cdot \left(1 + \|u\|_{W_2^1(\Omega)}\right)$.

Proof. For the function $\left(\tilde{V} - K\right)$ we have

$$\begin{cases} L\left(\tilde{V} - K\right) - c(x)\left(\tilde{V} - K\right) = c(x) \cdot K + f_1(x) \sum\limits_{i=1}^{N} \tilde{h}_i(x) D_i \left(\tilde{V} - K\right) + \\ + (\lambda_2 - \lambda_1) f_0(x), \ \ x \in \Omega, \\ \left(\tilde{V} - K\right)\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\varphi(x) - K, \ \ x \in \partial\Omega. \end{cases}$$

Further,

$$\begin{aligned} &c(x) K + (\lambda_2 - \lambda_1) f_0(x) \\ &= (\lambda_2 - \lambda_1)\left[c(x)\left(1 + \|u\|_{W_2^1(\Omega)}\right) + (f_1(x) - c(x) u(x))\right] \\ &= (\lambda_2 - \lambda_1)\left[c(x)\left(\|u\|_{W_2^1(\Omega)} - u(x)\right) + (c(x) + f_1(x))\right] \geq 0, \ \ x \in \Omega, \end{aligned}$$

$$\left(\tilde{V}-K\right)\Big|_{\partial\Omega}=(\lambda_2-\lambda_1)\left[\varphi(x)-\left(1+\|u\|_{W_2^1(\Omega)}\right)\right]$$
$$\leq(\lambda_2-\lambda_1)\left[\varphi(x)-\|u\|_{W_2^1(\Omega)}\right]\leq 0,\ \ x\in\partial\Omega.$$

Then from A.D.Alexandrov‘s paper [2] it follows that $\tilde{V}(x)\leq K$ in $x\in\overline{\Omega}$.
Analogous reasoning proves the inequality $\tilde{V}(x)\geq -K,\ x\in\overline{\Omega}$.
Lemma 1.4 is proved.

We continue the proof of Theorem 1.1. For that we consider the parametric family of problems (1.9). Note that by virtue of Lemma 1.3, the solution of problem (1.9) for $\lambda=1$ coincides with the solution of problem (1.8) and with the solution of problem (1.1).

Let $v_1(x)$ and $v_2(x)$ be the solutions of problem (1.9) corresponding to the values of λ_1 and $\lambda_2>\lambda_1$ of the parameter λ. Then by virtue of Lemma 1.4 we have

$$\|v_2-v_1\|_{W_2^1(\Omega)}\leq(\lambda_2-\lambda_1)\left(1+\|u\|_{W_2^1(\Omega)}\right). \tag{1.10}$$

On the other hand $\tilde{v}(x)=v_2(x)-v_1(x)$ function is the solution of the problem

$$\begin{cases} L\tilde{v}-c(x)\tilde{v}=f_1(x)\left[|Dv_2|^\mu-|Dv_1|^\mu\right]+(\lambda_2-\lambda_1)f_0(x), & x\in\Omega,\\ \tilde{v}\Big|_{\partial\Omega}=(\lambda_2-\lambda_1)\varphi(x), & x\in\partial\Omega.\end{cases}$$

Consequently,

$$\|L\tilde{v}-c(x)\tilde{v}\|_{p;\Omega}\leq 2^{\mu-1}\cdot\|f_1(x)\|_{q;\Omega}\cdot\|D\tilde{v}\|_{S;\Omega}^\mu$$
$$+2^\mu\cdot\|f_1\|_{q;\Omega}\cdot\|Dv_1\|_{S;\Omega}^\mu+(\lambda_2-\lambda_1)\|f_0\|_{p;\Omega}, \tag{1.11}$$
$$\|\tilde{v}\|_{W_p^{2-1/p}(\partial\Omega)}\leq(\lambda_2-\lambda_1)\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}, \tag{1.12}$$

for $1\leq\lambda_1<\lambda_2\leq 1$.

From the inequality (1.4) for the function $\tilde{v}(x)$ from $W_p^2(\Omega)$ with $p>\frac{2N}{N+2}$ it follows that

$$\|D\tilde{v}\|_{S;\Omega}\leq c_1\cdot\|\tilde{v}\|_{W_p^2(\Omega)}^{1/\mu}\cdot\|\tilde{v}\|_{W_2^1(\Omega)}^{1-1/\mu}+c_2\cdot\|\tilde{v}\|_{W_2^1(\Omega)}, \tag{1.13}$$

with positive constants c_1 and c_2 independent of the function $\tilde{v}(x)$ from $W_p^2(\Omega)$.

On the other hand, by virtue of the known linear theory of elliptic problems, the following inequality is fulfilled:

$$\|\tilde{v}\|_{W_p^2(\Omega)}\leq c_3\left(\|L\tilde{v}-c(x)\tilde{v}\|_{p;\Omega}+\|\tilde{v}\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

Here $c_3=c_3(\Omega,N,p,q)$. Then, using the inequalities (1.10)- (1.13), we get

$$\|\tilde{v}\|_{W_p^2(\Omega)}\leq c_3\cdot(\lambda_2-\lambda_1)\left(\|f_0\|_{p;\Omega}+\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right)+c_3\cdot 2^\mu\cdot\|f_1\|_{q;\Omega}\cdot\|Dv_1\|_{S;\Omega}^\mu$$
$$+c_3\cdot 2^{\mu-1}\cdot\|f_1\|_{q;\Omega}\cdot\left[c_1\cdot\|\tilde{v}\|_{W_p^2(\Omega)}^{1/\mu}\cdot\|\tilde{v}\|_{W_2^1(\Omega)}^{1-1/\mu}+c_2\cdot\|\tilde{v}\|_{W_p^2(\Omega)}\right]^\mu$$
$$\leq c_3\cdot 2^{2\mu-2}\cdot\|f_1\|_{q;\Omega}\cdot c_1^\mu\cdot\|\tilde{v}\|_{W_2^1(\Omega)}^{\mu-1}\cdot\|\tilde{v}\|_{W_p^2(\Omega)}$$
$$+c_3\cdot 2^{2\mu-2}\cdot\|f_1\|_{q;\Omega}\cdot c_2^\mu\cdot\|\tilde{v}\|_{W_2^1(\Omega)}^\mu+c_3\cdot 2^\mu\cdot\|f_1\|_{q;\Omega}\cdot\|Dv_1\|_{S;\Omega}^\mu$$
$$+c_3(\lambda_2-\lambda_1)\left(\|f_0\|_{p;\Omega}+\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right)$$

$$\leq c_3 \cdot 2^{2\mu-2} \cdot \|f_1\|_{q;\Omega} \cdot c_1^{\mu} \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot \left(1 + \|u\|_{W_2^1(\Omega)}\right)^{\mu-1} \cdot \|\tilde{v}\|_{W_p^2(\Omega)}$$

$$+c_3 \cdot 2^{2\mu-2} \cdot \|f_1\|_{q;\Omega} \cdot c_2^{\mu} \cdot (\lambda_2 - \lambda_1)^{\mu} \cdot \left(1 + \|u\|_{W_2^1(\Omega)}\right)^{\mu}$$

$$+c_3 \cdot 2^{\mu} \cdot \|f_1\|_{q;\Omega} \cdot \|Dv_1\|_{S;\Omega}^{\mu}$$

$$+c_3 \cdot (\lambda_2 - \lambda_1) \left(\|f_0\|_{p;\Omega} + \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

We require that the following inequalities be satisfied:

$$c_3 \cdot 2^{2\mu-2} \cdot \|f_1\|_{q;\Omega} \cdot c_1^{\mu} \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot \left(1 + \|u\|_{W_2^1(\Omega)}\right)^{\mu-1} \leq \frac{1}{2}.$$

Then we have

$$\|\tilde{v}\|_{W_p^2(\Omega)} \leq \Phi_1 + \Phi_2 \cdot \|Dv_1\|_{S;\Omega}^{\mu}. \tag{1.14}$$

for

$$0 < \lambda_2 - \lambda_1 \leq h, \tag{1.15}$$

where $\Phi_1,\, \Phi_2 \,:\, R_+ \to R_+$ are increasing functions determined by the known data. Here λ_1 and λ_2 are any numbers from the interval $[0,1]$ satisfying the inequality (1.15), and $v_1,\, v_2$ are solutions corresponding to them from $W_p^2\,(\Omega)$ of problem (1.9).

Denote $\lambda^{(k-1)} = \lambda_1,\, \lambda^{(k)} = \lambda_2$ and $v^{(k-1)} = v_1,\, v^{(k)} = v_2$, from the inequality (1.14) we obtain:

$$\|v^{(k)}\|_{W_p^2(\Omega)} \leq \|W^{(k-1)}\|_{W_p^2(\Omega)} + \Phi_1 + \Phi_2 \cdot \|Dv^{(k-1)}\|_{S;\Omega}^{\mu}$$

for $0 < \lambda^{(k)} - \lambda^{(k-1)} \leq h,\ \lambda^{(k-1)}, \lambda^{(k)} \in [0,1]$. Hence, by virtue of independence of h of the indicated values $\lambda^{(k)}$ and $\lambda^{(k-1)}$, from the inequality of the embedding $W_p^2\,(\Omega) \to W_2^1\,(\Omega)$ with $p > 2N/\,(N+2)$

$$\|Dv^{(k-1)}\|_{S;\Omega} \leq \text{const} \cdot \|v^{(k-1)}\|_{W_p^2(\Omega)}$$

for the final number of iterations we obtain the statements of the theorem. At the same time the first iteration$(k=1)$ corresponds to $\lambda^{(0)} = 0$ and $v^{(0)} = 0$.

Theorem 1.1 is proved.

As can be seen from Theorem 1.1, the conditions under which the inclusion theorem holds are of primary importance for this method. Since we are looking for the maximum value of μ, we obtain the following result:

$$\frac{1}{S} = \frac{1}{2} + \frac{1}{\mu}\left(\frac{1}{p} - \frac{N+2}{2N}\right),$$

$$\frac{q-p}{\mu pq} = \frac{1}{2} + \frac{1}{\mu p} - \frac{1}{\mu \cdot 2N}.$$

The solution of this equation is calculated explicitly in the form:

$$\mu = \frac{N+2}{N} - \frac{2}{q}.$$

To obtain a significantly nonlinear problem, we will assume that $\mu > 1$. This leads to the relation $q > N$, which will be assumed in the future.

2 Unimprovability of the indicator $\mu = \frac{N+2}{N} - \frac{2}{q}$

This paragraph provides an example of a boundary value problem of the form (1.1) for which all conditions of Theorem 1.1 are fulfilled, except for condition A.3). For this counterexample, the corresponding inequality is verified and it is shown that the statement of Theorem 1.1 is not true.Therefore, the indicator $\mu = \frac{N+2}{N} - \frac{2}{q}$ from condition A.3) cannot be replaced by $\mu > \frac{N+2}{N} - \frac{2}{q}$ without additional assumptions.To this end, consider the following counterexample (see [19]).

Let $N = 1,\ \Omega = [0, 1] \subset R$, and

$$\begin{cases} \Delta u = b(x) \cdot |Du|^{\mu}, \ \ x \in \Omega, \\ u(0) = 0, u(1) = (1+\varepsilon)^{\nu} - \varepsilon^{\nu} \end{cases} \tag{2.1}$$

with $b(x) = \nu^{1-\mu} \cdot (\nu - 1) \cdot (x + \varepsilon)^{\nu - 2 - \mu(\nu - 1)}$. Here $0 < \nu < 1$, $0 < \varepsilon < 1$, $\mu > \frac{N+2}{N} - \frac{2}{q} = 3 - \frac{2}{q}$, $p > 1 \ \ Du = u', \Delta u = u''$.

This problem for the given parameters has a unique solution.

$$u = (x + \varepsilon)^{\nu} - \varepsilon^{\nu}.$$

For the norm $\|b\|_{q;\Omega}$ we have

$$\|b\|_{q;\Omega} = \nu^{1-\mu} \cdot (1 - \nu) \left| \frac{(1+\varepsilon)^k - \varepsilon^k}{k} \right|^{1/q},$$

where $k = [(1 - \nu)\mu + \nu - 2] \cdot q + 1$. Hence, for $k > 0$, i.e. for

$$\mu > \frac{2 - \frac{1}{q} - \nu}{1 - \nu}, \tag{2.2}$$

and with the specified parameter values, the following inequality is satisfied

$$\|b\|_{q;\Omega} \le \nu^{1-\mu} \cdot 2^{k/q} \cdot k^{-1/q}$$

for all ε in the interval $(0, 1)$.

For the function $\tilde{\varphi}(x) = [(1+\varepsilon)^{\nu} - \varepsilon^{\nu}] \cdot x$ (such that $\tilde{\varphi}\Big|_{\partial\Omega} = u\Big|_{\partial\Omega}$) we have

$$\|\tilde{\varphi}\|_{W_p^2(\Omega)} = \|\tilde{\varphi}\|_{p;\Omega} + \|D\tilde{\varphi}\|_{p;\Omega} + \|D^2\tilde{\varphi}\|_{p;\Omega}$$

$$= [(1+\varepsilon)^{\nu} - \varepsilon^{\nu}] \cdot \left[1 + (p+1)^{-1/p}\right] \le 2^{1+\nu} < 4,$$

for the given values of parameters.

By virtue of inequality (2.2), for any $p > 1$ and $\mu > 3 - \frac{2}{q}$, $q > p$, there exists ν_* , $0 < \nu_* < 1$, such that inequality (2.2) is satisfied.

Then, for the given values parameters $p > 1$, $\mu > 3 - \frac{2}{q}$ and corresponding ν_* the inequalities

$$\|b\|_{q;\Omega} \le \nu_*^{1-\mu} \cdot 2^{k/q} \cdot k^{-1/q},$$

$$\|\varphi\|_{W_p^{2-1/p}(\partial\Omega)} \le c_0 \|\tilde{\varphi}\|_{W_p^2(\Omega)} \le 4c_0,$$

$$\|u\|_{W_2^1(\Omega)} = \|u\|_{2;\Omega} + \|Du\|_{2;\Omega} = \left(\int_0^1 |u|^2 dx \right)^{1/2}$$

$$+\left(\int_0^1 |Du|^2 dx\right)^{1/2} = \left(\int_0^1 \left((x+\varepsilon)^{2\nu} - 2\varepsilon^{\nu}(x+\varepsilon)^{\nu} + \varepsilon^{2\nu}\right) dx\right)^{1/2}$$

$$+\left(\int_0^1 \nu^2 (x+\varepsilon)^{2\nu-2} dx\right)^{1/2} \leq \sqrt{2+2^{2\nu}}, \tag{2.3}$$

uniformly relative to ε out of $(0,1)$. Here, the constant c_0 is independent of the functions $\tilde{\varphi}(x)$. Therefore, $\|u\|_{W_p^2(\Omega)} \to +\infty$ for $\varepsilon \to 0$ $(\varepsilon > 0)$. In this way the statement of Theorem 1.1 does not hold for any $\mu > 3 - \frac{2}{q}$. In fact, if the statement of this theorem was completed, then from the boundedness of the function Φ on the compact set defined by inequalities (2.3), would follow limitation $\|u\|_{W_p^2(\Omega)}$ is uniformly bounded with respect to ε from $(0,1)$.

3 Solvability theory

Let us consider boundary value problem (1.1) under the conditions A.1)-A.4) and the following Lipschitz condition.

A.5) Let

$$|f(x,u,\eta) - f(x,u,\xi)| \leq b_2(x,u,\xi,\eta) \cdot |\eta - \xi|$$

almost for all $x \in \Omega$ and for all $(u,\xi,\eta) \in R \times R^N \times R^N$, where the function $b_2(x,u,\xi,\eta)$ is measurable with respect to x for all $(u,\xi,\eta) \in R \times R^N \times R^N$, is continuous with respect to (u,ξ,η) almost for all $x \in \Omega$ and for any fixed $r > 0$

$$b_{2,r}(x) \equiv \sup\left\{ b_2(x,u,\xi,\eta) \,\Big|\, |u| \leq r,\ |\xi| \leq r,\ |\eta| \leq r \right\}$$

and $b_{2,r} \in L_q(\Omega)$ with $q > p$, $p > 2N/(N+2)$.

We recall the known definition.

Definition 3.1 *The function $u^+(x)$ from $W_p^2(\Omega)$ with $p > 2N/(N+2)$ is called an upper solution of the problem* (1.1) *if*

$$\begin{cases} -Lu^+ + f(x,u^+,Du^+) \geq 0, & x \in \Omega, \\ u^+\Big|_{\partial\Omega} \geq \varphi(x), & x \in \partial\Omega. \end{cases}$$

Definition 3.2 *The function $u^-(x)$ from $W_p^2(\Omega)$ with $p > 2N/(N+2)$ is called an upper solution of the problem* (1.1) *if*

$$\begin{cases} -Lu^- + f(x,u^-,Du^-) \leq 0, & x \in \Omega, \\ u^-\Big|_{\partial\Omega} \leq \varphi(x), & x \in \partial\Omega. \end{cases}$$

Theorem 3.1 *Let conditions A.1) - A.5) with some $p > 2N/(N+2)$ be fulfilled and the boundary function $\varphi(x)$ belong to the space $W_p^{2-1/p}(\Omega)$. Let there exist such upper $u^+(x)$ and lower $u^-(x)$ solutions of problem* (1.1) *from $W_p^2(\Omega)$ that $u^+(x) \geq u^-(x)$ in $\overline{\Omega}$. Then there exists a solution $u(x)$ from $W_p^2(\Omega)$ of the problem* (1.1) *and*

$$u^-(x) \leq u(x) \leq u^+(x), x \in \overline{\Omega}.$$

The proof of this theorem is based on the following lemmas.

Lemma 3.1 *Let the real-values function* $F_0(x,u,\xi)$ *determined on* $\Omega \times R \times R^N$ *satisfy the Caratheodory condition A.1) with* $f = F_0$ *and*

$$\sup_{(x,\xi)\in R\times R^N} |F_0(\cdot,u,\xi)| \in L_p(\Omega), \ \ p > 2N/(N+2). \tag{3.1}$$

Then the boundary value problem

$$\begin{cases} Lu = F_0(x,u,Du), & x\in\Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \ x\in\partial\Omega, \end{cases}$$

where $\varphi(x) \in W_p^{2-1/p}(\partial\Omega)$ *has the solution* $u \in W_p^2(\Omega)$.

Proof. Let us consider the following boundary value problem

$$\begin{cases} L_v u = F_0(x, v(x), Dv(x)), & x\in\Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \ x\in\partial\Omega \end{cases}$$

for the arbitrary function $v(x) \in C^1(\overline{\Omega})$. Then according to the known linear L_p theory of elliptic problems, this boundary value problem for any function $v(x)$ from $C^1(\overline{\Omega})$ has the solution $u(x)$ from $W_p^2(\Omega)$, $u = T(v)$ for which by virtue of (3.1) the following inequality is fulfilled:

$$\|u\|_{W_p^2(\Omega)} \le C, \tag{3.2}$$

where the constant C is independent of $v(x)$ from $C^1(\overline{\Omega})$. The operator $T : C^1(\overline{\Omega}) \to W_p^2(\Omega)$ is continuous and as a result of the compact embedding $W_p^2(\Omega) \to C^1(\overline{\Omega})$, $(p > 2N/(N+2))$ is a completely continuous operator from $C^1(\overline{\Omega})$ to $C^1(\overline{\Omega})$. Due to the estimate (3.2), there exists a sphere in the space $C^1(\overline{\Omega})$ that the operator takes to itself. Then, according to the well-known Schauder theorem, operator T has in space $C^1(\overline{\Omega})$ a fixed point $u(x)$, which, by definition of operator T, belongs to space $W_p^2(\Omega)$.

The lemma 3.1 is proved.

Determine we now for the function $u(x)$ from $W_p^2(\Omega)$ with $p > \frac{2N}{N+2}$ the cut off operator σ by the relation

$$\sigma u(x) = \begin{cases} u^+(x) & \text{for } u(x) > u^+(x), \\ u(x) & \text{for } u^-(x) \le u(x) \le u^+(x), \\ u^-(x) & \text{for } u(x) < u^-(x), \end{cases}$$

and consider the following boundary value problem

$$\begin{cases} Lu = f(x,\sigma u, Du), & x\in\Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \ x\in\partial\Omega. \end{cases} \tag{3.3}$$

Lemma 3.2 *Let the function* $f(x,\sigma u, Du)$ *satisfy conditions A.1) -A.5) with* $p > 2N/(N+2)$ *and* $\varphi \in W_p^{2-1/p}(\Omega)$. *Let* $u(x)$ *from* $W_p^2(\Omega)$ *be the solution of the problem* (3.3). *Then*

$$u^-(x) \le u(x) \le u^+(x), \ \ x\in\overline{\Omega} \tag{3.4}$$

Proof. For the function $\mathcal{W}(x) = u - u^+$ we have

$$L\mathcal{W} \geq f(x, \sigma u, Du) - f\left(x, u^+, Du^+\right), \quad x \in \Omega, \tag{3.5}$$

$$\mathcal{W}\Big|_{\partial\Omega} \leq 0, \quad x \in \partial\Omega. \tag{3.6}$$

We now assume the inverse statement of the lemma. Then set the $G = \left\{x \in \Omega \,\Big|\, \mathcal{W}(x) > 0\right\}$ is not empty, and

$$L\mathcal{W} \geq f\left(x, u^+, Du\right) - f\left(x, u^+, Du^+\right) \geq -b_{2,r}(x) \cdot |D\mathcal{W}|, \quad x \in \Omega,$$

where $b_{2,r}(x) = \sup\left\{b_2\left(x, u^+, Du, Du^+\right)\right\}$ and $b_{2,r} \in L_q(\Omega)$ with $q > p$ and $p > 2N/(N+2)$. Then it follows from A.D. Alexandrov's paper [2] that $\mathcal{W}(x)$ attains strong positive maximum on the boundary $\partial\Omega$. This contradicts the boundary condition (3.6). Thus, it is proved that $u(x) \leq u^+(x)$ in the domain Ω. Similarly, the inequality $u^-(x) \leq u(x)$ is proven in the domain Ω.

Lemma 3.2 is proved.

Now let $M = \max\limits_{\Omega}\left\{\|u^+\|_{W_2^1(\Omega)}, -\|u^-\|_{W_2^1(\Omega)}\right\}$. Then $\|u\|_{W_2^1(\Omega)} \leq M$ and by virtue of Theorem 1.1

$$\|u\|_{W_p^2(\Omega)} \leq \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

By virtue of the Sobolev embedding theorem [22] we have

$$W_p^2(\Omega) \subset W_2^1(\Omega), \; \|u\|_{W_2^1(\Omega)} \leq c_2 \cdot \|u\|_{W_p^2(\Omega)}, \; p > \frac{2N}{N+2}, \tag{3.7}$$

where the positive constant c_2 is independent of the function $u(x) \in W_p^2(\Omega)$. Then we obtain:

$$\|u\|_{W_p^2(\Omega)} \leq M_1, \quad M_1 = c_2 \cdot \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

Determine we now the function

$$F_1(x, u, \xi) = \begin{cases} f(x, u, \xi) & \text{for } |\xi| \leq M_2, \\ f\left(x, u, M_2, \frac{\xi}{|\xi|}\right) & \text{for } |\xi| > M_2, \end{cases}$$

where $M_2 = \max\left\{M_1, \|u^+\|_{W_2^1(\Omega)}, \|u^-\|_{W_2^1(\Omega)}\right\}$.

This function satisfies the conditions A.1) - A.3) with the appropriate inequality

$$|F_1(x, u, \xi)| \leq \begin{cases} b(x, u) + b_1(x, u)\,|\xi|^\mu & \text{for } |\xi| \leq M_2, \\ b(x, u) + b_1(x, u) \cdot |M_2|^\mu & \text{for } |\xi| > M_2. \end{cases}$$

The function F_1 satisfies also the condition A.5) with the appropriate function $b_2(x, u, \xi, \eta)$.

Let us consider the following boundary value problem

$$\begin{cases} Lu = F_1(x, \sigma u, Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \; x \in \partial\Omega. \end{cases} \tag{3.8}$$

The function

$$\overline{F}_1(x, u, \xi) = \begin{cases} F_1\left(x, u^+(x), \xi\right) & \text{for} \quad u > u^+(x), \\ F_1(x, u(x), \xi) & \text{for} \quad u^-(x) \leq u(x) \leq u^+(x), \\ F\left(x, u^-(x), \xi\right) & \text{for} \quad u < u^-(x), \end{cases}$$

satisfies the conditions of Lemma 1.3, and

$$\overline{F}_1(x,u(x),Du(x)) = F_1(x,\sigma u(x),Du(x)).$$

Consequently, to the problem (3.8) we can apply Lemma 3.1 by virtue of which there exists the solution $u(x)$ of this problem from the space $W_p^2(\Omega)$ with $p > \frac{2N}{N+2}$.

The function $u^+(x)$ is the upper solution and $u^-(x)$ is the down solution of the problem

$$\begin{cases} Lu = F_1(x,u,Du), \quad x \in \Omega, \\ u\Big|_{\partial\Omega} = \varphi(x), \ x \in \partial\Omega. \end{cases} \tag{3.9}$$

By the Lemma 3.2 we have:

$$u^-(x) \le u(x) \le u^+(x), \quad x \in \overline{\Omega},$$

and consequently, $\sigma u(x) = u(x)$ so that the obtained solution $u(x)$ of the problem (3.8) is the solution of problem (3.9) as well.

We now apply the theorem to the solution $u(x)$ of the problem (3.9) and obtain

$$\|u\|_{W_p^2(\Omega)} \le \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

Then from the inequality (3.7) it follows that

$$\|u\|_{W_p^2(\Omega)} \le M_1, \quad M_1 = c_2 \cdot \Phi\left(M, \|\varphi\|_{W_p^{2-1/p}(\partial\Omega)}\right).$$

We finally obtain that the solution $u(x)$ of problem (3.9) is the solution of problem (1.1). Theorem 3.1 is proved.

Choosing the function from various trial functions sets as upper and down solutions, we can obtain various theorems on the existence of solutions to boundary value problems of the form (1.1).

Let us take as a “trial” set of functions of the form

$$u(x) = \lambda,$$

where λ is a real number.

Theorem 3.2 *Let be satisfied conditions A.1)- A.5) with some $p > 2N/(N+2)$ and the boundary function $\varphi(x)$ belong to the space $W_p^{2-1/p}(\partial\Omega)$. Let the functions $f(x,u,Du)$ and $\varphi(x)$ be such that there exist two numbers λ^+ and λ^-, $\lambda^+ \ge \lambda^-$, such that*

$$\begin{cases} f(x,\lambda^-,0) \le 0 \le f(x,\lambda^+,0), \quad x \in \Omega, \\ \lambda^- \le \varphi(x) \le \lambda^+, \ x \in \partial\Omega. \end{cases}$$

Then there exists the solution $u(x)$ of the boundary value problem (1.1) *from the space $W_p^2(\Omega)$ and*

$$\lambda^- \le u(x) \le \lambda^+, \ x \in \Omega.$$

For prove this theorem, it suffices to apply theorem 3.1 $u^+ = \lambda^+$ and $u^- = \lambda^-$.

Now let us take as a “trial” set of functions of the form

$$u(x) = \lambda\frac{|x|^2}{2},$$

where λ is a real number.

Theorem 3.3 *Let conditions A.1)- A.5) be satisfied for some* $p > N$ *and the boundary function* $\varphi(x)$ *belong to the space* $W_p^{2-1/p}(\partial\Omega)$. *Let the functions* $f(x,u,Du)$ *and* $\varphi(x)$ *be such that there exist two numbers* λ^+ *and* λ^-, $\lambda^+ \geq \lambda^-$, *such that*

$$\begin{cases} -\sum\limits_{|\alpha|=2} a_\alpha\left(x, \lambda^+ \cdot \frac{|x|^2}{2}, \lambda^+ x\right)\lambda^+ + f\left(x, \lambda^+ \cdot \frac{|x|^2}{2}, \lambda^+ x\right) \geq, \;\; x \in \Omega, \\ -\sum\limits_{|\alpha|=2} a_\alpha\left(x, \lambda^- \frac{|x|^2}{2}, \lambda^- x\right) + f\left(x, \lambda^- \frac{|x|^2}{2}, \lambda^- x\right) \leq 0, \;\; x \in \Omega, \\ \lambda^- \cdot \frac{|x|^2}{2} \leq \varphi(x) \leq \lambda^+ \frac{|x|^2}{2}, \;\; x \in \partial\Omega. \end{cases}$$

Then there exists the solution $u(x)$ *to the boundary value problem* (1.1) *from the space* $W_p^2(\Omega)$, *and*

$$\lambda^- \cdot \frac{|x|^2}{2} \leq u(x) \leq \lambda^+ \cdot \frac{|x|^2}{2}, \;\; x \in \Omega.$$

To prove this theorem, it is enough to apply Theorem 3.1 with $u^+ = \lambda^+ \cdot \frac{|x|^2}{2}$ and $u^- = \lambda^- \cdot \frac{|x|^2}{2}$.

Similarly, we obtain the solvability theorem for boundary value problem (1.1), if we "trial" set of functions take functions of the form

$$u(x) = \lambda \cdot \frac{|x - x_0|^2}{2}$$

with appropriate $\lambda^-, \lambda^+ \in R$ and $x_0, x \in R^N$.

Let us take now as a "trial" set of functions the functions of the form

$$u(x) = u_0(x) + \lambda\psi_1(x),$$

where $u_0(x)$ is the solution of the problem

$$\begin{cases} Lu_0 = 0, \;\; x \in \Omega, \\ u_o\Big|_{\partial\Omega} = \varphi(x), \;\; x \in \partial\Omega, \end{cases}$$

λ is a real number and $\psi_1(x)$ is some first eigenfunction of the boundary value problem

$$\begin{cases} L\psi_1 + \lambda_1\psi_1 = 0, \;\; x \in \Omega, \\ \psi_1\Big|_{\partial\Omega} = 0, \end{cases}$$

and $\psi_1(x) > 0$ in Ω.

Theorem 3.4 *Let conditions A.1) - A.5) be satisfied for some* $p > 2N/(N+2)$ *and let the boundary function* $\varphi(x)$ *belong to the space* $W_p^{2-1/p}(\partial\Omega)$. *Let the functions* $f(x,u,Du)$ *and* $\varphi(x)$ *be such that there exist two numbers* λ^- *and* λ^+, $\lambda^+ \geq \lambda^-$ *such that*

$$f\left(x, u_0 + \lambda^+\psi_1, Du_0 + \lambda^+ D\psi_1\right) + \lambda_1\lambda^+\psi_1(x) \geq 0, \;\; x \in \Omega,$$

$$f\left(x, u_0 + \lambda^-\psi_1, Du_0 + \lambda^- D\psi_1\right) + \lambda_1\lambda^-\psi_1(x) \leq 0, \;\; x \in \Omega.$$

Then there exists the solution $u(x)$ *to the boundary value problem* (1.1) *from the space* $W_p^2(\Omega)$, *and*

$$u_0(x) + \lambda^-\psi_1(x) \leq u(x) \leq u_0(x) + \lambda^+\psi_1(x), \;\; x \in \Omega.$$

To prove this theorem, it is enough to apply Theorem 3.1 with $u^+ = u_0 + \lambda^+\psi_1$ and $u^- = u_0 + \lambda^-\psi_1$.

4 Application to second-order systems

In this part of the paper we prove a theorem on a priori estimation for a second order quasilinear elliptic system. The main attention is on the attainability of maximum degrees of the growth of the nonlinear component of the system and constraints in its summability index. The growth conditions for the right hand side obtained in this the paper and providing strong a priori estimations of the solutions to the quasilinear system through the maximum of the modulus of this solution, are a bit different from the conditions [4], [8], [7], [12], [13], [15], [16], [20], [21]. The conditions under which the generalized maximum principle holds for the system of the specified type, are given and a theorem on the existence of the solution is proved. When proving the main result on a strong a priori estimation, the interpolation method for obtaining such estimations and developed for elliptic equations by S.I. Pohojaev [19], [20] is used.

Let us consider a quasi-linear operator generated by the system

$$\vec{L}\vec{u} \equiv \left\{ \sum_{|\alpha|=2} \left[\sum_{j=1}^{n} a_{\alpha j}^{k} \left(x, \vec{u}, D\vec{u}\right) D^{\alpha} u_j \right], \;\; k = \overline{1,n} \right\},$$

where the coefficients $a_{\alpha j}^{k} \;\; (|\alpha| = 2), \; k, j = \overline{1,n}$ are real and continuous function in $\overline{\Omega} \times R^n \times R^{nN}$.

With the help of the described interpolation method, we will establish sufficient conditions that are enough a strong a priori estimate of solutions $\vec{u} \in \vec{W}_p^2\left(\Omega\right)$ of the problem

$$\begin{cases} \vec{L}\vec{u} = \vec{f}\left(x, \vec{u}, D\vec{u}\right), \;\; x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}\left(x\right), \;\; x \in \partial\Omega, \end{cases} \tag{4.1}$$

The following conditions are assumed to be fulfilled.

B.1) The functions $f_k\left(x, \vec{\xi}_0, \vec{\xi}_1\right), \; k = \overline{1,n}$ vector components $\vec{f}\left(x, \vec{\xi}_0, \vec{\xi}_1\right)$ determined on $\overline{\Omega} \times R^n \times R^{nN}$ and caratheodorian.

B.2) Let

$$\left|\vec{f}\left(x, \vec{\xi}_0, \vec{\xi}_1\right)\right| \le \sum_{k=1}^{n} \left|f_k\left(x, \vec{\xi}_0, \vec{\xi}_1\right)\right| \le \vec{b}\left(x, \vec{\xi}_0\right) + \vec{b}_1\left(x, \vec{\xi}_0\right) \cdot \left|\vec{\xi}_1\right|^{\mu},$$

almost for all $x \in \Omega, \vec{\xi}_0 \in R^n$ and $\vec{\xi}_1 \in R^{nN}$ with such non-negative caratheodorian functions $\vec{b}\left(x, \vec{\xi}_0\right)$ and $\vec{b}_1\left(x, \vec{\xi}_0\right)$ that for any $r \ge 0$

$$\vec{b}_r\left(x\right) \equiv \sup\left\{\vec{b}\left(x, \vec{\xi}_0\right) \Big| |\vec{\xi}_0| \le r\right\}$$

belongs to space $\vec{L}_p\left(\Omega\right)$ with $p > 1$ and $p > \frac{2N}{N+2}$ and the function

$$\vec{b}_{1,r}\left(x\right) \equiv \sup\left\{\vec{b}_1\left(x, \vec{\xi}_0\right) \Big| |\vec{\xi}_0| \le r\right\}$$

belongs to space $\vec{L}_q\left(\Omega\right)$ with $q > p$ and $p > \frac{2N}{N+2}$.

B. 3) Let

$$\mu = \frac{N+2}{N} - \frac{2}{q}.$$

B.4) Let $\vec{L}$ be a second-order quasilinear elliptic operator with real and continuous coefficients on $\overline{\Omega} \times R^n \times R^{nN}$. Let the operator $\vec{L}_{\vec{v}}\vec{u}$, which is linear with respect to $\vec{u}\,(x)$, be equal to

$$\vec{L}_v\vec{u} \equiv \sum_{|\alpha|=2} \left[\sum_{j=1}^{n} a^k_{\alpha j}\,(x, \vec{v}, D\vec{v})\, D^\alpha u_j, \right], \quad k = 1, n$$

for any function $\vec{v} \in \vec{C}^1\left(\overline{\Omega}\right)$ be a linear elliptic operator with such real coefficients that the linear (with respect $\vec{u}\,(x)$) boundary value problem

$$\begin{cases} \vec{L}_{\vec{v}}\vec{u} = \vec{g}\,(x)\,, & x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}\,(x)\,, & x \in \partial\Omega \end{cases} \tag{4.2}$$

for any function $\vec{v}\,(x) \in \vec{C}^1\left(\overline{\Omega}\right)$ is coercive in the space $\vec{W}^2_p\,(x)$, i.e. the a priori estimation

$$\|\vec{u}\|_{\vec{W}^2_p(\Omega)} \leq c\left(\|\vec{g}\|_{p;\Omega} + \|\vec{\varphi}\|_{\vec{W}^{2-1/p}_p(\partial\Omega)} + \|\vec{u}\|_{p;\Omega}\right)$$

with a positive constant c independent of $\vec{g} \in \vec{L}_p\,(\Omega)$ of $\vec{\varphi} \in \vec{W}^{2-1/p}_p\,(\partial\Omega)$ and of the solution $\vec{u} \in \vec{W}^2_p\,(\Omega)$ of the linear problem (4.2) is fulfilled. In this case, the constant c from continuity modules of coefficients $a^k_{\alpha j}$ $(|\alpha| = 2)$, $k, j = \overline{1, n}$ and module of continuity of the functions $\vec{v}$, $D\vec{v}$. Note that sufficient conditions for the coefficients of the operator $\vec{L}$ ensuring the coercivity of the linear (with respect to $\vec{u}\,(x)$) boundary value problem (4.2) are given in [1], [10].

Thus, boundary problem (4.1) is considered at the intersection of spaces $\vec{W}^2_p\,(\Omega) \cap \vec{W}^1_2\,(\Omega)$. Note that when $p \geq \frac{2N}{N+2}$ there is an insertion $\vec{W}^2_p\,(\Omega) \subset \vec{W}^1_2\,(\Omega)$ (see, [22]), so that the boundary function $\vec{\varphi}\,(x)$ is the trace $\vec{\varphi}\,(x) = \vec{\psi}\,(x)\Big|_{\partial\Omega}$ of function $\vec{\psi}\,(x)$ from $\vec{W}^2_p\,(\Omega)$, i.e. it belongs to space $\vec{W}^{2-1/p}_p\,(\partial\Omega)$ with norm $\|\vec{\varphi}\|_{\vec{W}^{2-1/p}_p(\partial\Omega)} \leq c_0 \cdot \|\vec{\psi}\|_{\vec{W}^2_p(\Omega)}$, where c_0 does not depend on $\vec{\psi}\,(x)$ (see [ ]).

In this work, it is assume that the nonlinear function $\vec{f}\,(x, \vec{u}, D\vec{u})$ belong to space $\vec{L}_p\,(\Omega)$ with arbitrarily fixed function $\vec{u}\,(x)$ from $\vec{W}^2_p\,(\Omega)$ with $p > \frac{2N}{N+2}$.

When investigating the problem (4.1), the embedding theorem of anisotropic spaces is used substantially $\vec{W}^2_p\,(\Omega)$ and the multiplicative inequality that follows from it are essential. Let us formulate the general embedding theorem [17] in the particular form we need.

Lemma 4.1 *Let $\vec{u}\,(x) \in \vec{W}^2_p\,(\Omega)$, $p > \frac{2N}{N+2}$ and the conditions have been fulfilled B.1) - B.3). Then*

$$\|D\vec{u}\|_{S;\Omega} \leq c_1 \cdot \|\vec{u}\|^{1/\mu}_{\vec{W}^2_p(\Omega)} \cdot \|\vec{u}\|^{1-1/\mu}_{\vec{W}^1_2(\Omega)} + c_2 \cdot \|\vec{u}\|_{\vec{W}^1_2(\Omega)} \tag{4.3}$$

with positive constants c_1 and c_2, independent of the function $\vec{u}\,(x)$ of $\vec{W}^2_p\,(\Omega)$, where

$$\begin{cases} \frac{1}{S} = \frac{1}{2} + \frac{1}{\mu}\left(\frac{1}{p} - \frac{N+2}{2N}\right), \\ 1 < \mu \leq \frac{N+2}{N}, \quad S = \frac{\mu p q}{q-p}. \end{cases} \tag{4.4}$$

Lemma 4.2 *Let conditions B.1) - B.3) be fulfilled. Then the operator $\vec{F}\,(\vec{u})\,(x) \equiv \vec{f}\,(x, \vec{u}\,(x)\,, D\vec{u}\,(x))$ acts completely continuously from $\vec{W}^2_p\,(\Omega)$ in $\vec{L}_p\,(\Omega)$ with $p > 1$ and $p > \frac{2N}{N+2}$.*

Proof. Let us evaluate $\|\vec{F}(\vec{u})\|_{p;\Omega}$ using conditions B.1) - B.3) and Hölder's inequality:

$$\|\vec{F}(\vec{u})\|_{p;\Omega} \leq \|\vec{b}_r\|_{p;\Omega} + \|\vec{b}_{1,r}\|_{q;\Omega} \cdot \|D\vec{u}\|^{\mu}_{S;\Omega} \tag{4.5}$$

with $r = \text{const} \cdot \|\vec{u}\|_{\vec{W}_2^1(\Omega)}$ and $S = \frac{\mu pq}{q-p}$. Then, based on equalities B.3) and interpolation inequalities (4.3) with the specified S and μ corresponding to them according to formula (4.4), we obtain

$$\|\vec{F}(\vec{u})\|_{p;\Omega} \leq \|\vec{b}_r\|_{p;\Omega} + \Phi_1\left(\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right) \cdot \|\vec{u}\|_{\vec{W}_p^2(\Omega)} + \Phi_2\left(\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right), \tag{4.6}$$

where $\Phi_1, \Phi_2 : R_+ \to R_+$ are the increasing functions determined by the known data. This proves the boundedness of operator $\vec{F}(\vec{u})(x)$. Since $p > 2N/(N+2)$. Under the restrictions of this theorem, the embedding operator $\vec{F_1} : \vec{W}_p^2(\Omega) \to \vec{L}_S(\Omega)$ is completely continuous. By virtue of estimate (4.6), the operator $\vec{F} : \vec{L}_S(\Omega) \to \vec{L}_p(\Omega)$ is bounded, and by the general properties of the superposition operator, it is continuous. Then operator $\vec{F} : \vec{W}_p^2(\Omega) \to \vec{L}_p(\Omega)$ is completely continuous as a composition of completely continuous and continuous operators.

Lemma 4.2 is proved.

Let $\vec{u} \in \vec{W}_p^2(\Omega)$ be the solution of problem (4.1) and let the conditions B.1)- B.4) be fulfilled. Then the problem (4.1) can be considered as a linear problem with a fixed right hand side $\vec{F}(\vec{u})(x) \equiv \vec{f}(x, \vec{u}(x), D\vec{u}(x)) \in \vec{L}_p(\Omega)$ with $p > \frac{2N}{N+2}$.

Theorem 4.1 *Let conditions B.1) - B.4) with some $p > 2N/(N+2)$ be fulfilled. Then there exists such a function $\Phi : R_+^2 \to R_+$ bounded in each compact that for any solution $\vec{u}(x)$ from $\vec{W}_p^2(\Omega)$ with $p > 2N/(N+2)$ of the problem* (4.1) *the inequality*

$$\|\vec{u}\|_{\vec{W}_p^2(\Omega)} \leq \Phi\left(M, \|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\right),$$

is fulfilled if $\|\vec{u}\|_{\vec{W}_2^1(\Omega)} \leq M$ and $R_+^2 = R_+ \times R_+$.

Proof. We have

$$\vec{L}\vec{u} = \vec{f}(x, \vec{u}, D\vec{u}) = \frac{\vec{f}(x, \vec{u}, D\vec{u})}{1 + |D\vec{u}|^{\mu}} \cdot (1 + |D\vec{u}|^{\mu}).$$

Hence

$$\vec{L}\vec{u} - c(x)\vec{u} = \frac{\vec{f}(x, \vec{u}, D\vec{u})}{1 + |D\vec{u}|^{\mu}} \cdot (1 + |D\vec{u}|^{\mu}) - c(x)u,$$

where $c(x) = b_r(x) \geq 0$. Then for the function $\vec{u}(x)$ we have

$$\begin{cases} \vec{L}\vec{u} - \vec{c}(x)\vec{u} = f_1(x) \cdot |D\vec{u}|^{\mu} + \vec{f_0}(x), & x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}(x), & x \in \partial\Omega. \end{cases} \tag{4.7}$$

Here $\vec{f_1}(x) = \frac{\vec{f}(x,\vec{u},D\vec{u})}{1+|D\vec{u}|^{\mu}}$, $\vec{f_0}(x) = \vec{f_1}(x) - \vec{c}(x)\vec{u}$. Our goal is to obtain an estimate of $\|\vec{u}\|_{\vec{W}_p^2(\Omega)}$ through values that bound $\|\vec{u}\|_{\vec{W}_2^1(\Omega)}$, $\|\vec{f_0}\|_{p;\Omega}$, $\|\vec{f_1}\|_{q;\Omega}$ and $\|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}$

from above. To this end, consider the boundary value problem in the space $\vec{W}_p^2(\Omega)$ with $p > 2N/(N+2)$.

$$\begin{cases} \vec{L}\vec{V}(x) - \vec{c}(x)\vec{V} = \vec{f}_1(x) \cdot |D\vec{V}|^{\mu} + \lambda \cdot \vec{f}_0(x), & x \in \Omega, \\ \vec{V}\Big|_{\partial\Omega} = \lambda \cdot \vec{\varphi}(x), & x \in \partial\Omega, \end{cases} \tag{4.8}$$

with the parameter $\lambda \in [0,1]$ and with the functions $\vec{c}(x)$, $\vec{f}_0(x)$ and $\vec{f}_1(x)$ determined above.

For problem (4.8), the following lemma on the uniqueness of the solution holds.

Lemma 4.3 *For any fixed λ, problem* (4.8) *has at most one solution from $\vec{W}_p^2(\Omega)$ with $p > 2N/(N+2)$.*

Proof. Let us assume the opposite. Then, for the difference $\vec{\mathcal{W}} = \vec{V} - \vec{Z}$ of two possible solutions $\vec{V}(x)$ and $\vec{Z}(x)$, we have

$$\begin{cases} \vec{L}\vec{\mathcal{W}} - \vec{c}(x)\vec{\mathcal{W}} = \vec{f}_1(x) \cdot \sum\limits_{i=1}^{N} \vec{h}_i(x) D_i\vec{\mathcal{W}}(x), & x \in \Omega, \\ \vec{\mathcal{W}}\Big|_{\partial\Omega} = 0, & x \in \partial\Omega. \end{cases}$$

Here $\vec{h}_i(x) = \int\limits_0^1 \vec{H}_i(x,\tau)\, d\tau$ where

$$\vec{H}_i(x,\tau) = \mu \left[\sum_{k=1}^{N} \left(\tau \cdot D_k\vec{\mathcal{W}} + D_k\vec{Z}\right)^2\right]^{\frac{\mu}{2}-1} \cdot \left(\tau \cdot D_i\vec{\mathcal{W}} + D_i\vec{Z}\right)(x),$$

for $\sum\limits_{k=1}^{N} \left(\tau \cdot D_k\vec{\mathcal{W}} + D_k\vec{Z}\right)^2(x) \neq 0$ and $\vec{H}_i(x,\tau) = 0$ for $\sum\limits_{k=1}^{N} \left(\tau \cdot D_k\vec{\mathcal{W}} + D_k\vec{Z}\right)^2(x) = 0$.

Since $\vec{c}(x) \geq 0$, $\vec{c}(x) \in L_p(x)$, $\vec{f}_1(x) \in \vec{L}_q(\Omega)$, $\vec{h}_i(x) \in \vec{L}_p(\Omega)$ $(i = 1, \ldots, N)$, $q > p$ and $p > 2N/(N+2)$, it follows from the results of A.D. Aleksandrov [2] that $\vec{\mathcal{W}}(x) \equiv 0$.

Lemma 4.3 is proved.

Let now $\vec{V}_1$ and $\vec{V}_2$ be the solutions of the problem (4.8) corresponding to the values of λ_1 and $\lambda_2 > \lambda_1$ of the parameter λ. Then for the difference $\vec{Y} = \vec{V}_2 - \vec{V}_1$ we have

$$\begin{cases} \vec{L}\vec{Y} - \vec{c}(x)\vec{Y} = \vec{f}_1(x) \cdot \sum\limits_{i=1}^{N} \vec{\tilde{h}}_i(x) D_i\vec{Y} + (\lambda_2 - \lambda_1)\vec{f}_0(x), & x \in \Omega, \\ \vec{Y}\Big|_{\partial\Omega} = (\lambda_2 - \lambda_1)\vec{\varphi}(x), & x \in \partial\Omega. \end{cases}$$

Here $\vec{\tilde{h}}_i(x) = \int\limits_0^1 \vec{\tilde{H}}_i(x,\tau)\, d\tau$ where

$$\vec{\tilde{H}}_i(x,\tau) = \mu \left[\sum_{k=1}^{N} \left(\tau \cdot D_k\vec{Y} + D_k\vec{V}_1\right)^2\right]^{\frac{\mu}{2}-1} \cdot \left(\tau \cdot D_i\vec{Y} + D_i\vec{V}_1\right)(x)$$

for $\sum\limits_{k=1}^{N}\left(\tau\cdot D_k\vec{Y}+D_k\vec{V}_1\right)^2(x)\neq 0$ and $\vec{\tilde{H}}_i(x,\tau)=0$ for $\sum\limits_{k=1}^{N}\left(\tau\cdot D_k\vec{Y}+D_k\vec{V}_1\right)^2(x)=0$.

Assume $K=(\lambda_2-\lambda_1)\left(1+\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right),\ \lambda_2>\lambda_1$.

Lemma 4.4 $\|\vec{V}_2-\vec{V}_1\|_{\vec{W}_2^1(\Omega)}\leq K.$

Proof. For the function $\left(\vec{Y}-K\right)$ we have

$$\begin{cases}\vec{L}\left(\vec{Y}-K\right)-\vec{c}(x)\left(\vec{Y}-K\right)=\vec{f}_1(x)\cdot\sum\limits_{i=1}^{N}\vec{\tilde{h}}_i(x)\,D_i\left(\vec{Y}-K\right)+\\ +(\lambda_2-\lambda_1)\,\vec{f}_0(x)+\vec{c}(x)\,K,\ \ x\in\Omega,\\ \left(\vec{Y}-K\right)\Big|_{\partial\Omega}=(\lambda_2-\lambda_1)\,\vec{\varphi}(x)-K,\ \ x\in\partial\Omega.\end{cases}$$

Further,

$$\vec{c}(x)\,K+(\lambda_2-\lambda_1)\,\vec{f}_0(x)=$$

$$=(\lambda_2-\lambda_1)\left[\vec{c}(x)\left(\|\vec{u}\|_{\vec{W}_2^1(\Omega)}-\vec{u}(x)\right)+\left(\vec{c}(x)+\vec{f}_1(x)\right)\right]\geq 0,\ \ x\in\Omega$$

$$\left(\vec{Y}-K\right)\Big|_{\partial\Omega}\leq(\lambda_2-\lambda_1)\left(\vec{\varphi}(x)-\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right)\leq 0,\ \ x\in\partial\Omega.$$

Then it follows from A.D. Alexandrov‘s paper [2] that $\vec{Y}(x)\leq -K$ in $\overline{\Omega}$.
Similar reasoning proves the inequality $\vec{Y}(x)\geq -K$ in $\overline{\Omega}$.
Lemma 4.4 is proved.

Let us continue the proof of Theorem 4.1. To do this, consider the parametric family of problems (4.8). Note that, by Lemma 4.3, the solution to problem (4.8) at $\lambda=1$ coincides with the solution to problem (4.7) and with the solution to problem (4.1).

Let $\vec{V}_1$ and $\vec{V}_2$ be solutions to problem (4.8) corresponding to the values λ_1 and $\lambda_2>\lambda_1$ of the parameter λ. Then by Lemma 4.4, we have

$$\|\vec{V}_2-\vec{V}_1\|_{\vec{W}_2^1(\Omega)}\leq(\lambda_2-\lambda_1)\left(1+\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right). \tag{4.9}$$

On the other hand, this function is the solution to the problem

$$\begin{cases}\vec{L}\vec{Y}-\vec{c}(x)\,\vec{Y}=\vec{f}_1(x)\left[|D\vec{V}_2|^{\mu}-|D\vec{V}_1|^{\mu}\right]+(\lambda_2-\lambda_1)\,\vec{f}_0(x)\,,\ \ x\in\Omega,\\ \vec{Y}\Big|_{\partial\Omega}=(\lambda_2-\lambda_1)\,\vec{\varphi}(x)\,,\ \ x\in\partial\Omega.\end{cases}$$

Consequently,

$$\|\vec{L}\vec{Y}-\vec{c}(x)\,\vec{Y}\|_{p;\Omega}\leq 2^{\mu-1}\cdot\|\vec{f}_1\|_{q;\Omega}\cdot\|D\vec{Y}\|_{S;\Omega}^{\mu}+$$

$$+2^{\mu}\cdot\|\vec{f}_1\|_{q;\Omega}\cdot\|D\vec{V}_1\|_{S;\Omega}^{\mu}+(\lambda_2-\lambda_1)\,\|\vec{f}_0\|_{p;\Omega}, \tag{4.10}$$

$$\|\vec{Y}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\leq(\lambda_2-\lambda_1)\,\|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}, \tag{4.11}$$

for $0\leq\lambda_1<\lambda_2\leq 1$.

From the inequality (4.3) for the function $\vec{Y}(x)$ from $\vec{W}_p^2(\Omega)$ with $p>2N/(N+2)$ it follows that

$$\|D\vec{Y}\|_{S;\Omega}\leq c_1\cdot\|\vec{Y}\|_{\vec{W}_p^2(\Omega)}^{1/\mu}\cdot\|\vec{Y}\|_{\vec{W}_2^1(\Omega)}^{1-1/\mu}+c_2\|\vec{u}\|_{\vec{W}_2^1(\Omega)}, \tag{4.12}$$

with positive constants c_1 and c_2 independent of the function $\vec{Y}(x)$ from $\vec{W}_p^2(\Omega)$. On the other hand, by virtue of the known theory of elliptic problems, the following inequality is fulfilled

$$\|\vec{Y}\|_{W_p^2(\Omega)} \leq c_3 \left(\|\vec{L}\vec{Y} - \vec{c}(x)\vec{Y}\|_{p;\Omega} + \|\vec{Y}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)} \right).$$

Here $c_3 = c_3(\Omega, N, p, q)$. Then, using inequalities (4.9)-(4.12) we obtain

$$\|\vec{Y}\|_{\vec{W}_p^2(\Omega)} \leq c_3(\lambda_2 - \lambda_1) \cdot \left(\|\vec{f}_0\|_{p;\Omega} + \|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)} \right)$$
$$+ c_3 \cdot c_2^\mu \cdot 2^{2\mu-2} \cdot \|\vec{f}_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^\mu \cdot \left(1 + \|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right)^\mu$$
$$+ c_3 \cdot 2^\mu \cdot \|\vec{f}_1\|_{q;\Omega} \cdot \|D\vec{V}_1\|_{S;\Omega}^\mu$$
$$+ c_3 \cdot 2^{2\mu-2} \cdot \|\vec{f}_1\|_{q;\Omega} \cdot c_1^\mu \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot \left(1 + \|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right)^{\mu-1} \cdot \|\vec{Y}\|_{\vec{W}_p^2(\Omega)}.$$

Hence it follows that

$$\|\vec{Y}\|_{\vec{W}_p^2(\Omega)} \leq 2c_3 \cdot (\lambda_2 - \lambda_1) \cdot \left(\|\vec{f}_0\|_{p;\Omega} + \|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)} \right)$$
$$+ c_3 \cdot c_2^\mu \cdot 2^{2\mu-2} \cdot \|\vec{f}_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^\mu \cdot \left(1 + \|\vec{u}\|_{\vec{W}_p^2(\Omega)}\right)^\mu$$
$$+ c_3 \cdot 2^{\mu+1} \cdot \|\vec{f}_1\|_{q;\Omega} \cdot \|D\vec{V}_1\|_{S;\Omega}^\mu,$$

for

$$c_3 \cdot 2^{2\mu-2} \cdot \|\vec{f}_1\|_{q;\Omega} \cdot (\lambda_2 - \lambda_1)^{\mu-1} \cdot \left(1 + \|\vec{u}\|_{\vec{W}_2^1(\Omega)}\right)^{\mu-1} \leq \frac{1}{2}.$$

Thus, we have

$$\|\vec{Y}\|_{\vec{W}_p^2(\Omega)} \leq \Phi_1 + \Phi_2 \cdot \|D\vec{V}_1\|_{S;\Omega}^\mu, \tag{4.13}$$

for

$$0 < \lambda_2 - \lambda_1 \leq h, \tag{4.14}$$

where

$$h = c_3^{-1/(\mu-1)} \cdot 2^{-(2\mu-1)/(\mu-1)} \cdot c_1^{-\mu/(\mu-1)} \cdot (1 + M)^{-1},$$

and Φ_1, $\Phi_2 : R_+ \to R_+$ are increasing functions determined by the known data. Here $\lambda_1 - \lambda_2$ are any numbers from the interval $[0, 1]$ satisfying the inequality (4.14) and $\vec{V}_1$, $\vec{V}_2$ are appropriate solutions from $\vec{W}_p^2(\Omega)$ of the problem (4.8).

Denote $\lambda^{(k-1)} = \lambda_1$, $\lambda^{(k)} = \lambda_2$ and $\vec{V}^{(k-1)} = \vec{V}_1$, $\vec{V}^{(k)} = \vec{V}_2$, from the inequality (4.13) we obtain

$$\|V^{(k)}\|_{\vec{W}_p^2(\Omega)} \leq \Phi_1 + \|V^{(k-1)}\|_{\vec{W}_p^2(\Omega)} + \Phi_2 \cdot \|D\vec{V}^{(k-1)}\|_{S;\Omega}^\mu,$$

for $0 < \lambda^{(k)} - \lambda^{(k-1)} \leq h$, $\lambda^{(k-1)}, \lambda^{(k)} \in [0, 1]$. Hence, due to the independence of h of the indicated values $\lambda^{(k-1)}$ and $\lambda^{(k)}$ and from the embedding inequality $\vec{W}_p^2(\Omega) \to \vec{W}_2^1(\Omega)$ with $p > 2N/(N+2)$

$$\|D\vec{V}^{(k-1)}\|_{S;\Omega} \leq \text{const} \cdot \|\vec{V}^{(k-1)}\|_{\vec{W}_p^2(\Omega)}$$

for finite number of iterations we obtain the theorem statement. Moreover, the first iterations $(k = 1)$ correspond to $\lambda^{(0)} = 0$ and $v^{(0)} = 0$.

Theorem 4.1 is proved.

5 Solvability theory

Let us consider the boundary value problem (4.1) under the conditions B.1)-B.4) and the following Lipschitz condition.

B.5) Let

$$\left|\vec{f}\,(x,\vec{u},\vec{\eta})-\vec{f}\left(x,\vec{u},\vec{\xi}\right)\right|\le\vec{b}_3\left(x,\vec{u},\vec{\xi},\vec{\eta}\right)\cdot|\vec{\eta}-\vec{\xi}|$$

for almost all $x\in\Omega$ and for all $\left(\vec{u},\vec{\xi},\vec{\eta}\right)\in R^n\times R^{nN}\times R^{nN}$, where the function $\vec{b}_3\left(x,\vec{u},\vec{\xi},\vec{\eta}\right)$ is measurable with respect to $x\in\Omega$ for all $\left(\vec{u},\vec{\xi},\vec{\eta}\right)\in R^n\times R^{nN}\times R^{nN}$, is continuous with respect to $\left(\vec{u},\vec{\xi},\vec{\eta}\right)$ almost for all $x\in\Omega$ and for any fixed $r\ge 0$

$$\vec{b}_{3,r}\,(x)\equiv\sup\left\{\vec{b}_3\left(\cdot,\vec{u},\vec{\xi},\vec{\eta}\right)\Big||\vec{u}|\le r,\ |\vec{\xi}|\le r,\ |\vec{\eta}|\le r\right\}$$

belong to $\vec{L}_q\,(\Omega)$ with $q>p$ and $p>2N/\,(N+2)$

Recall the known definitions.

Definition 5.1 *The function $\vec{u}^+\,(x)$ from $\vec{W}_p^2\,(\Omega)$ with $p>2N/\,(N+2)$ is said to be the upper solution of the problem* (4.1) *if*

$$\begin{cases}-\vec{L}\vec{u}^+ +\vec{f}\,(x,\vec{u}^+,D\vec{u}^+)\ge 0, & x\in\Omega,\\ \vec{u}^+\Big|_{\partial\Omega}\ge\vec{\varphi}\,(x)\,, & x\in\partial\Omega.\end{cases}$$

Definition 5.2 *The function $\vec{u}^-\,(x)$ from $\vec{W}_p^2\,(\Omega)$ with $p>2N/\,(N+2)$ is said to be the down solution of the problem* (4.1) *if*

$$\begin{cases}-\vec{L}\vec{u}^- +\vec{f}\,(x,\vec{u}^-,D\vec{u}^-)\le 0, & x\in\Omega,\\ \vec{u}^-\Big|_{\partial\Omega}\le\varphi\,(x)\,, & x\in\partial\Omega.\end{cases}$$

Theorem 5.1 *Let conditions B.1) - B.5) be satisfied with some $p>2N/\,(N+2)$ and the boundary function $\vec{\varphi}\,(x)$ belongs to the space $\vec{W}_p^{2-1/p}\,(\Omega)$. Let there exist upper $\vec{u}^+\,(x)$ and lower $\vec{u}^-\,(x)$ solutions to problem* (4.1) *of $\vec{W}_p^2\,(\Omega)$ such that $\vec{u}^+\,(x)\ge\vec{u}^-\,(x)$ in $\overline{\Omega}$. Then there exists a solution $\vec{u}\,(x)$ of $\vec{W}_p^2\,(\Omega)$ of problem* (4.1) *and*

$$\vec{u}^-\,(x)\le\vec{u}\,(x)\le\vec{u}^+\,(x)\,,\ x\in\overline{\Omega}.$$

The proof of this theorem is based on the following lemmas.

Lemma 5.1 *Let the real function $\vec{F}_0\left(x,\vec{u},\vec{\xi}\right)$ defined on $\Omega\times R^n\times R^{nN}$ satisfy the Caratheodori condition B.1) with $\vec{f}=\vec{F}_0$ and*

$$\sup_{\left(\vec{u},\vec{\xi}\right)\in\left(R^n\times R^{nN}\right)}\left|\vec{F}_0\left(\cdot,\vec{u},\vec{\xi}\right)\right|\in\vec{L}_p\,(\Omega)\,,\ \ p>2N/\,(N+2)\,. \tag{5.1}$$

Then the boundary value problem

$$\begin{cases}\vec{L}\vec{u}=\vec{F}_0\,(x,\vec{u},D\vec{u})\,, & x\in\Omega,\\ \vec{u}\Big|_{\partial\Omega}=\vec{\varphi}\,(x)\,, & x\in\partial\Omega,\end{cases}$$

where $\vec{\varphi}\,(x)\in\vec{W}_p^{2-1/p}\,(\partial\Omega)$ has the solution $\vec{u}\,(x)\in\vec{W}_p^2\,(\Omega)$.

Proof. We consider the boundary value problem

$$\begin{cases} \vec{L}_{\vec{v}}\vec{u} = \vec{F}_0\left(x, \vec{v}, D\vec{v}\right), & x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}\left(x\right), & x \in \partial\Omega \end{cases}$$

at an arbitrary function $\vec{v}\left(x\right) \in \vec{C}^1\left(\overline{\Omega}\right)$. Then by the well-known linear L_p -theory of elliptic problems, this boundary value problem for any function $\vec{v}\left(x\right)$ of $\vec{C}^1\left(\overline{\Omega}\right)$ has a solution $u\left(x\right)$ of $\vec{W}_p^2\left(\Omega\right)$, $\vec{u}\left(x\right) = \vec{T}\vec{v}\left(x\right)$, for which, by virtue of (5.1) the inequality

$$\|\vec{u}\|_{\vec{W}_p^2(\Omega)} \le C, \tag{5.2}$$

where the constant c is independent of $\vec{v}\left(x\right)$ of $\vec{C}^1\left(\overline{Q}\right)$. The operator $\vec{T} : \vec{C}^1\left(\overline{\Omega}\right) \to \vec{W}_p^2\left(\Omega\right)$ is continuous and as a result of compact embedding $\vec{W}_p^2\left(\Omega\right) \to \vec{C}^1\left(\overline{\Omega}\right)$ $\left(p > 2N/\left(N+2\right)\right)$ is a completely continuous operator from $\vec{C}^1\left(\Omega\right)$ to $\vec{C}^1\left(\overline{\Omega}\right)$. By virtue of estimation (5.2) there exists a sphere in the space $\vec{C}^1\left(\overline{\Omega}\right)$ that the operator , operator $\vec{T}$ takes into itself. Then by the Schauder‘s known theorem, the operator $\vec{T}$ has in the space $\vec{C}^1\left(\overline{\Omega}\right)$ a fixed point $\vec{u}\left(x\right)$ that by the definition of the operator $\vec{T}$ belongs to the space $\vec{W}_p^2\left(\Omega\right)$.

Lemma 5.1 is proved.

Let us now define for the function $\vec{u}\left(x\right)$ of $\vec{W}_p^2\left(\Omega\right)$ with $p > 2N/\left(N+2\right)$ the shear operator $\vec{\sigma}$ by the relation

$$\vec{\sigma}\vec{u}\left(x\right) = \begin{cases} \vec{u}^+\left(x\right) & \text{for } \vec{u}\left(x\right) > \vec{u}^+\left(x\right), \\ \vec{u}\left(x\right) & \text{for } \vec{u}^-\left(x\right) \le \vec{u}\left(x\right) \le \vec{u}^+\left(x\right), \\ \vec{u}^-\left(x\right) & \text{for } \vec{u}\left(x\right) < \vec{u}^-\left(x\right), \end{cases}$$

and consider the following boundary value problem

$$\begin{cases} \vec{L}\vec{u} = \vec{f}\left(x, \vec{\sigma}\vec{u}, D\vec{u}\right), & x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}\left(x\right), & x \in \partial\Omega. \end{cases} \tag{5.3}$$

Lemma 5.2 *Let the function $\vec{f}\left(x, \vec{\sigma}\vec{u}, D\vec{u}\right)$ satisfy conditions B.1) -B.5) with $p > 2N/\left(N+2\right)$ and $\vec{\varphi} \in \vec{W}_p^{2-1/p}\left(\Omega\right)$. Let $\vec{u}\left(x\right)$ of $\vec{W}_p^2\left(\Omega\right)$ be a solution to problem* (5.3). *Then*

$$\vec{u}^-\left(x\right) \le \vec{u}\left(x\right) \le \vec{u}^+\left(x\right), \quad x \in \overline{\Omega} \tag{5.4}$$

Proof. *For the function $\vec{\mathcal{W}} = \vec{u} - \vec{u}^+$ we have*

$$\vec{L}\vec{\mathcal{W}} \ge \vec{f}\left(x, \vec{\sigma}\vec{u}, D\vec{u}\right) - \vec{f}\left(x, \vec{u}^+, D\vec{u}^+\right), \quad x \in \Omega, \tag{5.5}$$

$$\vec{\mathcal{W}}\Big|_{\partial\Omega} \le 0, \quad x \in \partial\Omega. \tag{5.6}$$

We now assume the opposite statement of the lemma. Then the set $G = \left\{x \in \Omega \,\middle|\, \vec{\mathcal{W}}\left(x\right) > 0\right\}$ is not empty, and

$$\vec{L}\vec{\mathcal{W}} \ge \vec{f}\left(x, \vec{u}^+, D\vec{u}\right) - f\left(x, \vec{u}^+, D\vec{u}^+\right), \quad x \in \Omega.$$

Hence, due to condition B.5) we obtain

$$\vec{L}\vec{\mathcal{W}} \geq -\vec{b}_{3,r}(x) \cdot |D\vec{\mathcal{W}}|, \quad x \in \Omega,$$

where $\vec{b}_{3,r} \in \vec{L}_q(\Omega)$ with $q > p$ and $p > 2N/(N+2)$. Then from the results of A.D. Alexandrov's paper [2] it follows that $\vec{\mathcal{W}}(x)$ attains strong positive maximum on the boundary $\partial\Omega$. This contradicts the boundary condition (5.6). Thus, it is proved that $\vec{u}(x) \leq \vec{u}^+(x)$ in the domain Ω.
Inequality $\vec{u}^-(x) \leq \vec{u}(x)$ in the domain of Ω is proved similarly.

Lemma 5.2 is proved.

Assume the proof of Theorem 5.1. Let

$$M = \max_{\Omega}\left\{\|\vec{u}^+(x)\|_{\vec{W}_p^1(\Omega)}, -\|\vec{u}^-(x)\|_{\vec{W}_2^1(\Omega)}\right\}.$$

Then $\|\vec{u}\|_{\vec{W}_2^1(\Omega)} \leq M$ and by virtue of Theorem 4.1

$$\|\vec{u}\|_{\vec{W}_p^2(\Omega)} \leq \Phi\left(M, \|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\right).$$

By virtue of the Sobolev embedding theorem [22] we have

$$\|\vec{u}\|_{\vec{W}_2^1(\Omega)} \leq c_2 \cdot \|\vec{u}\|_{\vec{W}_p^2(\Omega)} \ \ p > 2N/(N+2), \tag{5.7}$$

where the positive constant c_2 is independent of the function $\vec{u}(x) \in \vec{W}_p^2(\Omega)$. Then we obtain

$$\|\vec{u}\|_{\vec{W}_p^2(\Omega)} \leq M_1,$$

where $M_1 = c_2 \cdot \Phi\left(M, \|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\right)$.

Now let's define the function

$$\vec{F}_1\left(x, \vec{u}, \vec{\xi}\right) = \begin{cases} \vec{f}\left(x, \vec{u}, \vec{\xi}\right) & \text{for } |\vec{\xi}| \leq M_2, \\ \vec{f}\left(x, \vec{u}, M_2, \frac{\vec{\xi}}{|\vec{\xi}|}\right) & \text{for } |\vec{\xi}| > M_2, \end{cases}$$

where

$$M_2 = \max_{\Omega}\left\{M_1, \|\vec{u}^+\|_{\vec{W}_2^1}, \|\vec{u}^-\|_{\vec{W}_2^1}\right\}.$$

This function satisfies condition B.1) -B.2) with the corresponding inequality

$$|\vec{F}_1\left(x, \vec{u}, \vec{\xi}\right)| \leq \begin{cases} b(x, \vec{u}) + b_1(x, \vec{u})\,|\vec{\xi}|^\mu & \text{for } |\vec{\xi}| \leq M_2, \\ b(x, \vec{u}) + b_1(x, \vec{u}) \cdot M_2^\mu & \text{for } |\vec{\xi}| > M_2. \end{cases}$$

The function $\vec{F}_1\left(x, \vec{u}, \vec{\xi}\right)$ also satisfies condition B.5) with the corresponding function b_3.

Let us now consider the boundary value problem

$$\begin{cases} \vec{L}\vec{u} = \vec{F}_1(x, \vec{\sigma}\vec{u}, D\vec{u}), & x \in \Omega, \\ \vec{u}\Big|_{\partial\Omega} = \vec{\varphi}(x), & x \in \partial\Omega. \end{cases} \tag{5.8}$$

The function

$$\vec{F}_2\left(x,\vec{u},\vec{\xi}\right)=\begin{cases}\vec{F}_1\left(x,\vec{u}^+(x),\vec{\xi}\right) & \text{for} \quad \vec{u}(x)>\vec{u}^+(x),\\ \vec{F}_1\left(x,\vec{u}(x),\vec{\xi}\right) & \text{for} \ \vec{u}^-(x)\le\vec{u}(x)\le\vec{u}^+(x),\\ \vec{F}\left(x,\vec{u}^-(x),\vec{\xi}\right) & \text{for} \quad \vec{u}(x)<\vec{u}^-(x),\end{cases}$$

satisfies the conditions of Lemma 5.2 and

$$\vec{F}_2\left(x,\vec{u}(x),D\vec{u}(x)\right)=\vec{F}_1\left(x,\vec{\sigma}\vec{u}(x),D\vec{u}(x)\right).$$

Hence, Lemma 5.2 applies to problem (5.8), by virtue of which there exists a solution $\vec{u}(x)$ of this problem from the space $\vec{W}_p^2(\Omega)$ with $p>2N/(N+2)$.

The function $\vec{u}^+(x)$ is the upper solution and $\vec{u}^-(x)$ is the lower solution to the problem

$$\begin{cases}\vec{L}\vec{u}=\vec{F}_1\left(x,\vec{u},D\vec{u}\right), \ \ x\in\Omega,\\ \vec{u}\Big|_{\partial\Omega}=\vec{\varphi}(x), \ x\in\partial\Omega.\end{cases} \tag{5.9}$$

From Lemma 5.1 we have

$$\vec{u}^-(x)\le\vec{u}(x)\le\vec{u}^+(x), \ \ x\in\overline{\Omega}$$

and hence $\vec{\sigma}\vec{u}(x)=\vec{u}(x)$, so that the solution $\vec{u}(x)$ of problem (5.8) is also a solution of problem (5.9).

Now applying the Theorem 4.1 to the solution $\vec{u}(x)$ of problem (5.9), we obtain

$$\|\vec{u}\|_{\vec{W}_p^2(\Omega)}\le\Phi\left(M,\|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\right).$$

Then it follows from the embedding inequality (5.7) that

$$\|\vec{u}\|_{\vec{W}_2^1(\Omega)}\le M_1,$$

where $M_1=c_2\cdot\Phi\left(M,\|\vec{\varphi}\|_{\vec{W}_p^{2-1/p}(\partial\Omega)}\right)$.

We finally obtain that the solution $\vec{u}(x)$ of problem (5.9) is the solution of problem (4.1).

Theorem 5.1 is proved.

By choosing functions from different “trial” sets of functions as upper and lower solutions, we can obtain various existence theorems for solutions of boundary value problems of the form (4.1).

Let us take as a “trial” set functions of the form

$$\vec{u}(x)=\vec{\lambda},$$

where $\vec{\lambda}$ is a real number vector.

Theorem 5.2 *Let conditions B.1)- B.5) with some $p>2N/(N+2)$ be fulfilled, and the function $\vec{\varphi}(x)$ belong to the space $\vec{W}_p^{2-1/p}(\partial\Omega)$. Let the functions $\vec{f}(x,\vec{u},D\vec{u})$ and $\vec{\varphi}(x)$ be such that there exist such two numbers $\vec{\lambda}^+$ and $\vec{\lambda}^-$, $\vec{\lambda}^+\ge\vec{\lambda}^-$ that*

$$\vec{f}\left(x,\vec{\lambda}^-,0\right)\le 0\le\vec{f}\left(x,\vec{\lambda}^+,0\right), \ \ x\in\Omega,$$
$$\vec{\lambda}^-\le\vec{\varphi}(x)\le\vec{\lambda}^+, \ x\in\partial\Omega.$$

Then there exists a solution $\vec{u}\,(x)$ of the boundary value problem (4.1) from the space $\vec{W}_p^2\,(\Omega)$ and

$$\vec{\lambda}^- \leq \vec{u}\,(x) \leq \vec{\lambda}^+,\ x \in \Omega.$$

To prove this theorem, it is enough to apply Theorem 5.1 $\vec{u}^+ = \vec{\lambda}^+$ and $\vec{u}^- = \vec{\lambda}^-$.

Let us now take as a "trial" set functions of the form

$$\vec{u}\,(x) = \vec{\lambda} \cdot \frac{|x|^2}{2},$$

where $\vec{\lambda}$ is a real number vector.

Theorem 5.3 *Let conditions B.1) - B.5) be satisfied with some* $p > 2N/\,(N+2)$ *and the boundary function* $\vec{\varphi}\,(x)$ *belongs to the space* $\vec{W}_p^{2-1/p}\,(\partial\Omega)$ *. Let the functions* $\vec{f}\,(x, \vec{u}, D\vec{u})$ *and* $\vec{\varphi}\,(x)$ *be such that there exist two numbers* $\vec{\lambda}^+$ *and* $\vec{\lambda}^-$*,* $\vec{\lambda}^+ \geq \vec{\lambda}^-$ *such that,*

$$\vec{f}\left(x, \vec{\lambda}^+ \cdot \tfrac{|x|^2}{2}, \vec{\lambda}^+\vec{x}\right) \geq \sum_{|\alpha|=2} \sum_{j=1}^{n} a_{\alpha j}^k \left(x, \vec{\lambda}^+ \cdot \tfrac{|x|^2}{2}, \vec{\lambda}^+\vec{x}\right) \vec{\lambda}^+,\ \ k = \overline{1,n},\ \ x \in \Omega,$$

$$\vec{f}\left(x, \vec{\lambda}^- \cdot \tfrac{|x|^2}{2}, \vec{\lambda}^-\vec{x}\right) \leq \sum_{|\alpha|=2} \sum_{j=1}^{n} a_{\alpha j}^k \left(x, \vec{\lambda}^- \cdot \tfrac{|x|^2}{2}, \vec{\lambda}^-\vec{x}\right) \vec{\lambda}^-,\ \ k = \overline{1,n},\ \ x \in \Omega,$$

$$\vec{\lambda}^- \cdot \tfrac{|x|^2}{2} \leq \vec{\varphi}\,(x) \leq \vec{\lambda}^+ \cdot \tfrac{|x|^2}{2},\ \ x \in \partial\Omega.$$

Then there exists a solution $\vec{u}\,(x)$ of the boundary value problem (5.1) from the space $\vec{W}_p^2\,(\Omega)$ and Then there exists the solution $\vec{u}\,(x)$ of the boundary value problem (4.1) from the space $\vec{W}_p^2\,(\Omega)$ and

$$\vec{\lambda}^- \cdot \frac{|x|^2}{2} \leq \vec{u}\,(x) \leq \vec{\lambda}^+ \cdot \frac{|x|^2}{2},\ \ x \in \Omega.$$

To prove this theorem, it suffices to apply Theorem 5.1 with $\vec{u}^+ = \vec{\lambda}^+ \cdot \frac{|x|^2}{2}$ and $\vec{u}^- = \vec{\lambda}^- \cdot \frac{|x|^2}{2}$.

Similarly, the theorem on solvability of the boundary value problem (4.1) is obtained if we take functions of the following form as a "trial" set of functions

$$\vec{u}\,(x) = \vec{\lambda}^+ \cdot \frac{|x - x_0|^2}{2},$$

where $\vec{\lambda}$ is a real number vector and $x, x_0 \in R^N$.

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