Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **46** (4), 1-8 (2026).

Embedding of total Morrey-Guliyev spaces associated with Laplace-Bessel operator

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Received: 07.01.2026 / Revised: 17.06.2026 / Accepted: 19.08.2026

Abstract. In this paper, we study some embeddings into the total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$. These spaces generalize the Morrey space associated with the Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$ so that $L_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx) \equiv L_{p,\lambda,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and the modified Morrey spaces associated with the Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$ so that $\widetilde{L}_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx) \equiv L_{p,\lambda,0}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$.

Keywords. Laplace-Bessel operator, total Morrey-Guliyev spaces, embedding.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

Let $\gamma_1 > 0, \dots, \gamma_k > 0$ and

$$\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}$$

be the Laplace-Bessel differential operator defined by setting, for suitable functions f and $x \in \mathbb{R}^n_{k,+}$.

Morrey spaces, introduced by Morrey [21], play an important role in the regularity theory of PDE, including heat equations and Navier-Stokes equations. In harmonic analysis, Morrey spaces are crucial for analyzing the behavior of integral operators and providing conditions for the global existence of solutions to nonlinear PDEs, such as the Schrödinger equation. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$, introduced by Guliyev [8], extend the Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ by including the second parameter μ, which can be seen as the intermediate spaces between Lebesgue spaces and Morrey spaces. The norm in these spaces

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is defined by a combination of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$, which allows a wider range of behavior. Let $0 < p < \infty$, $\lambda \in \mathbb{R}$, $\mu \in \mathbb{R}$, $[t]_1 = \min\{1,t\}$, $t > 0$. The total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n)$ are the set of all locally integrable functions f with the finite (quasi-)norm

$$\|f\|_{L_{p,\lambda,\mu}} = \sup_{x\in\mathbb{R}^n,\, t>0} [t]_1^{-\frac{\lambda}{p}} \, [1/t]_1^{\frac{\mu}{p}} \, \|f\|_{L_p(B(x,t))},$$

where $B(x,t)$ denotes the ball centered at x with radius $t > 0$. Here the norm in the case $\mu \le \lambda$ is equal to the maximum of the norms of $L_{p,\lambda}(\mathbb{R}^n)$ and $L_{p,\mu}(\mathbb{R}^n)$. Total Morrey-Guliyev spaces can be viewed as generalizations of both classical and modified Morrey spaces. In particular, the case where $\lambda = \mu$ corresponds to classical Morrey space, and the case where $\mu = 0$ corresponds to modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, see [1,2,5–7,9–20, 22–24].

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $x' = (x_1,\dots,x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1},\dots,x_n) \in \mathbb{R}^{n-k}$, $x = (x',x'') \in \mathbb{R}^n$, $n \ge 2$, $\mathbb{R}^n_{k,+} = \{x = (x',x'') \in \mathbb{R}^n;\, x_1 > 0,\dots, x_k > 0\}$, $1 \le k \le n$, $E(x,r) = \{y \in \mathbb{R}^n_{k,+}\,;\, |x-y| < r\}$, $E_r = E(0,r)$, $\gamma = (\gamma_1,\dots,\gamma_k)$, $\gamma_1 > 0,\dots,\gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1}\cdots x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \prod_{i=1}^k \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give some embeddings into the total Morrey-Guliyev spaces $L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$.

2 Preliminaries in the Laplace-Bessel setting on $\mathbb{R}^n_{k,+}$

Definition 2.1 *Let $L_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm*

$$\|f\|_{L_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the space $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ is defined by means of the usual modification

$$\|f\|_{L_\infty(\mathbb{R}^n_{k,+})} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n_{k,+}} |f(x)|.$$

Let $1 \le p < \infty$. We denote by $WL_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ the weak $L_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ space defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with the finite norms

$$\|f\|_{WL_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \sup_{r>0} r \left| \left\{ x \in \mathbb{R}^n_{k,+} \;:\; |f(x)| > r \right\} \right|_\gamma^{1/p}.$$

Definition 2.2 *Let $0 < p < \infty$, $\lambda, \mu \in \mathbb{R}$, $[t]_1 = \min\{1, t\}$, $t > 0$. We denote by $L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$ the total Morrey space ($\equiv$ total B–Morrey space), associated with the Laplace-Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norm*

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \left(\int_{E(x,t)} |f(y)|^p (y')^\gamma dy \right)^{1/p}.$$

We denote by $WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$ the weak total B–Morrey space the set of all classes of locally integrable functions f with the finite norm

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \sup_{r>0} r \left(\int_{\{y\in E(x,t): |f(y)|^p > r\}} (y')^\gamma dy \right)^{1/p}.$$

We note that

$$L_{p,\lambda}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset\!\!\succ WL_{p,\lambda}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \text{ and}$$
$$\|f\|_{WL_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \|f\|_{L_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Let us note that if $\lambda = \mu$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$ is the Morrey space associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$, $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$ is the weak Morrey space associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$, see [3,4,7] and if $\mu = 0$, then $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+})$ is the modified Morrey space associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$, $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+})$ is the weak modified Morrey space associated with Laplace-Bessel operator on $\mathbb{R}^n_{k,+}$, see [7].

We note that
$$L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_p(\mathbb{R}^n_{k,+}, (x')^\gamma dx),$$
and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$.

Note that

$$\begin{aligned}
&L_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,0}(\mathbb{R}^n_{k,+}) = L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_p(\mathbb{R}^n_{k,+}, (x')^\gamma dx),\\
&WL_{p,(0,0),\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0}(\mathbb{R}^n_{k,+}) = WL_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = WL_p(\mathbb{R}^n_{k,+}, (x')^\gamma dx),\\
&L_{p,(\lambda,\lambda),\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}), \quad L_{p,(\lambda,0),\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),\\
&\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \text{ therefore}\\
&L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)
\end{aligned}$$

and

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\lambda,\gamma}} \le \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}, \tag{2.1}$$

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset\!\!\succ L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}),\ \mu \le \lambda \text{ and } \|f\|_{L_{p,\mu,\gamma}} \le \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}, \tag{2.2}$$

and if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where $\Theta \equiv \Theta(\mathbb{R}^n_{k,+})$ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$.

Lemma 2.1 *If $0 < p < \infty$ and $0 \le \mu \le \lambda \le Q$, then*

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

Proof. Let $f \in L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and $0 \le \mu \le \lambda \le Q$. Then from (2.1) and (2.2) we have that $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$ and $\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} \le \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}$.

Now let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \sup_{x\in\mathbb{R}^n_{k,+},t>0} [t]_1^{-\frac{\lambda}{p}} [1/t]_1^{\frac{\mu}{p}} \left(\int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p}$$
$$= \max\left\{\sup_{x\in\mathbb{R}^n_{k,+},0<t\le 1} \left(t^{-\lambda}\int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p},\right.$$
$$\left.\sup_{x\in\mathbb{R}^n_{k,+},t>1} \left(t^{-\mu}\int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p}\right\} \le \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\}.$$

Therefore, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and the embedding

$$L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$$

is valid.

Thus

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\mu,\gamma}}\right\} = \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Corollary 2.1 *If $0 < p < \infty$, $0 \le \lambda \le Q$, then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \max\left\{\|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_p(\mathbb{R}^n_{k,+},(x')^\gamma dx)}\right\}.$$

Lemma 2.2 *If $0 < p < \infty$ and $0 \le \mu \le \lambda \le Q$, then*

$$WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx) = WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap WL_{p,\mu,\gamma}(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{WL_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \max\left\{\|f\|_{WL_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}, \|f\|_{WL_{p,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}\right\}.$$

Lemma 2.3 *If $0 < p < \infty$, $0 \le \lambda_2 \le \lambda_1 \le Q$ and $0 \le \mu_1 \le \mu_2 \le Q$, then*

$$L_{p,\lambda_1,\mu_1}(\mathbb{R}^n_{k,+},(x')^\gamma dx) \subset\!\!\succ L_{p,\lambda_2,\mu_2}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$$

and

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Proof. Let $f \in L_{p,\lambda_1,\mu_1}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$, $0<p<\infty$, $0 \le \lambda_2 \le \lambda_1 \le Q$, $0 \le \mu_1 \le \mu_2 \le Q$. Then

$$\|f\|_{L_{p,\lambda_2,\mu_2}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \max\Bigg\{ \sup_{x\in\mathbb{R}^n_{k,+},\,0<t\le 1} \left(t^{\lambda_1-\lambda_2} t^{-\lambda_1} \int_{E(x,t)} |f(y)|^p \,(y')^\gamma dy\right)^{1/p},$$

$$\sup_{x\in\mathbb{R}^n_{k,+},\,t\ge 1} \left(t^{\mu_1-\mu_2} t^{-\mu_1} \int_{E(x,t)} |f(y)|^p \,(y')^\gamma dy\right)^{1/p}\Bigg\} \le \|f\|_{L_{p,\lambda_1,\mu_1}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Lemma 2.4 *If* $0<p<\infty$, $0\le\lambda\le Q$ *and* $0\le\mu\le Q$, *then*

$$L_{p,Q,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx) \subset_{\succ} L_\infty(\mathbb{R}^n_{k,+}) \subset_{\succ} L_{p,\lambda,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$$

and

$$\|f\|_{L_{p,\lambda,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty(\mathbb{R}^n_{k,+})} \le \|f\|_{L_{p,Q,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Proof. Let $f \in L_\infty(\mathbb{R}^n_{k,+})$. Then for all $x \in \mathbb{R}^n_{k,+}$ and $0<t\le 1$

$$\left(t^{-\lambda} \int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty(\mathbb{R}^n_{k,+})},\ 0\le\lambda\le Q$$

and for all $x\in\mathbb{R}^n_{k,+}$ and $t>1$

$$\left(t^{-Q} \int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}.$$

Therefore $f \in L_{p,\lambda,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and

$$\|f\|_{L_{p,\lambda,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty(\mathbb{R}^n_{k,+})}.$$

Let $f \in L_{p,Q,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$. By the Lebesgue's Theorem we have

$$\lim_{t\to 0} |E(x,t)|^{-1}_\gamma \int_{E(x,t)} |f(y)|^p (y')^\gamma dy = |f(x)|^p \quad a.e.\ x\in\mathbb{R}^n_{k,+}.$$

Then for a.e. $x\in\mathbb{R}^n_{k,+}$

$$\begin{aligned} |f(x)| &= \left(\lim_{t\to 0} |E(x,t)|^{-1}_\gamma \int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p} \\ &\le \omega(n,k,\gamma)^{-1/p} \times \sup_{x\in\mathbb{R}^n_{k,+},\,0<t\le 1} \left(t^{-Q} \int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p} \\ &\le \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,Q,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}. \end{aligned}$$

Therefore $f\in L_\infty(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{L_\infty(\mathbb{R}^n_{k,+})} \le \omega(n,k,\gamma)^{-1/p} \|f\|_{L_{p,Q,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Corollary 2.2 *If* $0 < p < \infty$, *then*

$$L_{p,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx) = \widetilde{L}_{p,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx) = L_\infty(\mathbb{R}^n_{k,+})$$

and

$$\|f\|_{L_{p,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \|f\|_{\widetilde{L}_{p,Q}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} = \omega(n,k,\gamma)^{1/p} \, \|f\|_{L_\infty(\mathbb{R}^n_{k,+})}.$$

Lemma 2.5 *If* $0 \le \lambda < Q$, $0 \le \mu < Q$, $0 \le \alpha < Q-\lambda$ *and* $0 \le \beta < Q-\mu$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx) \subset\!\!\!\succ L_{1,Q-\alpha,Q-\beta}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ *the following inequality*

$$\|f\|_{L_{1,Q-\alpha,Q-\beta}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p'} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}$$

is valid, where $1/p + 1/p' = 1$.

Proof. Let $0 < \alpha < Q$, $0 \le \lambda < Q$, $f \in L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}$. By the Hölder's inequality we have

$$\begin{aligned}
\|f\|_{L_{1,(Q-\alpha,Q-\beta),\gamma}} &= \sup_{x\in\mathcal{G},\,t>0} [t]_1^{\alpha-Q}\,[1/t]_1^{Q-\beta} \int_{E(x,t)} |f(y)|(y')^\gamma dy \\
&\le \sup_{x\in\mathcal{G},\,t>0} [t]_1^{\alpha-Q}\,[1/t]_1^{Q-\beta} \left(\int_{E(x,t)} (|f(y)|)^p\,(y')^\gamma dy\right)^{1/p} |E(x,t)|_\gamma^{1/p'} \\
&\le \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathcal{G},\,t>0} \left([t]_1\,t^{-1}\right)^{-Q/p'} [t]_1^{\alpha-\frac{Q-\lambda}{p}}\,[1/t]_1^{Q-\beta-\frac{\mu}{p}} \\
&\times \left([t]_1^{-\lambda}\,[1/t]_1^{\mu} \int_{E(x,t)} |f(y)|^p (y')^\gamma dy\right)^{1/p} \\
&\le \omega(n,k,\gamma)^{1/p'}\,\|f\|_{L_{p,\lambda,\mu}} \sup_{t>0} \left([t]_1\,t^{-1}\right)^{\frac{Q-\mu}{p}-\beta}\,[t]_1^{\alpha-\frac{Q-\lambda}{p}}.
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_{t>0} \left([t]_1\,t^{-1}\right)^{\frac{Q-\mu}{p}-\beta}\,[t]_1^{\alpha-\frac{Q-\lambda}{p}} &= \max\left\{\sup_{0<t\le 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\beta-\frac{Q-\mu}{p}}\right\} < \infty \\
&\Longleftrightarrow \frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\beta}.
\end{aligned}$$

Therefore $f \in L_{1,Q-\alpha,Q-\beta}(\mathbb{R}^n_{k,+},(x')^\gamma dx)$ and

$$\|f\|_{L_{1,Q-\alpha,Q-\beta}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p'} \, \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

From Lemma 2.5 we get the following

Corollary 2.3 *If* $0 \le \mu \le \lambda < Q$, $0 \le \alpha < Q - \lambda$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q-\mu}{\alpha}$

$$L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset_{\succ} L_{1,Q-\alpha}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$$

and for $f \in L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+}, (x')^\gamma dx)$ *the following inequality*

$$\|f\|_{L_{1,Q-\alpha}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda,\mu}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}$$

is valid.

Corollary 2.4 *If* $0 \le \lambda < Q$ *and* $0 \le \alpha < Q - \lambda$, *then for* $p = \frac{Q-\lambda}{\alpha}$

$$L_{p,\lambda}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset L_{1,Q-\alpha}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \quad \textit{and}$$
$$\|f\|_{L_{1,Q-\alpha}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{L_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Corollary 2.5 *If* $0 \le \lambda < Q$ *and* $0 \le \alpha < Q - \lambda$, *then for* $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \subset L_{1,Q-\alpha}(\mathbb{R}^n_{k,+}, (x')^\gamma dx) \quad \textit{and}$$
$$\|f\|_{L_{1,Q-\alpha}(\mathbb{R}^n_{k,+},(x')^\gamma dx)} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n_{k,+},(x')^\gamma dx)}.$$

Remark 2.1 It should be noted that in the case $n = k = 1$ lemmas 2.1, 2.2, 2.3, 2.4 and 2.5 were proved in the paper [22].

Acknowledgements. The authors thanks the referees for his/her careful reading the paper and useful comments.

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