Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **46** (4), 1-18 (2026).

Commutators of (p, p)-admissible multilinear singular integral operators in product generalized local Morrey spaces

Amil A.Hasanov *

Received: 12.02.2026 / Revised: 23.07.2026 / Accepted: 21.08.2026

Abstract. In this paper we find the sufficient conditions on $(\varphi_1, \ldots, \varphi_m, \varphi)$ which ensures the boundedness of the commutators of $(\mathbf{p}, p)$-admissible multilinear singular integral operators $T_m^{\overrightarrow{b}}$ from product generalized local Morrey space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to generalized local Morrey space $LM_{p,\varphi}^{\{x_0\}}$. In all cases the conditions for the boundedness of $T_m^{\overrightarrow{b}}$ are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \ldots, \varphi_m, \varphi)$, which do not require any assumption on monotonicity of $\varphi_1, \ldots, \varphi_m, \varphi$ in r.

Keywords. product generalized local Morrey space; $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator; commutator.

Mathematics Subject Classification (2010): Primary 42B20, 42B35, 47G10.

1 Introduction

The multilinear theory has been well developed in the past twenty years. Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderón-Zygmund operators was done by Coifman and Meyer in [8] and was later systematically studied by Grafakos and Torres in [16, 17].

The classical Morrey spaces, introduced by Morrey [41] in 1938, have been studied intensively by various authors and together with Lebesgue spaces play an important role in the theory of partial differential equations. Although such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and that the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [19, 40, 42] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [1, 2, 13, 20, 21, 30, 44]).

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

* Corresponding author

A.A. Hasanov
Gandja State University, Gandja, Azerbaijan
E-mail: amilhasanov@gmail.com

Let $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m}(\mathbb{R}^n)$. The multi-sublinear maximal operator M_m is defined by

$$M_m(\overrightarrow{f})(x) = \sup_{r>0} \prod_{i=1}^{m} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i.$$

In [17] Grafakos and Torres studied the multilinear Calderón-Zygmund operator which can be written for $x \notin \cap_{j=1}^m \mathrm{supp} f_j$ as

$$K_m(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 dy_2 \ldots dy_m,$$

where $K(x, y_1, \ldots, y_m)$ is the kernel function defined of the diagonal $x = y_1 = \ldots y_m =$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|K(y_0, y_1, \ldots, y_m)| \leq c_1 \Big(\sum_{k,l=0}^{m} |y_k - y_l| \Big)^{-mn},$$

and whenever $2|y_j - y_j'| \leq \frac{1}{2} \max\limits_{0 \leq k \leq m} |y_j - y_k|$,

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \leq \frac{c_1 |y_j - y_j'|^{\epsilon}}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l| \Big)^{mn+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$. Grafakos and Torres [17] proved that the operator $K_m(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1 (i = 1, \ldots, m)$ and $1/p = 1/p_1 + \ldots + 1/p_m$, and bounded from $L_1(\mathbb{R}^n) \times \ldots \times L_1(\mathbb{R}^n)$ to $L_{\frac{1}{m},\infty}(\mathbb{R}^n)$.

It is well known that multi-sublinear maximal operator and multilinear Calderón-Zygmund operators play an important role in harmonic analysis (see [10,15,17,18,23,36,37]).

Let $\overrightarrow{b} = (b_1, \ldots, b_m)$ be a finite family of locally integrable functions. The commutators generated by the m-th fractional integral K_m and $\overrightarrow{b}$ are defined by

$$K_m^{\overrightarrow{b}}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^{m} \big(b_i(x) - b_i(y_i)\big) f_i(y_i)\, dy_1 dy_2 \ldots dy_m.$$

Let T_m be a multi-sublinear operator.

Definition 1.1 *($(\mathbf{p},p)$-admissible multi-sublinear singular integral operator). Let multi-sublinear operator T_m will be called $(\mathbf{p},p)$-admissible multi-sublinear singular integral operator, if:*

1) *T_m satisfies the size condition of the form*

$$\begin{aligned} &\chi_{B(x,r)}(z) \Big| T_m \Big(f_1 \chi_{\mathbb{R}^n \setminus B(x,2r)}, \ldots, f_m \chi_{\mathbb{R}^n \setminus B(x,2r)} \Big)(z) \Big| \\ &\leq C \chi_{B(x,r)}(z) \int_{\left(\mathbb{R}^n \setminus B(x,2r)\right)^m} \frac{\big| f_1(y_1) \ldots f_m(y_m) \big|}{|(z - y_1, \ldots, z - y_m)|^{nm}} dy \end{aligned} \tag{1.1}$$

for $x \in \mathbb{R}^n$ and $r > 0$;

2) *T_m is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$.*

Definition 1.2 *(weak $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator). Let multi-sublinear operator T_m will be called the weak $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator, if:*

1) *T_m satisfies the size condition* (1.2).

2) *T_m is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to the weak $WL_p(\mathbb{R}^n)$.*

Let $T_m^{\overrightarrow{b}}$ be a multi-sublinear operator.

Definition 1.3 *(commutator of $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator). Let multi-sublinear operator $T_m^{\overrightarrow{b}}$ will be called commutator of $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operator, if:*

1) *$T_m^{\overrightarrow{b}}$ satisfies the size condition of the form*

$$\chi_{B(x,r)}(z)\left|T_m^{\overrightarrow{b}}\left(f_1\chi_{\mathbb{R}^n\setminus B(x,2r)}, \ldots, f_m\chi_{\mathbb{R}^n\setminus B(x,2r)}\right)(z)\right|$$

$$\leq C\chi_{B(x,r)}(z)\int_{\left(\mathbb{R}^n\setminus B(x,2r)\right)^m}\frac{\prod\limits_{i=1}^{m}\left|b_i(x)-b_i(y_i)\right|\,|f_i(y_i)|}{|(z-y_1,\ldots,z-y_m)|^{nm}}dy \tag{1.2}$$

for $x \in \mathbb{R}^n$ and $r > 0$;

2) *$T_m^{\overrightarrow{b}}$ is bounded from product $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$.*

In [36] (see also [34]), we prove the boundedness of the $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators T_m from product generalized local Morrey space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{q,\varphi}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/p_1 + \cdots + 1/p_m = 1/p$. Also in [36] (see also [34]) we prove the boundedness of the weak $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators T_m from the space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to the weak space $WLM^{\{x_0\}}_{p,\varphi}$, if $1 \leq p_1, \ldots, p_m < \infty$, $1/p_1 + \cdots + 1/p_m = 1/p$ and at least one exponent p_i equals one.

In this paper, in the case $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$, for $0 < \lambda_i < 1/n$, $i = 1, 2, \ldots, m$, we find the sufficient conditions on $(\varphi_1, \ldots, \varphi_m, \varphi)$ which ensures the boundedness of the commutator of $(\mathbf{p}, p)$-admissible multilinear singular integral operators $T_m^{\overrightarrow{b}}$ from $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$, $1 < p, p_i, q_i < \infty$, for $i = 1, 2, \ldots, m$ such that $1/p = 1/p_1 + \ldots + 1/p_m + 1/q_1 + \ldots + 1/q_m$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Preliminaries

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, by $|B(x, r)|$ the Lebesgue measure of the ball $B(x, r)$, and by ${}^{\complement}B(x, r)$ its complement. Morrey spaces $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ introduced by C. Morrey [41] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}}\|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda < n$, $1 \leq p < \infty$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space. These spaces play an important role in the study of local properties of the solutions of partial differential equations, together with weighted Lebesgue spaces.

We define the generalized local Morrey spaces as follows.

Definition 2.1 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi}$ the generalized Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.2 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} \, |B(0,r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM^{\{x_0\}}_{p,\lambda}$ and weak local Morrey space $WLM^{\{x_0\}}_{p,\lambda}$ under the choice $\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}$:

$$LM^{\{x_0\}}_{p,\lambda}=LM^{\{x_0\}}_{p,\varphi}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}},\qquad WLM^{\{x_0\}}_{p,\lambda}=WLM^{\{x_0\}}_{p,\varphi}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [45,46] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [4] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q+1/q'=1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [12] observed that the space B_q can be described by

$$\|f\|_{B_q}=\sup_{k\geq 0}2^{-\frac{kn}{q}}\|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{2.1}$$

where χ_0 is the characteristic function of the unit ball $\{x\in\mathbb{R}^n : |x|\leq 1\}$, χ_k is the characteristic function of the annulus $\{x\in\mathbb{R}^n : 2^{k-1}<|x|\leq 2^k\}$, $k=1,2,\ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q}=\sum_{k=0}^{\infty}2^{-\frac{kn}{q'}}\|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{2.2}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k\in\mathbb{Z}$ in (2.1) and (2.2) instead of $k\geq 0$ there.

If $\lambda<0$ or $\lambda>n$, then $LM^{\{x_0\}}_{p,\lambda}(\mathbb{R}^n)=\Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n)=L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n)=\dot{B}_p(\mathbb{R}^n)$.

$$\dot{B}_{p,\mu}=LM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}},\qquad W\dot{B}_{p,\mu}=WLM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez, Guzman-Partida and Lakey [3] in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n)\equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu\in[-\frac{1}{p},0]$. If $\mu<-\frac{1}{p}$ or $\mu>0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n)=\Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n)=L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n)=\dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n)\equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

The following statements, containing results obtained in [40,42] was proved in [19,21] (see also [2,20,29,33]).

Theorem A. [19,20] *Let $x_0\in\mathbb{R}^n$, $1\leq p<\infty$ and (φ_1,φ_2) satisfy the condition*

$$\int_r^{\infty}\varphi_1(x_0,t)\,\frac{dt}{t}\leq C\,\varphi_2(x_0,r), \tag{2.3}$$

where C does not depend on x_0 and r. Then the Calderón-Zygmund operator $K\equiv K_1$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi_2}$ for $p>1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi_2}$.

The following statements, containing results in Theorem A was proved in [2], see also [22,29,33].

Theorem B. *Let $x_0 \in \mathbb{R}^n$, $1 \le p < \infty$ and (φ_1, φ) satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0,s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \frac{dt}{t} \le C\,\varphi(x_0,r), \tag{2.4}$$

where C does not depend on x_0 and r. Let the operator K is bounded from $LM^{\{x_0\}}_{p,\varphi_1}$ to $LM^{\{x_0\}}_{p,\varphi}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1}$ to $WLM^{\{x_0\}}_{1,\varphi}$.

In [36] we give the boundedness of multi-sublinear singular integral operators in product generalized local Morrey spaces.

Theorem 2.1 *Let $x_0 \in \mathbb{R}^n$, $m \ge 2$, $1 \le p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \dots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0,s)s^{\frac{n}{p_i}}}{t^{\frac{n}{p}+1}} dt \lesssim \varphi(x_0,r), \tag{2.5}$$

where the implicit constant does not depend on r.

If T_m be a $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$ for $p_i > 1$, $i = 1, \dots, m$.

If T_m be a weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $WLM^{\{x_0\}}_{p,\varphi}$ for $p_i \ge 1$, $i = 1, \dots, m$, $\min\{p_1, \dots, p_m\} = 1$.

From Theorem 2.1 we get the following corollary proven in [26] (see also [27]) about boundedness of multilinear singular integral operators on product generalized local Morrey space.

Corollary 2.1 *Let $x_0 \in \mathbb{R}^n$, $m \ge 2$, $1 \le p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \dots, \varphi_m, \varphi)$ satisfies the condition* (3.15). *Then the operator K_m is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$ for $p_i > 1$, $i = 1, \dots, m$, and from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $WLM^{\{x_0\}}_{p,\varphi}$ for $p_i \ge 1$, $i = 1, \dots, m$, $\min\{p_1, \dots, p_m\} = 1$.*

From Theorem 2.1 we get the following corollary about boundedness of $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators on product generalized Morrey space.

Corollary 2.2 *Let $m \ge 2$, $1 \le p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \dots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x,s)s^{\frac{n}{p_i}}}{t^{\frac{n}{p}}} \frac{dt}{t} \lesssim \varphi(x,r), \tag{2.6}$$

where the implicit constant does not depend on x and r.

If T_m be a $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$, $i = 1, \dots, m$.

If T_m be a weak $(\mathbf{p},p)$-admissible multi-sublinear singular integral operators, then the operator T_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{p,\varphi}$ for $p_i \ge 1$, $i = 1, \dots, m$, $\min\{p_1, \dots, p_m\} = 1$.

From Corollary 2.2 we get the following corollary proven in [26] (see also [27]) about boundedness of multilinear singular integral operators on product generalized Morrey space.

Corollary 2.3 *Let $m \geq 2$, $1 \leq p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition (2.6). Then the operator K_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$, and from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{p,\varphi}$ for $p_i \geq 1$, $i = 1, \ldots, m$, $\min\{p_1, \ldots, p_m\} = 1$.*

3 Commutators of (p, p)-admissible multilinear fractional integral operators in the product spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$

Let T be a linear operator. For a function b we define the commutator $[b, T]$ by

$$[b, T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f. If $\widetilde{T}$ is a Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss [9] states that the commutator $[b, \widetilde{T}]f = b\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [6,7,11,14,28,29,31–33,35,43]). In [5], Chanillo proved that the commutator $[b, I_\alpha]f = b\,I_\alpha f - I_\alpha(bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $(1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n})$ if and only if $b \in BMO(\mathbb{R}^n)$.

The definition of local Campanato space as follows.

Definition 3.1 *Let $1 \leq q < \infty$ and $0 \leq \lambda < \frac{1}{n}$. A function $f \in L_q^{\mathrm{loc}}(\mathbb{R}^n)$ is said to belong to the $CBMO^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$ (central Campanato space), if*

$$\|f\|_{CBMO^{\{x_0\}}_{q,\lambda}} = \sup_{r>0} \Big(\frac{1}{|B(x_0,r)|^{1+\lambda q}} \int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|^q dy \Big)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(y) dy.$$

Define

$$CBMO^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n) = \{ f \in L_q^{\mathrm{loc}}(\mathbb{R}^n) \;:\; \|f\|_{CBMO^{\{x_0\}}_{q,\lambda}} < \infty \}.$$

In [38], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = CBMO^{\{0\}}_{q,0}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n) \subset CBMO^{\{x_0\}}_q(\mathbb{R}^n)$, $1 \leq q < \infty$. The space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \leq q < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{r>0} \Big(\frac{1}{|B|} \int_B |f(y) - f_B|^q dy \Big)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ depends on q. If $q_1 < q_2$, then $CBMO^{\{x_0\}}_{q_2}(\mathbb{R}^n) \subsetneqq CBMO^{\{x_0\}}_{q_1}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 3.1 *[24,25,39] Let b be a function in $CBMO_{q,\lambda}^{\{x_0\}}(\mathbb{R}^n)$, $1 \le q < \infty$, $0 \le \lambda < \frac{1}{n}$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{|B(x_0,r_1)|^{1+\lambda q}}\int_{B(x_0,r_1)} |b(y) - b_{B(x_0,r_2)}|^q dy\right)^{\frac{1}{q}} \le C\left(1 + \left|\ln\frac{r_1}{r_2}\right|\right)\|b\|_{CBMO_{q,\lambda}^{\{x_0\}}},$$

where $C > 0$ is independent of b, r_1 and r_2.

Lemma 3.2 *Let $x_0 \in \mathbb{R}^n$, $m \ge 2$, $1 < p, p_i, q_i < \infty$, $b_i \in CBMO_{q_i,\lambda_i}^{\{x_0\}}$ for $0 \le \lambda_i < \frac{1}{n}$, $i = 1, 2, \ldots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} + \frac{1}{q_1} + \ldots + \frac{1}{q_m}$. Then, for the commutator of $(\mathbf{p}, p)$-admissible multilinear fractional integral operators $T_m^{\mathbf{b}}$ the following inequality*

$$\|T_m^{\mathbf{b}}(\mathbf{f})\|_{L^p(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{CBMO_{q_i,\lambda_i}^{\{x_0\}}} r^{\frac{n}{p}} \int_{2r}^{\infty}\left(1 + \ln\frac{t}{r}\right)^m \times t^{n\sum\limits_{i=1}^m \lambda_i + n\sum\limits_{i=1}^m \frac{1}{q_i} - \frac{n}{p} - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt$$

holds for all $B(x_0, r)$ and all $f_i \in L_{loc}^{p_i}(\mathbb{R}^n)$, $i = 1, 2, \ldots, m$.

Proof. Without loss of generality, it is suffice for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$, $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where $f_i^0 = f_i\chi_{2B}$ and $f_i^\infty = f_i\chi_{(2B)^c}$, for $i = 1.2$. Thus, we have

$$|T_2^{(b_1,b_2)}(f_1, f_2)(x)| \le |T_2^{(b_1,b_2)}(f_1^0, f_2^0)(x)| + |T_2^{(b_2,b_2)}(f_1^0, f_2^\infty)(x)| + |T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)(x)| + |T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)(x)|.$$

So,

$$\begin{aligned}\|T_2^{(b_1,b_2)}(f_1, f_2)\|_{L_p(B)} &\le \|T_2^{(b_1,b_2)}(f_1^0, f_2^0)\|_{L_p(B)} + \|T_2^{(b_1,b_2)}(f_1^0, f_2^\infty)\|_{L_p(B)} \\ &\quad + \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)\|_{L_p(B)} + \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)\|_{L_p(B)} \\ &=: I + II + III + IV.\end{aligned}$$

Let us estimate I, II, III and IV, respectively.
Since

$$\begin{aligned}&(b_1(x) - b_1(y))(b_2(x) - b_2(y)) \\ &= (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) - (b_1(x) - (b_1)_B)(b_2(y) - (b_2)_B) \\ &- (b_1(y) - (b_1)_B)(b_2(x) - (b_2)_B) + (b_1(y) - (b_1)_B)(b_2(y) - (b_2)_B), \end{aligned} \tag{3.1}$$

then

$$\|T_2^{(b_1,b_2)}(f_1^0,f_2^0)\|_{L_p(B)} \le \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^0,f_2^0)\|_{L_p(B)}$$
$$+\|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_p(B)}$$
$$+\|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^0,f_2^0)_{L_p(B)}$$
$$+\|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_p(B)}$$
$$=: I_1+I_2+I_3+I_4. \tag{3.2}$$

Let $1<\overline{p},\overline{q}<\infty$, such that $\frac{1}{\overline{p}}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{\overline{q}}=\frac{1}{q_1}+\frac{1}{q_2}$. Then, using the Hölder's inequality and by the boundedness of T_2 from product $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)$ to $L_{\overline{p}}(\mathbb{R}^n)$ for each $p_i>1 (i=1,2)$, we have

$$\begin{aligned}
I_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_B)\|_{L_{\overline{q}}(B)}\,\|T_2(f_1^0,f_2^0)\|_{L_{\overline{p}}(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_1}(2B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\, r^{\left(\frac{1}{p_1}+\frac{1}{p_2}\right)n}\\
&\times\int_{2r}^{\infty}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{\left(\frac{1}{p_1}+\frac{1}{p_2}\right)n+1}}\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\, r^{\frac{n}{p}}\\
&\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))dt}.
\end{aligned}\tag{3.3}$$

Let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$. Then similar to the estimate of (3.3), we have

$$\begin{aligned}
I_2 &\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\,\|T_2(f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_\tau(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\,\|f_1^0\|_{L_{p_1}(\mathbb{R}^n)}\,\|(b_2-(b_2)_{2B})f_2^0\|_{L_s(\mathbb{R}^n)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\,\|b_2-(b_2)_B\|_{L_{q_2}(2B)}\,\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_2}(2B)},
\end{aligned}\tag{3.4}$$

where $1<s<\frac{2n}{\alpha}$, such that $\frac{1}{s}=\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{\tau}-\frac{1}{p_1}+\frac{\alpha}{n}$.

From Lemma 3.1, it is easy to see that

$$\|b_i-(b_i)_B\|_{L_{q_i}(B)}\le C\, r^{\frac{n}{q_i}+n\lambda_i}\,\|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}},$$

and

$$\begin{aligned}
\|b_i-(b_i)_B\|_{L_{q_i}(B)} &\le \|b_i-(b_i)_{2B}\|_{L_{q_i}(2B)}+\|(b_i)_B-(b_i)_{2B}\|_{L_{q_i}(2B)}\\
&\le C\, r^{\frac{n}{q_i}+n\lambda_i}\,\|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}},
\end{aligned}\tag{3.5}$$

for $i=1,2$.

Then,

$$\begin{aligned}
I_2 &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\, r^{\frac{n}{p}}\\
&\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}\,dt.
\end{aligned}$$

Similarly,

$$I_3 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Moreover, let $1<\tau_1,\tau_2<\infty$, such that $\frac{1}{\tau_1}=\frac{1}{p_1}+\frac{1}{q_1}$ and $\frac{1}{\tau_2}=\frac{1}{p_2}+\frac{1}{q_2}$. It is easy to see that $\frac{1}{p}=\frac{1}{\tau_1}+\frac{1}{\tau_2}$. Then by the boundedness of T_2 from product $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)$ to $L_{\overline{p}}(\mathbb{R}^n)$ with $1/\overline{p}=1/p_1+1/p_2$ for each $p_i>1 (i=1,2)$, Hölder's inequality and inequality (3.5), we obtain

$$\begin{aligned}
I_4 &\lesssim \|(b_1-(b_1)_B)f_1^0\|_{L_{\tau_1}(\mathbb{R}^n)}\|(b_2-(b_2)_B)f_2^0\|_{L_{\tau_2}(\mathbb{R}^n)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(2B)}\|b_2-(b_2)_B\|_{L_{q_2}(2B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_2}(2B)}\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{q}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2\\
&\times t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))dt}. \qquad (3.6)
\end{aligned}$$

Therefore, combining the estimates of T_1, T_2, T_3 and T_4, we have

$$I \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))dt}.$$

Let us estimate II.

Similarly with (4.2), we have

$$\begin{aligned}
&\|T_2^{(b_1,b_2)}(f_1^0,f_2^\infty)\|_{L_p(B)}\\
&=\|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^0,f_2^\infty)\|_{L_p(B)}\\
&+\|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)}\\
&+\|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^0,f_2^\infty)_{L_p(B)}\\
&+\|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)}\\
&=: II_1+II_2+II_3+II_4. \qquad (3.7)
\end{aligned}$$

Let $1<\overline{p},\overline{q}<\infty$, such that $\frac{1}{\overline{p}}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{\overline{q}}=\frac{1}{q_1}+\frac{1}{q_2}$. Then, using Hölder's inequality we have

$$\begin{aligned}
II_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2)_{2B})\|_{L_{\overline{q}}(B)}\|T_2(f_1^0,f_2^\infty)\|_{L_{\overline{p}}(B)}\\
&\lesssim \|b_1-(b_1)_B\|_{L_{q_2}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_1}(2B)}\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{(\frac{1}{q_1}+\frac{1}{q_2})n+(\lambda_1+\lambda_2)n} r^{\big(\frac{1}{p_1}+\frac{1}{p_2}\big)n}\\
&\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{-\big(\frac{1}{p_1}+\frac{1}{p_2}\big)n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}\\
&\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \qquad (3.8)
\end{aligned}$$

Moreover, we have

$$|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \\ \lesssim \int_{2B}|f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2.$$

It's obvious that

$$\int_{2B}|f_1(y_1)|dy_1\lesssim \|f_1\|_{L_{p_1}(2B)}\,|2B|^{1-1/p_1} \tag{3.9}$$

and

$$\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2 \\ \lesssim \int_{(2B)^c}|b_2(y_2)-(b_2)_B||f_2(y_2)|\Big[\int_{|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big]dy_2 \\ \lesssim \int_{2r}^\infty\|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\frac{dt}{t^{2n+1}} \\ +\int_{2r}^\infty|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}|\,\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}} \\ \lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty|B(x_0,t)|^{\frac{1}{q_2}+\lambda_2}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\frac{dt}{t^{2n+1}} \\ +\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)|B(x_0,t)|^{\lambda_2}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}} \\ \lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{-n+n\lambda_2-\frac{n}{p_2}-1}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \tag{3.10}$$

Therefore, from (3.9) and (3.10) it follows that

$$|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \\ \lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\|f_1\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{-n+n\lambda_2-\frac{n}{p_2}-1}\|f_2\|_{L_{p_2}(B(x_0,t))}dt \\ \lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Thus, let $1<\tau<\infty$, such that $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{\tau}$, then similarly to the estimate of (3.3), we have

$$II_2=\|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\ \lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_\tau(B)} \\ \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,|B|^{\lambda_1+\frac{1}{q_1}+\frac{1}{\tau}} \\ \times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\,\|f_2\|_{L_{p_2}(B(x_0,t))}dt \\ \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 \\ \times t^{n\left(\lambda_1+\lambda_2\right)+n\left(\frac{1}{p_1}+\frac{1}{p_2}\right)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \tag{3.11}$$

Similarly, we have

$$II_3 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Let us estimate II_4.

Since,

$$|T_2((b_1-(b_1)_B)f_1^0, (b_2-(b_2)_B)f_2^{\infty})(x)|$$
$$\lesssim \int_{2B} |b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1 \int_{(2B)^c} \frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}} dy_2,$$

and

$$\int_{2B} |b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1 \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} |B|^{\lambda_1+1-\frac{1}{p_1}} \|f_1\|_{L_{p_1}(2B)}. \tag{3.12}$$

Then, by (3.10) and (3.13), we get

$$|T_2((b_1-(b_1)_B)f_1^0, (b_2-(b_2)_B)f_2^{\infty})(x)| \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Therefore,

$$II_4 = \|T_2((b_1-(b_1)_B)f_1^0, (b_2-(b_2)_B)f_2^{\infty})\|_{L_p(B)}$$
$$\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Combining the estimates of II_1-II_4, we have

$$II \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Similarly,

$$III \lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

For IV we have

$$\|T_2^{(b_1,b_2)}(f_1^{\infty}, f_2^{\infty})\|_{L_p(B)}$$
$$\le \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^{\infty}, f_2^{\infty})\|_{L_p(B)}$$
$$+ \|(b_1-(b_1)_B)T_2(f_1^{\infty}, (b_2-(b_2)_B)f_2^{\infty})\|_{L_p(B)}$$
$$+ \|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^{\infty}, f_2^{\infty})_{L_p(B)}$$
$$+ \|T_2((b_1-(b_1)_B)f_1^{\infty}, (b_2-(b_2)_B)f_2^{\infty})\|_{L_p(B)}$$
$$=: IV_1 + IV_2 + IV_3 + IV_4.$$

Let us estimate IV_1, IV_2, IV_3 and IV_4, respectively.

Let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{\tau}$. Then, from Hölder's inequality and inequality (3.5), we get

$$\begin{aligned}
IV_1 &\lesssim \|(b_1-(b_1)_B)\|_{L_{q_1}(B)}\|(b_2-(b_2)_B)\|_{L_{q_2}(B)}\,\|T_2(f_1^\infty,f_2^\infty)\|_{L_\tau(B)}\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,|B|^{(\lambda_1+\lambda_2)+(\frac{1}{q_1}+\frac{1}{q_2})+\frac{1}{\tau}}\\
&\times\int_{2r}^\infty \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}t^{-n(\frac{1}{p_1}+\frac{1}{p_2})-1}dt\\
&\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\,\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\\
&\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\,\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty(x)|\\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|}{(|x_0-y_1|+|x_0-y_2|)^{2n}}dy_1dy_2\\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}|f_1(y_1)||b_2(y_2)-(b_2)_B||f_2(y_2)|\Big(\int_{|x_0-y_1|+|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big)dy_1dy_2\\
&\lesssim \int_{2r}^\infty\Big(\int_{B(x_0,t)}|f_1(y_1)|dy_1\Big)\Big(\int_{B(x_0,t)}|b_2(y_2)-(b_2)||f_2(y_2)|dy_2\Big)\frac{dt}{t^{2n+1}}.
\end{aligned}$$

Since,

$$\int_{B(x_0,t)}|f_1(y_1)|dy_1\lesssim\|f_1\|_{L_{p_1}(B(x_0,t))}t^{n(1-\frac{1}{p_1})},$$

and

$$\begin{aligned}
&\int_{B(x_0,t)}|b_2(y_2)-(b_2)_B||f_2(y_2)|\,dy_2\\
&\lesssim \|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))}\|f_2\|_{L_{p_2}}\,|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\\
&+|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}|\|f_2\|_{L_{p_2}}\,|B(x_0,t)|^{1-\frac{1}{p_2}}\\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}|B(x_0,t)|^{\frac{1}{q_2}+\lambda_2}\|f_2\|_{L_{p_2}}\,|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\\
&+\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\Big(1+\ln\frac{t}{r}\Big)|B(x_0,t)|^{\lambda_2}\|f_2\|_{L_{p_2}}\,|B(x_0,t)|^{1-\frac{1}{p_2}}\\
&\lesssim \|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-\frac{n}{p_2}+n}\,\|f_2\|_{L_{p_2}(B(x_0,t))}.
\end{aligned}$$

Then,

$$\begin{aligned}
&|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)(x)|\lesssim\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2\\
&\times t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\,\|f_2\|_{L_{p_2}(B(x_0,t))}\,dt. \qquad (3.13)
\end{aligned}$$

Let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$. Then, from Hölder's inequality and inequality (3.13), we have

$$\begin{aligned} IV_2 &\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\,\|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L_\tau(B)}\\ &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\\ &\quad\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Similarly,

$$\begin{aligned} IV_3 &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\,\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\\ &\quad\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Similar to the estimate of (3.13), we have

$$\begin{aligned} &|T_2(b_1-(b_1)_B)f_1^\infty,(b_2-(b_2)_B)f_2^\infty(x)|\\ &\lesssim \int_{(2B)^c}\int_{(2B)^c}|b_1(y_1)-(b_1)_B||b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|\\ &\times\Big(\int_{|x_0-y_1|+|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big)dy_1dy_2\lesssim\int_{2r}^\infty\Big(\int_{B(x_0,t)}|f_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\Big)\\ &\times\Big(\int_{B(x_0,t)}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\Big)\frac{dt}{t^{2n+1}}\lesssim\|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\\ &\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)-n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Thus,

$$\begin{aligned} IV_4 &\lesssim \|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\,r^{\frac{n}{p}}\\ &\quad\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Then, from the estimate of IV_1-IV_4, we deduced that

$$\begin{aligned} IV &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\\ &\quad\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

So, combining the estimates of I, II, III and IV, we have

$$\begin{aligned} &\|T_2^{(b_1,b_2)}(f_1,f_2)\|_{L_p(B)}\lesssim\|b_1\|_{CBMO^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{CBMO^{\{x_0\}}_{q_2,\lambda_2}}\,r^{\frac{n}{p}}\\ &\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\big(\lambda_1+\lambda_2\big)+n\big(\frac{1}{p_1}+\frac{1}{p_2}\big)-1}\,\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \qquad (3.14) \end{aligned}$$

Therefore, we complete the proof of Lemma 3.2.

Now we give the boundedness of the commutator of $(\mathbf{p},p)$-admissible multilinear singular integral operators $T_m^{\mathbf{b}}$ in product generalized local Morrey spaces.

Theorem 3.1 *Let $x_0 \in \mathbb{R}^n$, $m \geq 2$, $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m + 1/q_1 + \ldots + 1/q_m$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right)^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m \frac{1}{p_i}} \operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0, s) s^{\frac{n}{p_i}} \frac{dt}{t} \lesssim \varphi(x_0, r), \tag{3.15}$$

where the implicit constant does not depend on r.

Then the commutator of $(\mathbf{p}, p)$-admissible multilinear singular integral operator $T_m^{\mathbf{b}}$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$.

Proof. Let $x_0 \in \mathbb{R}^n$, $1 < p_1, \ldots, p_m < \infty$ and $\overrightarrow{f} \in LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$. By Theorem 1 in [24] and Lemma 3.2 with $v_2(r) = \varphi(x_0, r)^{-1}$, $v_1(r) = \varphi(x_0, r)^{-1}$, we have

$$\begin{aligned}
\|T_m(\overrightarrow{f})\|_{LM^{\{x_0\}}_{q,\varphi}} &\lesssim \prod_{i=1}^m \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}} \sup_{r>0} \varphi(x_0, r)^{-1} \int_{2r}^\infty \left(1 + \ln\frac{t}{r}\right)^m \\
&\times t^{n\sum\limits_{i=1}^m \lambda_i + n\sum\limits_{i=1}^m \frac{1}{q_i} - \frac{n}{p} - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt \\
&\lesssim \prod_{i=1}^m \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}} \sup_{r>0} \prod_{i=1}^m \varphi_i(x_0, r)^{-1} r^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,r))} \\
&= \prod_{i=1}^m \|b_i\|_{CBMO^{\{x_0\}}_{q_i,\lambda_i}} \prod_{i=1}^m \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.
\end{aligned}$$

From Theorem 3.1 we get the boundedness of $(\mathbf{p}, p)$-admissible multi-sublinear singular integral operators in product generalized Morrey spaces.

Corollary 3.1 *Let $m \geq 2$, $1 < p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m + 1/q_1 + \ldots + 1/q_m$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^\infty \left(1 + \ln\frac{t}{r}\right)^m t^{n\sum\limits_{i=1}^m \lambda_i - n\sum\limits_{i=1}^m \frac{1}{p_i}} \operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x, s) s^{\frac{n}{p_i}} \frac{dt}{t} \lesssim \varphi(x, r), \tag{3.16}$$

where the implicit constant does not depend on x and r.

Then the commutator of $(\mathbf{p}, p)$-admissible multilinear singular integral operator $T_m^{\mathbf{b}}$ is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$.

From Theorem 3.1 we get the boundedness of multilinear singular integral operators in product generalized local Morrey spaces.

Corollary 3.2 *Let $x_0 \in \mathbb{R}^n$, $m \geq 2$, $1 < p_1, \ldots, p_m < nm/\alpha$ with $1/p = 1/p_1 + \ldots + 1/p_m + 1/q_1 + \ldots + 1/q_m$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ for $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition* (3.15). *Then the operator $K_m^{\overrightarrow{b}}$ is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$.*

From Corollary 3.2 we get the boundedness of multilinear singular integral operators in product generalized local Morrey spaces.

Corollary 3.3 *Let* $m \geq 2$, $1 < p_1, \ldots, p_m < nm/\alpha$ *with* $1/p = 1/p_1 + \ldots + 1/p_m + 1/q_1 + \ldots + 1/q_m$, $b_i \in CBMO^{\{x_0\}}_{q_i,\lambda_i}$ *for* $0 \leq \lambda_i < \frac{1}{n}$, $1 \leq i \leq m$ *and* $(\varphi_1, \ldots, \varphi_m, \varphi)$ *satisfies the condition* (3.16). *Then the operator* $K_m^{\vec{b}}$ *is bounded from product space* $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ *to* $M_{p,\varphi}$.

References

1. Akbulut, A., Celik, S., Omarova M.N.: *Fractional maximal operator associated with Schrodinger operator and its commutators on vanishing generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **44** (1), 3-19 (2024).
2. Akbulut, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the maximal operator and singular integral operator in generalized Morrey spaces*, Math. Bohem. **137** (1), 27-43 (2012).
3. Alvarez, J., Guzman-Partida, M., Lakey, J.: *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*, Collect. Math. **51**, 1-47 (2000).
4. Beurling, A.: *Construction and analysis of some convolution algebras*, Ann. Inst. Fourier (Grenoble) **14**, 1-32 (1964).
5. Chanillo, S.: *A note on commutators*, Indiana Univ. Math. J. **23**, 7-16 (1982).
6. Chiarenza, F., Frasca, M., Longo, M.: *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40**, 149-168 (1991).
7. Chiarenza, F., Frasca, M., Longo, M.: *$W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336**, 841-853 (1993).
8. Coifman, R.R., Meyer, Y.: *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212**, 315-331 (1975).
9. Coifman, R.R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2), 611-635 (1976).
10. Coifman, R.R., Grafakos L.: *Hardy spaces estimates for multilinear operators I*, *Rev. Math. Iberoamericana*, **8**, 45-68 (1992).
11. Fazio, G.Di, Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241-256 (1993).
12. Feichtinger, H.: *An elementary approach to Wiener's third Tauberian theorem on Euclidean n-space*, In: *Harmonic Analysis, Symmetric Spaces and Probability Theory, Cortona/Italy 1984, Symp. Math.* pp. 267-301, Academic Press, London (1987).
13. Eroglu A., Hasanov A.A., Omarova M.N. *(p, q)-admissible multilinear fractional integral operators and their commutators in product generalized local Morrey spaces*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 36 (2016), no. 4, Mathematics, 77-98.
14. Eroglu, A., Omarova, M.N., Muradova, Sh.: *Elliptic equations with measurable coefficients in generalized weighted Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **43** (2), 197-213 (2017).
15. Grafakos L.: *Classical and Modern Fourier Analysis*, Pearson Education, Inc. Upper Saddle River, New Jersey, 2004.
16. Grafakos L., Torres R.H.: *Multilinear Calderón-Zygmund theory*, Adva. Math. **8** (1), 124-164 (2002).
17. Grafakos L., Torres R.H.: *On multilinear singular integrals of Calderón-Zygmund type*, Publ. Mat. 57-91 (2002).
18. Grafakos L., Kalton, N.: *Multilinear Calderón-Zygmund operators on Hardy spaces*, Collect. Math. **52**, 169-179 (2001).

19. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, (in Russian) Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 329 pp. (1994).
20. Guliyev, V.S.: *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications*, (in Russian) Casioglu, Baku, 332 pp. (1999).
21. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp. (2009).
22. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.S.: *Boundedness of sublinear operators and commutators on generalized Morrey Space*, Integral Equations Operator Theory **71** (3), 327-355 (2011).
23. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 33-61 (2012).
24. Guliyev, V.S.: *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2), 79-94 (2013).
25. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2), 211-227 (2013).
26. Guliyev, V.S., Ismayilova, A.F.: *Multi-sublinear maximal operator and multilinear singular integral operators on generalized Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2), 65-77 (2014).
27. Guliyev, V.S., Omarova M.N.: *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math. **5** (1), 104-132 (2015).
28. Guliyev, V.S., Muradova, Sh., Omarova, M.N., Softova, L.: *Gradient estimates for parabolic equations in generalized weighted Morrey spaces*, Acta Math. Sin. (Engl. Ser.) **32** (8), 911-924 (2016).
29. Guliyev, V.S., Omarova, M.N., Ragusa, M.A., Scapellato, A.: *Commutators and generalized local Morrey spaces*, J. Math. Anal. Appl. **457** (2), 1388-1402 (2018).
30. Guliyev, V.S., Guliyev, R.V., Omarova, M.N.: *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. **17** (1), 56-71 (2018).
31. Guliyev, V.S., Omarova, M.N., Softova, L.: *The Dirichlet problem in a class of generalized weighted Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **45** (2), 270-285 (2019).
32. Guliyev, V.S., Ekincioglu, I., Ahmadli, A., Omarova, M.: *Global regularity in Orlicz-Morrey spaces of solutions to parabolic equations with VMO coefficients*, J. Pseudo-Differ. Oper. Appl. **11** (4), 1963-1989 (2020).
33. Guliyev V.S., Omarova M.N., Ragusa M.A., Scapellato A.: *Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces*, Anal.Math.Phys. **11**, Article number 13 (2021).
34. Hasanov, A.A., Muradova, Sh.A., Gadjieva, E.A.: *Commutators of multilinear singular integral operators on generalized local Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **35** (4), Mathematics, 61-74 (2015).
35. Guliyev, V.S., Omarova, M.N.: *Estimates for operators on generalized weighted Orlicz-Morrey spaces and their applications to non-divergence elliptic equations*, Positivity **26** (2), Paper No. 40, 27 pp. (2022).
36. Hasanov, A.A.: $(\boldsymbol{p}, p)$*-admissible multilinear singular integral operators in product generalized local Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **40** (4), 85-96 (2020).
37. Hasanov, S.G.: *Multi-sublinear maximal operators on product total Morrey-Guliyev spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **46** (1), 56-63 (2026).

38. Lu, S.Z., Yang, D.C.: *The central BMO spaces and Littlewood-Paley operators*, Approx. Theory Appl. (N.S.) **11**, 72-94 (1995).
39. Lu, S.Z., Wu, Q.: *CBMO estimates for commutators and multilinear singular integrals*, Math. Nachr. **276**, 75-88 (2004).
40. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189 (1991).
41. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
42. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95-103 (1994).
43. Omarova, M.N.: *Estimates for operators on generalized weighted Orlicz-Morrey spaces and their applications to non-divergence elliptic equations*, J. Math. Ineq. **18** (3), 865-893 (2024).
44. Y. Sawano, *A thought on generalized Morrey spaces*, J. Indonesian Math. Soc. 25 (3) (2019), 210-281.
45. Wiener, N.: *Generalized Harmonic Analysis*, Acta Math. **55**, 117-258 (1930).
46. Wiener, N.: *Tauberian theorems*, Ann. Math. **33**, 1-100 (1932).