Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **46** (4), 1-19 (2026).

Schauder type estimates up to the boundary for m-th order elliptic equations in grand Sobolev spaces

Bilal T. Bilalov * · **Sabina R. Sadigova** · **Nijat M. Suleymanov** · **Vafa G. Alili**

Received: 02.02.2026 / Revised: 28.07.2026 / Accepted: 16.08.2026

Abstract. *It is considered an m-th order elliptic equation with continuous coefficients in the principal part. Let $L_{p)}(\Omega),\ 1 < p < +\infty$, be the grand Lebesgue space on a bounded domain $\Omega \subset R^n$ and let $W^m_{p)}(\Omega)$ be the corresponding grand Sobolev space. It has been established that if the boundary $\partial\Omega$ is sufficiently smooth, then, under certain conditions, a Schauder-type estimate holds for the coefficients of the elliptic operator under consideration up to the boundary. Such estimates play a key role in establishing the Fredholm property of boundary value problems (BVPs) for elliptic equations.*

Keywords. grand Sobolev space, elliptic equation, Schauder type estimates, BVP.

Mathematics Subject Classification (2010): 35A01, 35J05, 35K05

1 Introduction

There are various methods for studying the solvability of BVPs for elliptic equations and the main ones can be found, for example, in the monograph [1]. The most universal method is the establishment of so called Schauder-type estimates (both interior and up to the boundary) for the considered elliptic equation. These estimates originate from the classic work of S.N. Bernstein and were further developed by Y. Schauder. For higher-order elliptic equations, such estimates was established by S. Agmon, A.Douglis and L. Nirenberg [2]. To

* Corresponding author

This work is supported by the Azerbaijan Science Foundation-Grant No: AEF-BQM-Az-Türk-4-2025-6(59)-07/06/1-M-06 and Grant No: AEF-MGC-2024-2(50)-16/02/1-M-02.

B.T. Bilalov
Yildiz Technical University, Istanbul, Turkiye
Institute of Mathematics, The Ministry of Science and Education, Baku, Azerbaijan, Baku, Azerbaijan
Azerbaijan University of Architecture and Construction, Baku, Azerbaijan
E-mail: bilal.bilalov@yildiz.edu.tr, bilalov.bilal@gmail.com, b_bilalov@mail.ru

S.R. Sadigova
Institute of Mathematics, The Ministry of Science and Education of the Republic of Azerbaijan, Baku, Azerbaijan
E-mail: sadigova.sr@gmail.com; s_sadigova@mail.ru

N.M. Suleymanov
Odlar Yurdu University, Baku, Azerbaijan
E-mail: nijat.suleyman@gmail.com

V.G. Alili
Baku Slavic University, Baku, Azerbaijan
E-mail: alilivefa@mail.ru

solve the elliptic differential equation under consideration in other Banach function space (such as Morrey, Orlicz, grand Lebesgue spaces, etc.), it is necessary to establish Schauder-type estimates corresponding to these spaces. For Morrey spaces these problems have been studied by various authors (see, e.g., [3–7] and the references therein). For grand Lebesgue spaces Schauder-type estimates for second order elliptic equations were established in [8–12]. We also note the related works [13–18,21].

In this work we consider an m-th order elliptic equation with nonsmooth coefficients. Let $W^m_{p)}(\Omega)$, $1 < p < +\infty$, be the grand Sobolev space on a bounded domain $\Omega \subset R^n$, generated by the norm of the grand Lebesgue space $L_{p)}(\Omega)$. It is established that a Schauder-type estimate up to the boundary, corresponding to the BVP for this elliptic operator, holds in the space $W^m_{p)}(\Omega)$, provided that the boundary $\partial\Omega$ is sufficiently smooth. Such estimates play an essential role in establishing the Fredholmness of BVPs for elliptic equations.

2 Preliminaries

21 Notations

Assume the following notations, which will be used throughout this paper.

- Z_+ is the set of non-negative integers; C is the complex plane;
- $x \in R^n \Leftrightarrow x = (x_1; ...; x_n)$, $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $R^n_+ = \{x \in R^n : x_n > 0\}$;
- $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$, $B_r(0) \equiv B_r$, $B_r^+ = \{x \in B_r : x_n \geq 0\}$, $B'_r = \left\{x \in R^{n-1} : |x| < r\right\}$;
- $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $\Omega_r = \Omega_r(0)$;
- $mesM = |M|$ denotes the Lebesgue measure ($n-$dimentional) of a set $M \subset R^n$;
- $\partial\Omega$ is the boundary of the domain $\Omega \subset R^n$, $\overline{\Omega} = \Omega \bigcup \partial\Omega$;
- $\rho(x; M) = \inf\limits_{y \in M} |x - y|$, $d_\Omega = diam\Omega = \sup\limits_{x;y \in \Omega} |x - y|$, $f/_M$ denotes the restriction of f to M;
- $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω;
- $C^\infty\left(\overline{\Omega}\right)$ is the set of infinitely differentiable functions on $\overline{\Omega}$;
- $C^m(\Omega)$ is the set of m-times order continuously differentiable functions in Ω;
- $C_0^m(\Omega)$ is the set of m-times continuously differentiable functions with compact support in Ω;
- B-space denotes a Banach space;
- R_T is the range of the operator T, $Ker\,T$ is the kernel of the operator T;
- $[X; Y]$ denotes the Banach space of bounded linear operators acting from X to Y, $[X] = [X; X]$;
- $\alpha = (\alpha_1; ...; \alpha_n) \in Z^n_+$ is a multiindex,$\partial_i = \frac{\partial}{\partial x_i}$, $\partial^\alpha = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$; $\xi = (\xi_1; ...\xi_n) \in R^n \Rightarrow \xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$;
- S^{n-1} is the unit sphere in R^n, $d\sigma$ is an area element of S^{n-1};
- p' the conjugate to number p : $\frac{1}{p'} + \frac{1}{p} = 1$;
- $\delta(\cdot)$ is the Dirac delta.

Throughout the paper, by C and c we denote constants that may vary from line to line.

22 Grand Sobolev Space $W^m_{p)}(\Omega)$. Trace operator, Trace space

Let us define the grand Lebesgue space $L_{p)}(\Omega)$, $1 < p < +\infty$, on a bounded domain $\Omega \subset R^n$. This is the Banach space of measurable (in Lebesgue sense) functions f on Ω

equipped with the norm

$$\|f\|_{L_{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{\Omega} |f(x)|^{p-\varepsilon} \, dx \right)^{\frac{1}{p-\varepsilon}} .$$

The space $L_{p)}(\Omega)$ is non separable and the following continuous embeddings hold

$$L_p(\Omega) \subset L_{p)}(\Omega) \subset L_{p-\varepsilon_0}(\Omega),$$

for $\forall \varepsilon_0 \in (0, p-1)$.

Let $S \subset \bar{\Omega}$ be an $(n-1)$-dimensional differentiable surface. We define the grand Lebesgue space $L_{p)}(S)$ on the measure space $(S; d\sigma)$ (where $d\sigma$ denotes the area element of S) with the norm

$$\|f\|_{L_{p)}(S)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{S} |f|^{p-\varepsilon} \, d\sigma \right)^{\frac{1}{p-\varepsilon}} .$$

The norm of $L_{p)}(\Omega)$ generates the following grand Sobolev space

$$W_{p)}^{m}(\Omega) = \left\{ f \in L_{p)}(\Omega) \ : \ \partial^{\alpha} f \in L_{p)}(\Omega), \ \forall \alpha \in Z_+ : \ |\alpha| \le m \right\},$$

where $\partial^{\alpha} f$ denotes the generalized (in Sobolev sense) derivatives of f. Equipped with the norm

$$\|f\|_{W_{p)}^{m}(\Omega)} = \sum_{\alpha:|\alpha|\le m} \|\partial^{\alpha} f\|_{L_{p)}(\Omega)},$$

the space $W_{p)}^{m}(\Omega)$ is a nonseparable B-space.

Note that in [16] the authors consider the separable subspace $N_{p)}(\Omega)$ of $L_{p)}(\Omega)$, in which $C_0^{\infty}(\Omega)$ is dense. Based on this subspace, they introduce the corresponding grand Sobolev space $N_{p)}^{m}(\Omega)$ $\left(N_{p)}^{0}(\Omega) \equiv N_{p)}(\Omega)\right)$. In that work an extension theorem for functions from $N_{p)}^{m}(\Omega)$ is established. The proof of this theorem also applies to the general space $W_{p)}^{m}(\Omega)$. Therefore, it is valid the following theorem.

Theorem 2.1 *Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega \in C^{(m)}$. Then, for every domain $\Omega' \subset R^n : \Omega \subset\subset \Omega'$, there exists an operator $\theta \in \left[W_{p)}^{m}(\Omega) \ ; \ W_{p)}^{m}(\Omega')\right]$, such that : 1) $(\theta f)/_{\Omega} = f$, $\forall f \in W_{p)}^{m}(\Omega)$; 2) $\theta f = f/_{\Omega}$, $\forall f \in C^{\infty}\left(\overline{\Omega}\right)$.*

The definition of the class $C^{(m)}$ will be given in Subsection 3.3.

Let us define the trace operator $\Gamma_S \in \left[W_{p)}^{m}(\Omega) \ ; \ L_{p)}(S)\right]$, corresponding to the surface $S \subset \overline{\Omega} : S \in C^{(m)}$. Let $\Gamma_S^{(m)} \in [W_1{}^m(\Omega) \ ; \ L_1(S)]$ be a trace operator, corresponding to the classical Sobolev space $W_1{}^m(\Omega)$, where $L_p(S)$ is a Lebesgue space on measure space $(S; d\sigma)$ with a usual Lebesgue norm. Using the continuous embedding $W_{p)}^{m}(\Omega) \subset W_1{}^m(\Omega)$, accept $\Gamma_S = \Gamma_S^{(1)}/_{W_{p)}^{1}(\Omega)}$. We have

$$\|\Gamma_S f\|_{L_1(S;d\sigma)} \le c\, \|f\|_{W_1^m(\Omega)} \le c\, \|f\|_{W_{p)}^{m}(\Omega)}, \forall f \in W_{p)}^{m}(\Omega), \forall m \in N,$$

where c is a constant independent of f. Hence $\Gamma_S \in \left[W_{p)}^{m}(\Omega) \ ; \ L_1(S)\right]$. Define

$$W_{p)}^{m;0}(\Omega) = \left\{ u \in W_{p)}^{m}(\Omega) \ : \ \Gamma_S u = 0 \right\}.$$

It is true the following

Lemma 2.1 ([16]) *$W_{p)}^{m;0}(\Omega)$ is a subspace of $W_{p)}^{m}(\Omega)$, i.e. $W_{p)}^{m;0}(\Omega)$ is the closure in $W_{p)}^{m}(\Omega)$.*

Let $W_{p)}^{m}(S) = \Gamma_S\left(W_{p)}^{m}(\Omega)\right)$ and assume

$$\|\varphi\|_{W_{p)}^{m}(S)} = inf\{\|u\|_{W_{p)}^{m}(\Omega)} : u \in W_{p)}^{m}(\Omega) \& \Gamma_S u = \varphi\}, \forall \varphi \in W_{p)}^{m}(S). \tag{2.1}$$

Along with this, we also consider the factor space

$$\mathscr{F}_{p)}^{m}(S) = W_{p)}^{m}(\Omega)\,|_{W_{p)}^{m;0}(\Omega)},$$

equipped with the factor norm

$$\|F\|_{\mathscr{F}_{p)}^{m}(S)} = \inf_{f \in F} \|f\|_{W_{p)}^{m}(\Omega)}\ ,\ \forall F \in \mathscr{F}_{p)}^{m}(S)\ . \tag{2.2}$$

It is clear that $\mathscr{F}_{p)}^{m}(S)$ equipped with the norm (2.2) is a B-space. Let $F \in \mathscr{F}_{p)}^{m}(S)$ and $f_1; f_2 \in F$. Consequently, $f_1 - f_2 \in W_{p)}^{m;0}(\Omega) \Rightarrow \Gamma_S f_1 = \Gamma_S f_2$. Accept $\hat{\Gamma}_S F = \Gamma_S f\ ,\ f \in F$. From the above, it follows that this definition of $\hat{\Gamma}_S F$, is well-defined for every $F \in \mathscr{F}_{p)}^{m}(S)$. Moreover, it is evident that $\hat{\Gamma}_S : \mathscr{F}_{p)}^{m}(S) \to W_{p)}^{m}(S)$.

The following statement is true.

Statement 2.2 ([16]) *The space $W_{p)}^{m}(S)$ equipped with the norm* (2.1) *is a B-space. Moreover, the spaces $W_{p)}^{m}(S)$ and $\mathscr{F}_{p)}^{m}(S)$ are isometrically isomorphic, and $\hat{\Gamma}_S \in \left[\mathscr{F}_{p)}^{m}(S)\ ;\ W_{p)}^{m}(S)\right]$ realizes this isomorphism.*

In [16] it is also proved the following

Statement 2.3 ([16]) *Let $S \subset \bar{\Omega}\ :\ S \in C^{(m)}$ be some surface. Let $W_{p)}^{m}(S)$ be the corresponding trace space defined by* (2.1) *and let $\Gamma_S \in \left[W_{p)}^{m}(S)\ ;\ L_1(S)\right]$ is corresponding trace operator. Then $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ W_{p)}^{m}(S)\right]$ and the continuous embedding $W_{p)}^{m}(S) \subset L_{p)}(S)$ holds.*

We also introduce the trace space $N_{p)}^{m}(S) = \Gamma_S\left(N_{p)}^{m}(\Omega)\right)$. Since, $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ W_{p)}^{m}(S)\right]$ and $N_{p)}^{m}(\Omega)$ is a subspace of $W_{p)}^{m}(\Omega)$, then it is evident that $N_{p)}^{m}(S)$ is a subspace of $W_{p)}^{m}(S)$. Consequently, $\Gamma_S \in \left[N_{p)}^{m}(\Omega)\ ;\ N_{p)}^{m}(S)\right]$. Moreover, since $C^{\infty}\left(\bar{\Omega}\right)$ is dense in $N_{p)}^{m}(\Omega)$, then it is obvious that $\Gamma_S\left(C^{\infty}\left(\bar{\Omega}\right)\right)$ is also dense in $N_{p)}^{m}(S)$, i.e. for every $\varphi \in N_{p)}^{m}(S)$, there exists $\{u_n\} \subset C^{\infty}\left(\bar{\Omega}\right)$, such that $\Gamma_S u_n \to \varphi\,,\ n \to \infty$, in $W_{p)}^{m}(S)$. Thus it is valid the following

Corollary 2.1 *Let all conditions of Statement 2.3 hold and let $\Gamma_S \in \left[W_{p)}^{m}(\Omega)\ ;\ W_{p)}^{m}(S)\right]$ be the trace operator. Assume $N_{p)}^{m}(S) = \Gamma_S\left(N_{p)}^{m}(\Omega)\right)$. Then $N_{p)}^{m}(S)$ is a subspace of $W_{p)}^{m}(S)$ and $\Gamma_S\left(C^{\infty}\left(\bar{\Omega}\right)\right)$ is dense in $N_{p)}^{m}(S)$.*

Before moving on to the next section, let us define some function spaces.

By ν we denote outward normal to $\partial\Omega$. Let $\Gamma = \Gamma_{\partial\Omega} \in \left[W^m_{p)}(\Omega)\ ;\ L_{p)}(\partial\Omega)\right]$ be the trace operator and define

$$\Gamma_k u = \Gamma\left(\frac{\partial^k u}{\partial\nu^k}\right)\ ,\ k = \overline{0, m-1}.$$

Accept also $\hat{s}_l = (s_k)_1^l\ :\ 0 \le s_1 < ... < s_l \le m-1,\ s_k \in Z_+\ ,\ \forall\, k = \overline{1,l}$. Let

$$\left(L_{p)}(\partial\Omega)\right)^l = \underbrace{L_{p)}(\partial\Omega) \times ... \times L_{p)}(\partial\Omega)}_{l}.$$

According to $\hat{s}_l$ define

$$W^m_{p)}(\partial\Omega; \hat{s}_l) = \left\{(\varphi_k)_1^l \in L_{p)}(\partial\Omega)^l : \exists\, u \in W^m_{p)}(\Omega)\ \Rightarrow\ \Gamma_{s_k} u = \varphi_k\ ,\ k = \overline{1,l}\right\}\ .$$

We call $W^m_{p)}(\partial\Omega; \hat{s}_l)$ the trace space, corresponding to $\hat{s}_l$.

Analogously to the above, we define the separable trace space $N_{p)}(\partial\Omega;\ \hat{s}_l)$, corresponding to $\hat{s}_l$. Let

$$\left(N_{p)}(\partial\Omega)\right)^{\,l} = \underbrace{N_{p)}(\partial\Omega) \times ... \times N_{p)}(\partial\Omega)}_{l}.$$

We set

$$N^m_{p)}(\partial\Omega;\ \hat{s}_l) = \left\{(\varphi_k)_1^l \in \left(N_{p)}(\partial\Omega)\right)^l :\ \exists\, u \in N^m_{p)}(\Omega) \Rightarrow \Gamma_{s_k} u = \varphi_k,\ k = \overline{1,l}\right\}.$$

Along with this, we also introduce the trace spaces $W^m_{p)}(\partial\Omega;\ \{k\})\ ,\ \forall\, k = \overline{0, m-1}$. It is obvious that

$$W^m_{p)}(\partial\Omega;\ \hat{s}_l) = \bigcap_{k=1}^{l} W^m_{p)}(\partial\Omega;\ \{s_k\})\,, \tag{2.3}$$

and moreover, it holds

$$\left\|(\varphi_k)_1^l\right\|_{W^m_{p)}(\partial\Omega;\hat{s}_l)} \le l \max_{k=\overline{1,l}} \|\varphi_k\|_{W^m_{p)}(\partial\Omega;\{s_k\})}\,. \tag{2.4}$$

Analogously to the trace space $W^m_{p)}(\partial\Omega)$ case, for $\hat{s}_l = (s_k)_1^l$ assume

$$\left\|(\varphi_k)_1^l\right\|_{W^m_{p)}(\Omega';\hat{s}_l)} = \sum_{k=1}^{l} \|\varphi_k\|_{W^m_{p)}(\Omega';\{s_k\})}\,,$$

where $\Omega' \subset R^{n-1}$ is some bounded domain and

$$\|\varphi\|_{W^m_{p)}(\Omega';\{k\})} = \inf\left\{\|u\|_{W^{m-k}_{p)}(\Omega)} :\ \forall\, u \in W^m_{p)}(\Omega)\ \Rightarrow\ \Gamma u = \varphi\right\}.$$

According to $\hat{s}_l$, define the trace operator

$$\hat{\Gamma}_{\hat{s}_l} = (\Gamma_{s_k})_{\,1}^{\,l} :\ \ W^m_{p)}(\Omega) \to W^m_{p)}(\partial\Omega;\ \hat{s}_l)\,.$$

It is valid the following

Statement 2.4 ([16]) *Let $\Omega \subset R^n : \partial\Omega \in C^{(m)}$ be a bounded domain, $W^m_{p)}(\partial\Omega\,;\,\hat{s}_l)$ be the trace space defined by* (2.3) *and $\hat{\Gamma}_{\hat{s}_l}$ be a corresponding trace operator. Then it holds $\hat{\Gamma}_{\hat{s}_l} \in \left[W^m_{p)}(\Omega)\,;\,W^m_{p)}(\partial\Omega;\,\hat{s}_l)\right]$ and moreover, $\hat{\Gamma}_{\hat{s}_l}\left(C^\infty\left(\bar{\Omega}\right)\right)$ is dense in $N^m_{p)}(\partial\Omega\,;\,\hat{s}_l)$.*

Introduce for consideration the following subspace of $W^m_{p)}(\Omega'\,;\,\hat{s}_l)$. Denote by $\hat{N}^m_{p)}(\Omega'\,;\,\hat{s}_l)$ the closure of $\left(C_0^\infty(\Omega')\right)^l = \underbrace{C_0^\infty\left(\Omega'\right) \times ... \times C_0^\infty\left(\Omega'\right)}_{l}$ in $W^m_{p)}(\Omega'\,;\,\hat{s}_l)$. Since, $\left(C_0^\infty(\Omega')\right)^l \subset N^m_{p)}(\Omega';\,\hat{s}_l)$, then it is evident that $\hat{N}_p(\Omega'\,;\,\hat{s}_l)$ is also a subspace of $N^m_{p)}(\Omega'\,;\,\hat{s}_l)$.

We present a result from the monograph [4], which will be used to derive the main result.

Theorem 2.5 ([4]) *Let R be a sublinear operator and $R \in [L_p(\Omega)] \bigcap [L_{p-\varepsilon_0}(\Omega)]$, for some $p \in (1, +\infty)$ & $\varepsilon_0 \in (0, p-1)$. Then $R \in \left[L_{p)}(\Omega)\right]$.*

At the end of this subsection, we state a lemma from [9], which will also be needed in the subsequent discussion.

Lemma 2.2 ([9]) *Let $\Omega \subset R^n$ be a bounded domain with sufficiently smooth boundary. Then for $\forall\varepsilon > 0$, there exists a constant $c = c(\varepsilon) > 0$, such that*

$$\|\partial^\alpha u\|_{L_{p)}(\Omega)} \le \varepsilon \left\|\partial^\beta u\right\|_{L_{p)}(\Omega)} + c\,\|u\|_{L_{p)}(\Omega)}\,,$$

$$\forall\alpha;\beta \in Z^n_+ :\ \alpha < \beta,\ \ |\beta| \le m,\ \ \forall u \in W^m_{p)}(\Omega)\,,$$

where $\alpha < \beta$ means $\alpha = (\alpha_1, ..., \alpha_n) \le \beta = (\beta_1, ..., \beta_n) \Leftrightarrow \alpha_k \le \beta_k,\ \ \forall k = \overline{1,n}$, and $\alpha < \beta \Leftrightarrow \alpha \le \beta \& \alpha \ne \beta$.

23 Elliptic equations. Complementing Condition

Let L be a uniformly elliptic operator of order m of the form

$$L = L(x;\partial) = \sum_{|\alpha| \le m} a_\alpha(x)\ \partial^\alpha,$$

with real-valued coefficients $a_\alpha \in L_\infty(\Omega)$, i.e. $\exists\ \delta > 0$, such that

$$\delta\,|\xi|^m \le \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \le \delta^{-1}\,|\xi|^m\,,\ \ \forall\xi \in R^n,\ \text{a.e. } x \in \Omega.$$

Assign to L a “tangential operator”

$$L_{x_0} = \sum_{|\alpha|=m} a_\alpha(x_0)\ \partial^\alpha,$$

at point $x_0 \in \Omega$ and without loss of generality, we assume that the operator L_{x_0} is elliptic at the point x_0 (in what follows, this condition is assumed to hold). Denote by $J_{x_0}(\cdot)$ the fundamental solution of the equation $L_{x_0}\varphi = 0$ with constant coefficients $a_\alpha(x_0)$. The function $J_{x_0}(\cdot)$ serves as a parametrix for the equation $L\varphi = 0$ with a singularity at the point x_0. Let

$$\left(S_{x_0}\varphi\right)(x) = \psi(x) = \int J_{x_0}(x-y)\,\varphi(y)\,dy,$$

and assume

$$T_{x_0} = S_{x_0} \left(L_{x_0} - L \right).$$

When $x_0 = 0 \in \Omega$ the operators L_{x_0}, S_{x_0} and T_{x_0} will be denoted by L_0, S_0 and T_0, respectively.

Regarding the coefficients of the operator L, we assume the following conditions:

Condition (A). *i)* $a_\alpha \in C\left(\overline{\Omega}\right)$, $\forall\, \alpha \in Z_+^n$: $|\alpha| = m$; *ii)* $a_\alpha \in L_\infty\left(\Omega\right)$, $\forall\, \alpha \in Z_+^n$: $|\alpha| < m$.

Let us also give the definition of the complementing condition for a BVP associated with an elliptic equation. Let

$$L\left(x; \partial\right) = L'\left(x; \partial\right) + L''\left(x; \partial\right),$$

be an elliptic operator of m-th order ($m = 2l \,:\, l \in N$), where

$$L'\left(x; \partial\right) = \sum_{|\alpha|=m} a_\alpha\left(x\right)\, \partial^\alpha,$$

is its principal part and $L'\left(x; \xi\right)$, $\xi \in R^n$, denotes the corresponding characteristic form. Consider the following equation

$$Lu = f,$$

with boundary conditions

$$B_i u = \varphi_i\, , \; i = \overline{1, l}\, ,$$

where $f \in L_{p)}\left(\Omega\right)$ and $\varphi_i \in L_{p)}\left(\partial\Omega\right)$, $i = \overline{1, l}$; are given functions. Here B_i , $i = \overline{1, l}$; are boundary differential operators of the form

$$B_i u = B_i' u + B_i'' u, i = \overline{1, l},$$

where

$$B_i' u = \sum_{|\beta|=m_i} b_{\beta;i} u, i = \overline{1, l},$$

is a principal part of B_i of m_i-th order : $0 \le m_1 < ... < m_l \le m - 1$. We adopt the following complementing condition.

Condition (B). *Let n_ξ denote the outer normal at the point $\xi \in \partial\Omega$ and let $\nu_\xi \neq 0$ be a vector tangent to $\partial\Omega$ at point ξ. We require that the polynomials $B_i'\left(\xi; \nu_\xi + \tau\, n_\xi\right)$, $i = \overline{1, l}$; considered as polynomials in the variable τ are linearly independent modulo the polynomial*

$$\prod_{k=1}^{l} \left(\tau - \tau_k^+\left(\nu_\xi\right)\right),$$

where $\tau_k^+\left(\nu_\xi\right)\, \left(\tau_k^-\left(\nu_\xi\right)\right)$ are the roots of the polynomial $L'\left(\xi; \nu_\xi + \tau\, n_\xi\right)$ with the positive (negative) imaginary parts.

Accept

$$M^\pm\left(\nu; \tau\right) = \prod_{k=1}^{l} \left(\tau - \tau_k^\pm\left(\nu\right)\right) = \sum_{k=0}^{m} a_k^\pm\left(\nu\right) \tau^{m-k}.$$

In other words, let for a given $\xi \in R^{n-1} :\; \xi \neq 0$, the polynomial

$$B_j'\left(\xi; \tau\right) = \sum_{k=1}^{l} b_{jk}\left(\xi\right) \tau^{k-1} \equiv B_j\left(\xi; \tau\right) \left(mod M^+\right),$$

is a remainder obtained from dividing $B_j(\xi;\tau)$ by $M^+(\xi;\tau)$ (where B_j and M^+ as a polynomials regarding τ). Then the complementing condition means that

$$d(\xi) = \det\left(b_{jk}(\xi)\right)_{j;k=1}^{l} \neq 0, \;\; \forall \xi \in R^{n-1} : \xi \neq 0.$$

Denote

$$\varDelta = \min_{|\xi|=1} |d(\xi)|. \tag{2.5}$$

If the complementing condition holds, then $\varDelta > 0$.

24 On solvability of BVP for Elliptic Equation with Constant Coefficients

Consider the following elliptic operator with constant coefficients

$$L_0 u = \sum_{|\alpha|=m} a_\alpha^0 \partial^\alpha u,$$

and boundary operators

$$b_i^0 u = \sum_{|\alpha|=m_i} b_{\alpha;i}^0 \partial^\alpha u, i = \overline{1,l}\,; \;\; 0 \leq m_1 < ... < m_l \leq m-1\,,$$

with constant coefficients $\left\{b_{\alpha;i}^0\right\}$. Let $\varOmega' \subset R^{n-1}$ be some bounded domain and $T \in (0,+\infty)$ be some number. Assume $\varOmega = \varOmega' \times (0,T) \subset R^n$ and for every point $P = (x;t) \in \varOmega : \; x \in \varOmega' \,\&\, t \in (0,T)$, define the distance $|P| = \sqrt{|x|^2 + t^2}$, $|x| = \sqrt{x_1^2 + ... + x_{n-1}^2}$, where $x = (x_1, ..., x_{n-1})$. Let $(\varphi_i)_1^l \in \left(L_{p)}(\varOmega')\right)^l$ be given functions and consider the following BVP

$$\left.\begin{array}{l} L_0 u = 0\,, \\ b_i^0 u = \varphi_i\,, \;\; i = \overline{1,l}\,. \end{array}\right\} \tag{2.6}$$

By the solution of the BVP (2.6), we mean the function $u \in N_{p)}^m(\varOmega)$, for which $b_i^0 u = \varphi_i\,, \;\; i = \overline{1,l}\,$, and it holds $(L_0 u)(P) = 0$, a.e. $P \in \varOmega$.
In [16] it is proved the following

Theorem 2.6 ([16]) *Let $(\varphi_k)_1^l \subset W_{p)}^m(\varOmega'; \hat{s}_l)$, $1 < p < +\infty$, and assume that the system of operators $\left\{L_0;\, b_i^0,\, i = \overline{1,l}\right\}$ satisfies Condition (B). Let $\{K_i(\cdot;\cdot)\}_{i=\overline{1,l}}$ be the corresponding Poisson's kernels for the BVP* (2.6) *. Then the function*

$$u(x;t) = \sum_{i=1}^{l} \int_{\varOmega'} K_i(x-y;t)\, \varphi_i(y)\, dy = \sum_{i=1}^{l} (K_i * \varphi_i)\,(x;t)\,, \tag{2.7}$$

is a solution of the BVP (2.6) *in $W_{p)}^m(\varOmega)$.*

3 Main results

Let $\Omega' \subset R^{n-1}$ be a bounded domain with sufficiently smooth boundary $\partial\Omega'$. We will sometimes use the notation $P = (x'; x_n)$, where $x' \in \Omega'$ and $x_n \in (0, T)$. Let $\hat{s}_l = (s_k)_1^l \in Z_+^l : \quad 0 \le s_1 < s_2 < ... < s_l \le m-1$. Consider the following BVP

$$L(P; \partial) u(P) = f(P), \ \ P \in \Omega, \tag{3.1}$$

$$B_j(\xi; \partial) u(\xi) = \varphi_j(\xi), \ \ j = \overline{1, l}, \tag{3.2}$$

for an m-th order elliptic operator (where m is even & $l = \frac{m}{2}$)

$$L(\cdot; \partial) u(\cdot) = L'(\cdot; \partial) u(\cdot) + L''(\cdot; \partial) u(\cdot),$$

where

$$L'(\cdot; \partial) u(\cdot) = \sum_{|\beta|=m} a_\beta(\cdot) \partial^\beta u(\cdot); L''(\cdot; \partial) u(\cdot) = \sum_{|\beta| \le m-1} a_\beta(\cdot) \partial^\beta u(\cdot).$$

The boundary operators have the form

$$B_j(\cdot; \partial) u(\cdot) = B_j'(\cdot; \partial) u(\cdot) + B_j''(\cdot; \partial) u(\cdot),$$

where

$$B_j'(\cdot; \partial) u(\cdot) = \sum_{|\alpha|=s_j} b_{\alpha;j}(\cdot) \partial^\alpha u(\cdot) \ \ ; \ \ B_j''(\cdot; \partial) u(\cdot) = \sum_{|\alpha|<s_j} b_{\alpha;j}(\cdot) \partial^\alpha u(\cdot).$$

Assume that these operators satisfy the following conditions.

***(i) Condition on** L. The operator L is uniformly elliptic on $\bar{\Omega}$, i.e. $\exists \delta \in (0, 1]$ such that*

$$\delta \ |\xi|^m \le |L'(P; \xi)| \le \delta^{-1} |\xi|^m, \ \forall P \in \bar{\Omega}; \ \ \forall \xi \in R^n.$$

Moreover, in the case $n = 2$, assume that for every point $P = (\xi; 0) \in \Omega'$ with $\xi \in R$: $\xi \ne 0$, exactly half of the roots of the polynomial $L'(P; \xi, \tau)$ with $\tau \in C$ lie in the upper part of the complex plane.

***(ii) Complementing condition of boundary operators with respect to the operator** L. For every point $P^* = (x^*; 0) \in \Omega'$ the system with constant coefficients, consisting of the elliptic operator $L'(P^*; \partial)$ and the boundary operators $B_j'(P^*; \partial)$, $j = \overline{1, l}$, satisfies the complementing Condition (B) uniformly on $P^* \in \Omega'$. Specifically, if Δ_{P^*} is a constant from relation* (2.5) *then $\Delta_{P^*} \ge \Delta > 0$, for some fixed Δ.*

***(iii) Condition on coefficients.** Regarding the coefficients of the operators L and $(B_j)_1^l$, we will assume that the following conditions hold:*

α) $a_\beta \in C(\bar{\Omega})$ and $b_{\alpha,j} \in C(\Omega')$, for $\forall \beta : |\beta| = m \& \forall \alpha : |\alpha| = s_j, j = \overline{1, l}$;

$$\beta) \, a_\beta \in L_\infty(\Omega), \ \ \forall \beta : |\beta| < m; \ \& \, b_{\alpha,j} \in L_\infty(\Omega'), \forall \alpha : |\alpha| < s_j, \, j = \overline{1, l}.$$

***(iv) Condition on** $\partial\Omega$. We say that $\partial\Omega \in C^{(m)}$, if $\partial\Omega$ can be covered by a finite number (n-dimensional) of neighborhoods U_i such that each $\bar{U}_i \bigcap \bar{\Omega}$ can be bijectively mapped onto $\bar{B}_{r_i}^+$ in R^n using the mapping T_i, which, together with its inverse has bounded derivatives up to order m. With this mapping, $\bar{U}_i \bigcap \partial\Omega$ will be mapped onto the flat part of the boundary of $\partial B_{r_i}^+$. Moreover, regarding the equation and boundary conditions, we assume that the operator L is uniformly elliptic and that for each of the mapping T_i, the problem* (3.1), (3.2) *transforms into the problem* (3.17), (3.18) *for $B_{r_i}^+$.*

To establish a Schauder-type estimate for the BVP (3.1), (3.2), we divide it into three parts.

31 The constant coefficient case

Consider the following elliptic operator

$$L_0 u = \sum_{|\alpha|=m} a^0_\alpha \partial^\alpha u,$$

and boundary conditions

$$b^0_i u = \sum_{|\alpha|=s_i} b^0_{\alpha;i} \partial^\alpha u, \ i = \overline{1,l}; \quad 0 \le s_1 < ... < s_l \le m-1,$$

with constant coefficients. The following theorem is true.

Theorem 3.1 *Let the system of operators* $\left\{ L_0; b^0_j, \, j = \overline{1,l} \right\}$ *satisfy the ellipticity and complementing conditions (i) and (ii). Let* $f \in L_{p)}(B^+_r)$ *and* $(\varphi_k)^l_1 \in W^m_{p)}(B'_r; \hat{s}_l)$, $1 < p < +\infty$. *Then the BVP*

$$\left. \begin{array}{l} (L_0 u)(P) = f(P), \quad P \in B^+_r, \\ \left(b^0_j u\right)(\xi) = \varphi_j(\xi), \ j = \overline{1,l}, \, \xi \in B'_r, \\ \operatorname{supp} u \subset B^+_r, \end{array} \right\} \tag{3.3}$$

has a unique solution $u \in W^m_{p)}(B^+_r)$ *and the following estimate holds for the solution*

$$\|u\|_{W^m_{p)}(B^+_r)} \le C \left(\|f\|_{L_{p)}(B^+_r)} + \left\| (\varphi_j)^l_1 \right\|_{W^m_{p)}(B'_r;\hat{s}_l)} \right), \tag{3.4}$$

where the constant $C > 0$ *is independent of the right-hand side of* (3.3).

Proof. First, let's prove the uniqueness of the solution of the BVP (3.3). Let $\varepsilon_0 \in (0, p-1)$ be any fixed number. It is evident that $f \in L_{p-\varepsilon_0}(B^+_r)$ and $(\varphi_j)^l_1 \in W^m_{p-\varepsilon_0}(B'_r; \hat{s}_l)$, where $W^m_{p-\varepsilon_0}(B'_r; \hat{s}_l)$ is the trace space corresponding to the surface $S = B'_r \subset \partial B^+_r$ with respect to the Sobolev space $W^m_{p-\varepsilon_0}(B^+_r)$. From the continuous embedding $W^m_{p)}(B^+_r) \subset W^m_{p-\varepsilon_0}(B^+_r)$ it follows that u is also a solution of the BVP (3.3) in the space $W^m_{p-\varepsilon_0}(B^+_r)$. Since, supp $u \subset B^+_r$ then it is obvious that the function

$$\tilde{u}(x) = \begin{cases} u(x), & x \in B^+_r, \\ 0, & R^+_n \backslash B^+_r, \end{cases}$$

is a solution of the following BVP

$$\left. \begin{array}{l} (L_0 \tilde{u})(P) = \tilde{f}(P), \quad P \in R^n_+, \\ \left(b^0_j \tilde{u}\right)(\xi) = \tilde{\varphi}_j(\xi), \ \xi \in R^{n-1}, \ j = \overline{1,l}, \\ \operatorname{supp} \tilde{u} \ \ is \ compact \ in \ R^n_+, \end{array} \right\} \tag{3.5}$$

in $W^m_{p-\varepsilon_0}\left(R^n_+\right)$, where

$$\tilde{f}(P) = \begin{cases} f(P), & P \in B^+_r, \\ 0, & P \in R^+_n \backslash B^+_r, \end{cases} \quad \tilde{\varphi}_j(\xi) = \begin{cases} \varphi_j(\xi), & \xi \in B'_r, \\ 0, & \xi \in R^{n-1} \backslash B'_r, \, j = \overline{1,l}. \end{cases}$$

From Theorem 14.1' of the work [2] (see, p.127) it follows that the BVP (3.5) has a unique solution. In fact, let $\tilde{f} \equiv \tilde{\varphi} \equiv 0$. Then, from this theorem, it follows that $\partial^\alpha \tilde{u} \equiv 0, \ \forall \alpha \in$

$Z_+^n : |\alpha| = m$. Consequently, $\tilde{u}$ is a polynomial of order $\leq m-1$. Since, $\tilde{u}$ has compact support, it is evident that $\tilde{u} \equiv 0$. Thus, the uniqueness of the solution of the BVP (3.3) is proved.

Let's prove the validity of the estimate (3.4) for the solution. Denote by $J_0(\cdot)$ the fundamental solution of the equation $L_0 u = 0$ and assume

$$\vartheta(x) = \int_{R^n} J_0(x-y)\,\tilde{f}(y)\,dy\,.$$

As established in works [8,9] the function $\vartheta(\cdot)$ belongs to the space $W_{p)}^m(B_r^+)$ and for the m-th order derivatives of $\vartheta(\cdot)$ the following relation holds

$$\partial^\alpha \vartheta(x) = \int_{R^n} \partial_x^\alpha J_0(x-y)\,\tilde{f}(y)\,dy + C_\alpha f(x)\,, \quad \text{a.e. } x \in B_r^+, \quad \forall \alpha \in Z_+^n : |\alpha| = m, \tag{3.6}$$

where C_α is a constant independent of f. Since, $\partial^\alpha J_0(\cdot)$ for all $\alpha \in Z_+^n : |\alpha| = m$, is a Calderon-Zygmund kernel (see, e.g., [1]) and a singular integral operator with a Calderon-Zygmund kernel is bounded in $L_{p)}(B_r^+)\,,\ 1 < p < +\infty$ (see, e.g., the monographs [19,20]), it follows from the expression (3.6) that $\vartheta \in W_{p)}^m(B_r^+)$. Assume $\psi_j(\xi) = \left(b_j^0 \vartheta\right)(\xi)\,,\ \ \xi \in R_{n-1}, j = \overline{1,l}$. We have

$$L_0 \vartheta = \tilde{f} \Rightarrow (L\vartheta)(x) = f(x)\,,\ \ a.e.\ \ x \in B_r^+,$$

$$\left(b_j^0 \vartheta\right)(\xi) = \psi_j(\xi)\,,\ \xi \in B_r',\ \ j = \overline{1,l}.$$

Consider the function $w(x) = \vartheta(x) - u(x)\,,\ x \in B_r^+$. Then, regarding the function $w(\cdot)$ we obtain

$$\left.\begin{array}{l} (Lw)(x) = 0,\ \ a.e.\ \ x \in B_r^+, \\ \left(b_j^0 \vartheta\right)(\xi) = \psi_j(\xi) - \varphi_j(\xi)\,,\ \xi \in B_r',\ \ j = \overline{1,l}. \end{array}\right\} \tag{3.7}$$

Let $K_j(x;t)\,,\ \ j = \overline{1,l}$; be the Poisson kernels corresponding to the BVP (3.7) (see, the monograph [2], § 2, p.25). Then, by Theorem 3.1 of the work [16], the solution of the BVP (3.7) in $W_{p)}^m(B_r^+)$ has the form

$$w(x;t) = \sum_{j=1}^{l} \int_{B_r'} K_j(x-\xi;t)\left(\psi_j(\xi) - \varphi_j(\xi)\right) d\xi,$$

and consequently

$$u(x) = \vartheta(x) + \sum_{j=1}^{l} \int_{B_r'} K_j(x-\xi;t)\left(\psi_j(\xi) - \varphi_j(\xi)\right) d\xi.$$

From here it directly follows that

$$\partial^\alpha u(x) = \partial^\alpha \vartheta(x) + \sum_{j=1}^{l} \int_{B_r'} \partial_x^\alpha K_j(x-\xi;t)\left(\psi_j(\xi) - \varphi_j(\xi)\right) d\xi,\ \ x \in B_r^+, \tag{3.8}$$

$$\forall \alpha \in Z_+^n :\ \ |\alpha| = m.$$

Since, $\partial^{\alpha} J_0(\cdot)$ for all $\alpha \in Z_+^n : \ |\alpha| = m$, is a Calderon-Zygmund kernel, then from (3.6) we obtain

$$\|\partial^{\alpha}\vartheta\|_{L_{p)}(B_r^+)} \leq C \|f\|_{L_{p)}(B_r^+)},$$

where $C > 0$ is a constant independent of f. Applying Theorem 3.1 of the work [16] (estimate (3.16)) to the integral part of the expression (3.8), we obtain

$$\|\partial^{\alpha}u\|_{L_{p)}(B_r^+)} \leq C \left(\|f\|_{L_{p)}(B_r^+)} + \sum_{j=1}^{l} \|\psi_j - \varphi_j\|_{W_{p)}^m(B_r';\{s_j\})} \right)$$

$$\leq C \left(\|f\|_{L_{p)}(B_r^+)} + \sum_{j=1}^{l} \|\varphi_j\|_{W_{p)}^m(B_r';\{s_j\})} \right) + C \sum_{j=1}^{l} \|\psi_j\|_{W_{p)}^m(B_r';\{s_j\})}. \tag{3.9}$$

Let's estimate the last term on the right-hand side of this inequality. We have

$$\|\psi_j\|_{W_{p)}^m(B_r';\{s_j\})} = \left\|b_j^0\vartheta\right\|_{W_{p)}^m(B_r';\{s_j\})} \leq C \sum_{|\alpha|=s_j} \|\partial^{\alpha}\vartheta\|_{W_{p)}^m(B_r';\{s_j\})}$$

$$\leq C \sum_{|\alpha|=s_j} \|\partial^{\alpha}\vartheta\|_{W_{p)}^{m-s_j}(B_r^+)} \leq C \|\vartheta\|_{W_{p)}^m(B_r^+)} \leq C \|f\|_{L_{p)}(B_r^+)},$$

where $C > 0$ is a constant independent of f. Taking this estimate into account in (3.9) we conclude the proof.

Theorem is proved.

32 Semisphere domain case

Using the above results, let's establish an a priori estimate for the solution of the BVP regarding the semisphere. We follow the scheme from the work [1]. Consider the BVP (3.1), (3.2) in the case when $\Omega = B_r^+$ and $\operatorname{supp} u \subset B_r^+$, for some $r > 0$. Denote by $L'(0;\partial)\,u(P)$ and $B_j'(0;\partial)\,u(x';0)$, $\ j = \overline{1,l}$; the following operators

$$L'(0;\partial)\,u(P) = \textstyle\sum_{|\beta|=m} a_{\beta}(0)\,\partial^{\beta}u(P),\ P \in B_r^+,$$

$$B_j'(0;\partial)\,u(x';0) = \textstyle\sum_{|\alpha|=s_j} b_{\alpha;j}(0)\,\partial^{\alpha}u(x';0),\ j = \overline{1,l};$$

$$(x';0) \in B_r' = \left\{x' \in R^{n-1} : |x'| \leq r\right\}.$$

Now, represent the BVP (3.1), (3.2) in the form

$$L'(0;\partial)\,u(P) = f(P) + \left(L'(0;\partial) - L'(P;\partial)\right)\,u(P)$$
$$-L''(P;\partial)\,u(P) \equiv F(P),\ P \in B_r^+; \tag{3.10}$$

and

$$B_j'(0;\partial)\,u\left(x';0\right) = \varphi_j\left(x'\right) + \left(B_j'(0;\partial) - B_j'\left(x';\partial\right)\right)u\left(x';0\right)$$
$$-B_j''\left(x';\partial\right)u\left(x';0\right) \equiv \Phi_j\left(x'\right),\ \ j = \overline{1,l};\ \ x' \in B_r'. \tag{3.11}$$

Let's estimate the $\|F\|_{L_{p)}(B_r^+)}$ and $\|\Phi_j\|_{W_{p)}^m(B_r';\{s_j\})}$. We have

$$\left\|\left(L'(0;\partial) - L'(P;\partial)\right)u(P)\right\|_{L_{p)}(B_r^+)}$$

$$\leq \max_{|\beta|=m} \|a_\beta(0) - a_\beta(P)\|_{L_\infty(B_r^+)} \|u\|_{W_{p)}^m(B_r^+)} = o(r) \|u\|_{W_{p)}^m(B_r^+)};$$

$$\left\|L''(P;\partial) u(P)\right\|_{L_{p)}(B_r^+)} \leq C \|u\|_{W_{p)}^{m-1}(B_r^+)}, \tag{3.12}$$

where the constant $C > 0$ is independent of u.

Also, taking into account the definition of the norm in $W_{p)}^m(B_r'; \hat{s}_l)$ for the boundary conditions, we obtain

$$\left\|(B_j'(0;\partial) - B_j'(x';\partial))u(x';0)\right\|_{W_{p)}^m(B_r';\{s_j\})}$$

$$\leq \max_{|\alpha|=s_j} \left\|b_{\alpha;j}(0) - b_{\alpha;j}(x')\right\|_{L_\infty(B_r')} \left\|\partial^\alpha u(x';0)\right\|_{W_{p)}^m(B_r';\{s_j\})}$$

$$\leq o(r) \|\partial^{s_j} u(x)\|_{W_{p)}^{m-s_j}(B_r^+)} \leq o(r) \|u\|_{W_{p)}^m(B_r^+)};$$

$$\begin{aligned} &\left\|B_j''(x';\partial) u(x';0)\right\|_{W_{p)}^m(B_r';\{s_j\})} \leq C \sum\nolimits_{|\alpha|<s_j} \|\partial^\alpha u(x';0)\|_{W_{p)}^m(B_r';\{s_j\})} \\ &\leq C \sum\nolimits_{|\alpha|<s_j} \|\partial^\alpha u(x)\|_{W_{p)}^{m-s_j}(B_r^+)} \leq C \|u\|_{W_{p)}^{m-1}(B_r^+)}, \end{aligned} \tag{3.13}$$

where the constant $C > 0$ is independent of u.

Now, let's apply Theorem 3.1 to the BVP (3.10), (3.11). By this theorem, we have that there exists a constant $C > 0$, such that

$$\|u\|_{W_{p)}^m(B_r^+)} \leq C \left(\|F\|_{L_{p)}(B_r^+)} + \left\|(\Phi_j)_1^l\right\|_{W_{p)}^m(B_r';\hat{s}_l)} \right). \tag{3.14}$$

Taking into account the expressions (3.10), (3.11) and the estimates (3.12), (3.13) for $F; (\Phi_j)_1^\nu$, we obtain

$$\|F\|_{L_{p)}(B_r^+)} \leq \|f\|_{L_{p)}(B_r^+)} + \left\|\left(L'(0;\partial) - L'(P;\partial)\right) u(P)\right\|_{L_{p)}(B_r^+)}$$

$$+ \left\|L''(P,\partial) u(P)\right\|_{L_{p)}(B_r^+)} \leq \|f\|_{L_{p)}(B_r^+)} + o(r) \|u\|_{W_{p)}^m(B_r^+)} + C \|u\|_{W_{p)}^{m-1}(B_r^+)},$$

$$\begin{aligned} &\left\|(\Phi_j)_1^l\right\|_{W_{p)}^m(B_r';\hat{s}_l)} \leq \left\|(\varphi_j)_1^l\right\|_{W_{p)}^m(B_r';\hat{s}_l)} \\ &+ \left\| \left(\left(B_j'(0;\partial) - B_j'(x';\partial) \right) u(x';0) \right)_1^l \right\|_{W_{p)}^m(B_r';\hat{s}_l)} \\ &+ \left\| \left(B_j''(x';\partial) \right) u(x';0)_1^l \right\|_{W_{p)}^m(B_r';\hat{s}_l)} \leq \left\|(\varphi_j)_1^l\right\|_{W_{p)}^m(B_r';\hat{s}_l)} \\ &+ o(r) \|u\|_{W_{p)}^m(B_r^+)} + C \|u\|_{W_{p)}^{m-1}(B_r^+)}. \end{aligned}$$

As a result, from (3.14) it follows

$$\|u\|_{W_{p)}^m(B_r^+)} \leq C \left(\|f\|_{L_{p)}(B_r^+)} + \left\|(\varphi_j)_1^l\right\|_{W_{p)}^m(B_r';\hat{s}_l)} \right)$$

$$+ o(r) \|u\|_{W_{p)}^m(B_r^+)} + C \|u\|_{W_{p)}^{m-1}(B_r^+)}. \tag{3.15}$$

Let $\varepsilon > 0$ be an arbitrary number. By Lemma 2.1, there exists a constant $C = C_\varepsilon > 0$, such that

$$\|u\|_{W_{p)}^{m-1}(B_r^+)} \leq \varepsilon \|u\|_{W_{p)}^m(B_r^+)} + C \|u\|_{L_{p)}(B_r^+)}.$$

Let $r_0 > 0$ and $\varepsilon > 0$ be sufficiently small such that the following inequality holds

$$o(r) + \varepsilon \leq \frac{1}{2}, \quad \forall r \in (0, r_0).$$

Taking these relations into account in (3.15) we obtain

$$\|u\|_{W^m_{p)}(B^+_r)} \leq C \left(\|f\|_{L_{p)}(B^+_r)} + \left\| (\varphi_j)^l_1 \right\|_{W^m_{p)}(B'_r;\hat{s}_l)} + \|u\|_{L_{p)}(B^+_r)} \right). \tag{3.16}$$

Thus, the following theorem is true.

Theorem 3.2 *Let the system of operators* $\left\{ L(\cdot;\partial); B_j(\cdot;\partial), \quad j = \overline{1,l} \right\}$ *satisfy the conditions (i)- (iii) regarding the semisphere* B^+_r. *Then,* $\exists r_0; \ C > 0$, *such that for the solution* $u(\cdot) = W^m_{p)}(B^+_r)$ *:* $\operatorname{supp} u \subset B^+_r$, *of the following BVP*

$$L(P;\partial) u(P) = f(P), \quad P \in B^+_r, \tag{3.17}$$

$$B_j(\xi;\partial) u(\xi) = \varphi_j(\xi), \quad \xi \in B'_r, \quad j = \overline{1,l}, \tag{3.18}$$

it holds the estimate

$$\|u\|_{W^m_{p)}(B^+_r)} \leq C \left(\|f\|_{L_{p)}(B^+_r)} + \left\| (\varphi_j)^l_1 \right\|_{W^m_{p)}(B'_r;\hat{s}_l)} + \|u\|_{L_{p)}(B^+_r)} \right),$$

$$\textit{for } \forall r \in (0, r_0).$$

33 General Case

Now, consider the general case of a bounded domain $\Omega \subset R^n$. Before moving on to the next presentation, let's prove the following lemma, which is essential for proving the main theorem.

Lemma 3.1 *Let the function* ψ *be defined on* $\partial\Omega$ *and assume that* $\operatorname{supp}\psi \subset \gamma \subset \partial\Omega$. *Suppose there exists a neighborhood (n-dimensional)* U *of* γ *such that we can bijectively map the set* $\bar{U} \bigcap \bar{\Omega}$ *onto the closure of the semisphere* $\overline{B^+_r}$ *using a mapping* T, *so that* $T(\tau) = B'_r$, *where* $\tau = \bar{U} \bigcap \partial\Omega$. *Moreover,* T *and its inverse* T^{-1} *are continuously differentiable up to order* m. *Let* $\varphi(x) = \psi\left(T^{-1}(x';0)\right), \ x' \in B'_r$. *Then, there exists a constant* $\delta > 0$, *independent of* ψ *(but depending on the distance* $\rho(\gamma;\partial U)$ *) such that*

$$\delta \|\psi\|_{W^m_{p)}(\tau;\{j\})} \leq \|\varphi\|_{W^m_{p)}(B'_r;\{j\})} \leq \delta^{-1} \|\psi\|_{W^m_{p)}(\tau;\{j\})}, \quad \forall j = \overline{0, m-1}.$$

Proof. The proof is carried out in exactly the same way as in the work [2] (see, p. 126, Lemma 14.2). Let $\Gamma_\tau \in \left[W^m_{p)}(U); W^m_{p)}(\tau; d\sigma) \right]$ and $\Gamma_{B'_r} \in \left[W^m_{p)}(B^+_r); W^m_{p)}(B'_r; d\sigma) \right]$ be trace operators. For $\varphi \in W^m_{p)}(B'_r;\{j\})$ take $\vartheta \in W^m_{p)}(B^+_r): \ \Gamma_{B'_r}\vartheta = \varphi$, such that

$$\|\vartheta\|_{W^m_{p)}(B^+_r)} \leq 2 \|\varphi\|_{W^m_{p)}(B'_r;\{j\})}.$$

Let $\omega \in C^\infty(B^+_r) : \omega/_{T(\gamma)} \equiv 1$ and $\operatorname{supp}\omega \subset B^+_r$. Set $u = T(\omega\vartheta)$. It is evident that $\operatorname{supp} u \subset B^+_r$ and $(\Gamma_\tau\vartheta)/_\gamma \equiv \psi$. By the assumption on T, we have

$$\|u\|_{W^k_{p)}(U)} \leq C \|\vartheta\|_{W^k_{p)}(B^+_r)}, \quad \forall k = \overline{0, m-1},$$

where $C > 0$ is independent of $\vartheta; u$ (but depends on $\rho(\gamma; \partial U)$). We have

$$\|\psi\|_{W^m_{p)}(\tau;\{j\})} \leq \|u\|_{W^{m-j}_{p)}(U)} \leq C\,\|\vartheta\|_{W^{m-j}_{p)}\left(B^+_r\right)} \leq C\,\|\varphi\|_{W^m_{p)}(B'_r;\{j\})}.$$

By the same reasoning, it is proved that there exists $\exists C > 0$ such that

$$\|\varphi\|_{W^m_{p)}(B'_r;\{j\})} \leq C\,\|\psi\|_{W^m_{p)}(\tau;\{j\})}.$$

Lemma is proved.

Now, let's prove the following main theorem.

Theorem 3.3 *Let $\Omega \subset R^n$ be a bounded domain with boundary $\partial\Omega \in C^{(m)}$ and let the elliptic operator $L(P;\partial)$ and the boundary operators $B_j(P;\partial)$, $j = \overline{1,l}$; satisfy the conditions (i)-(iv). Then there exists a constant $C > 0$ such that*

$$\|u\|_{W^m_{p)}(\Omega)} \leq C\left(\|Lu\|_{L_{p)}(\Omega)} + \left\|(B_k u)_1^l\right\|_{W^m_{p)}(\Omega';\hat{s}_l)} + \|u\|_{L_{p)}(\Omega)}\right),\ \forall u \in W^m_{p)}(\Omega).$$

Proof. Taking into account condition (iv) on the boundary $\partial\Omega$, we assume that $\sum_{k=1}^l \omega_k \equiv 1$ is a partition of unity in $\bar{\Omega}$, where the functions $\{\omega_k\}_1^l$ have the following properties. Each ω_k, $k = \overline{1,l}$, is infinitely differentiable and if $\operatorname{supp}\omega_k \not\subset \Omega$, then $\operatorname{supp}\omega_k \subset U_{i(k)}$ for some neighborhood from (iv). Moreover, we assume that $T_{i(k)}(\operatorname{supp}\omega_k) \subset B^+_{r_0}$ for $r_0 > 0$ from Theorem 3.2.

Now, consider the case when $\operatorname{supp}\omega_k \not\subset \Omega$. By assumption $\operatorname{supp}\omega \subset B^+_{r_0}$, where $\omega = T_{i(k)} \circ \omega_k$. Set $\vartheta = T_{i(k)}u$. From the conditions on $T_{i(k)}$, it directly follows that $\exists\delta > 0$:

$$\delta\,\|\omega\vartheta\|_{W^k_{p)}\left(B^+_{r_0}\right)} \leq \|\omega_k u\|_{W^k_{p)}\left(U_{i(k)}\right)} \leq \delta^{-1}\,\|\omega u\|_{W^k_{p)}\left(B^+_{r_0}\right)},\ \forall k = \overline{1,m}. \tag{3.19}$$

Let's denote by $\tilde{L}$ and $\tilde{B}_j$, $j = \overline{1,l}$; the images of the operators L and B_j, $j = \overline{1,l}$, under the mapping $T_{i(k)}$, respectively . Then, according to Theorem 3.2, we have

$$\|\omega\vartheta\|_{W^m_{p)}\left(B^+_{r_0}\right)} \leq C\left(\left\|\tilde{L}(\omega\vartheta)\right\|_{L_{p)}\left(B^+_{r_0}\right)} + \left\|\tilde{B}_j(\omega\vartheta)_1^l\right\|_{W^m_{p)}\left(B'_{r_0};\hat{s}_l\right)} + \|\omega\vartheta\|_{L_{p)}\left(B^+_{r_0}\right)}\right). \tag{3.20}$$

Taking into account the expression of the operator L, we obtain

$$L(u) = \sum_{|\alpha|\leq m} a_\alpha(x)\,\partial_x^\alpha(u(x)) \Big/_{x=\theta(y)}$$

$$= \sum_{|\alpha|\leq m} a_\alpha(\theta(y))\,\partial_y^\alpha(u(\theta(y))) = \sum_{|\alpha|\leq m} \tilde{a}_\alpha(y)\,\partial_y^\alpha\vartheta(y),$$

where $\tilde{a}_\alpha(y) \equiv a_\alpha(\theta(y))$, for $\forall\alpha : |\alpha| = m$. The other coefficients $\tilde{a}_\alpha(\cdot)$, for $\forall\alpha : |\alpha| < m$, are expressed through the coefficients $a_\alpha(\theta(\cdot))$ and it holds that $\tilde{a}_\alpha \in L_\infty\left(B^+_{r_0}\right)$.

Consequently, for $q \geq 1$ it is true

$$\int_{U_{i(k)}} |Lu(x)|^q\,dx = |x = \theta(y)| = \int_{B^+_{r_0}} \left|\tilde{L}\vartheta(y)\right|^q \left|\det\theta'(y)\right| dy,$$

where $|\det \theta'(y)|$ is the Jacobian of $\theta(\cdot)$. From here, it directly follows that $\exists\delta>0$:

$$\delta \|Lu\|_{L_{p)}\left(U_{i(k)}\right)} \leq \left\|\tilde{L}\vartheta\right\|_{L_{p)}\left(B_{r_0}^{+}\right)} \leq \delta^{-1}\|Lu\|_{W_{p)}\left(U_{i(k)}\right)}, \forall u \in W_{p)}^{m}\left(U_{i(k)}\right).$$

Analogously, we establish

$$\|\omega\vartheta\|_{L_{p)}\left(B_{r_0}^{+}\right)} \leq const\,\|u\|_{L_{p)}\left(U_{i(k)}\right)}.$$

From (3.19) and (3.20), we obtain

$$\|\omega_k u\|_{W_{p)}^{m}\left(U_{i(k)}\right)} \leq$$

$$\leq C\left(\|L(\omega_k u)\|_{L_{p)}\left(U_{i(k)}\right)} + \|u\|_{L_{p)}\left(U_{i(k)}\right)} + \left\|\tilde{B}_j(\omega\vartheta)_1^l\right\|_{W_{p)}^{m}\left(B'_{r_0};\hat{s}_l\right)}\right). \qquad (3.21)$$

Let's estimate the term $\left\|\tilde{B}_j(\omega\vartheta)_1^l\right\|_{W_{p)}^{m}\left(B'_{r_0};\hat{s}_l\right)}$. Set $\tilde{\psi}_j=\tilde{B}_j(\omega\vartheta)\,\&\tilde{\varphi}_j=B_j(\omega_k u)$. Then, by the definition of the trace we have that $\exists\tilde{u}\in W_{p)}^{m}\left(U_{i(k)}\right)\&\exists\tilde{\vartheta}\in W_{p)}^{m}\left(B_{r_0}^{+}\right)$, such that $\Gamma_{\tau_i}\tilde{u}=\tilde{\varphi}_j\&\Gamma_{B'_{r_0}}\tilde{\vartheta}=\tilde{\psi}_j$, where $\tau_i=U_{i(k)}\bigcap\partial\Omega$ and $\Gamma_{\tau_i};\Gamma_{B'_{r_0}}$ are the corresponding trace operators. Applying Lemma 3.1 we have

$$\left\|\tilde{B}_j(\omega\vartheta)\right\|_{W_{p)}^{m}\left(B'_{r_0};\{s_j\}\right)} \leq C\|B_j(\omega_k u)\|_{W_{p)}^{m}(\tau_i;\{s_j\})},$$

where $\tau=T_{i(k)}^{-1}(B'_{r_0})\subset\partial\Omega$. It is easy to see that (see, also [2], p. 135)

$$B_j(\omega_k u)=\omega_k\varphi_j+\sum_{\substack{|\beta|+|\alpha|\leq s_j\\|\alpha|<s_j}} C_{j;\beta;\alpha}\partial^{\beta}\omega_k\partial^{\alpha}u,$$

where $C_{j;\beta;\alpha}\in C^{m-s_j}\left(\bar{U}_{i(k)}\right)$. Consequently

$$\left\|\sum_{\substack{|\beta|+|\alpha|\leq s_j\\|\alpha|<s_j}} C_{j;\beta;\alpha}\partial^{\beta}\omega_k\partial^{\alpha}u\right\|_{W_{p)}^{m}(\tau;\{s_j\})}$$

$$\leq \left\|\sum_{\substack{|\beta|+|\alpha|\leq s_j\\|\alpha|<s_j}} C_{j;\beta;\alpha}\partial^{\beta}\omega_k\partial^{\alpha}u\right\|_{W_{p)}^{m-s_j}\left(U_{i(k)}\right)} \leq C\|u\|_{W_{p)}^{m-1}\left(U_{i(k)}\right)}.$$

Let $\vartheta_j\in W_{p)}^{m}\left(U_{i(k)}\right)$, such that $\Gamma_{\tau}\vartheta_j=\varphi_j$ and

$$\|\vartheta_j\|_{W_{p)}^{m-s_j}\left(U_{i(k)}\right)} \leq 2\|\varphi_j\|_{W_{p)}^{m}(\tau;\{s_j\})}.$$

Then we have

$$\|\omega_k\varphi_j\|_{W_{p)}^{m}(\tau;\{s_j\})} \leq \|\omega_k\vartheta_j\|_{W_{p)}^{m-s_j}\left(U_{i(k)}\right)} \leq C\|\vartheta_j\|_{W_{p)}^{m-s_j}\left(U_{i(k)}\right)} \leq c\|\varphi_j\|_{W_{p)}^{m}(\tau;\{s_j\})}.$$

Taking these relations into account in (3.21), we finally obtain

$$\|\omega_k u\|_{W^m_{p)}(U_{i(k)})} \le C \left(\|L(\omega_k u)\|_{L_{p)}(U_{i(k)})} + \|u\|_{L_{p)}(U_{i(k)})} \right.$$

$$\left. + \left\| (\varphi_j)_1^l \right\|_{W^m_p(\tau;\hat{s}_l)} + \|u\|_{W^{m-1}_{p)}(U_{i(k)})} \right).$$

On the other hand, we have

$$L(\omega_k u) = \omega_k L(u) + \sum_{|\alpha| \le m-1} \tilde{a}_\alpha(\cdot)\, \partial^\alpha u,$$

where $\tilde{a}_\alpha \in L_\infty(\Omega)$, $\forall \alpha : |\alpha| \le m-1$. From this, it directly follows that

$$\|L(\omega_k u)\|_{L_{p)}(U_{i(k)})} \le C \left(\|f\|_{L_{p)}(U_{i(k)})} + \|u\|_{W^{m-1}_{p)}(U_{i(k)})} \right).$$

By Lemma 2.1 for $\forall \varepsilon > 0$, there exists $C = C_\varepsilon > 0$ such that

$$\|u\|_{W^{m-1}_{p)}(U_{i(k)})} \le \varepsilon \|u\|_{W^m_{p)}(U_i(k))} + C \|u\|_{L_{p)}(U_{i(k)})}.$$

Therefore

$$\|\omega_k u\|_{W^m_{p)}(U_{i(k)})} \le C \left(\|f\|_{L_{p)}(\Omega)} + \varepsilon \|u\|_{W^m_{p)}(\Omega)} + \|u\|_{L_{p)}(\Omega)} + \left\| (\varphi_j)_1^l \right\|_{W^m_{p)}(\partial\Omega;\hat{s}_l)} \right). \tag{3.22}$$

So, if $\operatorname{supp} \omega_k \not\subset \Omega$, then the estimate (3.22) holds. On the other hand, for $\Omega_k = \operatorname{supp} \omega_k \subset \Omega$ by the results of the work [9], the interior Schauder-type estimates hold

$$\|u\|_{W^m_{p)}(\Omega_k)} \le C \left(\|f\|_{L_{p)}(\Omega)} + \|u\|_{L_{p)}(\Omega)} \right).$$

It is evident that

$$\|\omega_k u\|_{W^m_{p)}(\Omega_k)} \le C \|u\|_{W^m_{p)}(\Omega_k)},$$

for some $C > 0$. Taking these relations into account, we have

$$\begin{aligned} \|u\|_{W^m_{p)}(\Omega)} &= C \left\| \textstyle\sum_{k=1}^l \omega_k u \right\|_{W^m_{p)}(\Omega)} \le \textstyle\sum_{k=1}^l \|\omega_k u\|_{W^m_{p)}(\Omega_k)} \\ &\le C \left(\|f\|_{L_{p)}(\Omega)} + \varepsilon \|u\|_{W^m_{p)}(\Omega)} + \left\| (\varphi_j)_1^l \right\|_{W^m_{p)}(\partial\Omega;\hat{s}_l)} + \|u\|_{L_{p)}(\Omega)} \right). \end{aligned}$$

Taking $\varepsilon > 0$ sufficiently small such that $C\varepsilon < \frac{1}{2}$, we obtain

$$\|u\|_{W^m_{p)}(\Omega)} \le C \left(\|f\|_{L_{p)}(\Omega)} + \left\| (\varphi_j)_1^l \right\|_{W^m_{p)}(\partial\Omega;\hat{s}_l)} + \|u\|_{L_{p)}(\Omega)} \right).$$

Theorem is proved.

References

1. Bers, L., John, F., Schechter, M.: *Partial differential equations*. Mir, Moscow (1966). (in Russian)
2. Agmon, S., Douglis, A., Nirenberg, L.: *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, Comm. Pure Appl. Math. **12**, 623–727 (1959).
3. Fario, G.D., Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241–256 (1993).
4. Caso, L., D'Ambrosio, R., Softova, L.: *Generalized Morrey spaces over unbounded domains*, Azerb. J. Math. **10**(1), 193–208 (2020).
5. Bilalov, B.T., Sadigova, S.R., Softova, L.G.: *Higher order elliptic equations in weighted Banach spaces*, Annali dell'Universit di Ferrara, **70**, 1351-1373 (2024). https://doi.org/10.1007/s11565-024-00505-9
6. Bilalov, B.T., Ahmadov, T.M., Zeren, Y., Sadigova, S.R.: *Solution in the small and interior Schauder-type estimate for the m-th order elliptic operator in Morrey-Sobolev spaces*, Azerb. J. Math. **12**(2), 190-219 (2022).
7. Palagachev, D.K., Softova, L.G.: *Elliptic systems in generalized Morrey spaces*, Azerb. J. Math. **11**(2), 153-162 (2021).
8. Bilalov, B.T., Sadigova, S.R.: *On solvability in the small of higher order elliptic equations in grand-Sobolev spaces*, Complex Var. Elliptic Equ. **66**(12), 2117–2130 (2021).
9. Bilalov, B.T., Sadigova, S.R.: *Interior Schauder-type estimates for higher-order elliptic operators in grand-Sobolev spaces*, Sahand Communications in Mathematical Analysis, **1**(2), 129-148 (2021).
10. Bilalov, B.T., Sadigova, S.R.: *On the Fredholmness of the Dirichlet problem for a second-order elliptic equation in grand Sobolev spaces*, Ric. Mat. **73**, 283–322 (2024).
11. Bilalov, B.T., Sadigova, S.R.: *On local solvability of higher order elliptic equations in rearrangement invariant spaces*, Siberian Math. J. **63**(3), 425–437 (2022).
12. Bilalov, B.T., Zeren, Y., Sadigova, S.R., Cetin, S.: *Solvability in the small of m-th order elliptic equations in weighted grand Sobolev spaces*, Turkish J. Math. **46**(6), 2078-2095 (2022).
13. Mammadov, T.J.: *Strong solvability of a nonlocal problem for the Laplace equation in weighted grand Sobolev spaces*, Azerb. J. Math. **13**(1), 188-204 (2023).
14. Nasibova, N.P., Safarova, A.R.: *On the weak solvability of a nonlocal boundary value problem for the Laplace equation in an unbounded domain*, Lobachevskii J. Math. **44**, 2810-2821 (2023).
15. Bilalov, B.T., Ahmedzade, N.R., Kasumov, Z.A.: *The weighted grand Lebesgue class of harmonic functions and the Dirichlet problem*, Lobachevskii J. Math. **46**, 377–387 (2025).
16. Bilalov, B.T., Sadigova, S.R., Suleymanov, N., Haciyeva, G.V.: *On solvability of boundary value problem for m-th order elliptic equations in grand Sobolev spaces*. (submitted)
17. Gasimov, T.B., Ahmadli, B.Q., Behbudova, J.H.: *On strong solvability of one nonlocal boundary value problem for Poisson's equation in grand Sobolev space in rectangle*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **45** (4), Mathematics 68-75 (2025).
18. Ismailov, M.I., Hagverdi, T.X.: *On stability of a basis consisting of an perturbed exponential system in the weighted grand Lebesgue space*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **46** (1), Mathematics 64-73 (2026).
19. Kokilashvili, V., Meskhi, A., Rafeiro, H., Samko, S.: *Integral operators in non-standard function spaces. Vol. 1: Variable exponent Lebesgue and amalgam spaces*.

Springer (2016).
20. Kokilashvili, V., Meskhi, A., Rafeiro, H., Samko, S.: *Integral operators in non-standard function spaces. Vol. 2: Variable exponent Holder, Morrey-Campanato and grand spaces*. Springer (2016).
21. Bilalov, B.T., Sadigova, S.R., Sezer, Y., Nasibova, N.P.: *On solvability of polyharmonic Dirichlet problem in symmetric Sobolev spaces*, Math. Methods Appl. Sci. **47**(18), 14386-14401 (2024).