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Control of the vertical motion and hovering of a quadcopter under equality control constraints

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Abstract. *This paper addresses the control of the vertical motion and hovering of a quadcopter. To this end, a linear quadratic optimal control problem with boundary conditions is considered, in which some components of the control input are prescribed as known constants over a specified subinterval of the time horizon. By employing the fundamental matrices of the corresponding Euler-Lagrange equations, the resulting equations are solved. Subsequently, the optimal open-loop control and the corresponding state trajectory are constructed on each subinterval of the time horizon*

Keywords. quadcopter, vertical motion control, hovering, linear quadratic optimal control, Euler–Lagrange equations, fundamental matrix.

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1 Introduction

A quadcopter is an unmanned aerial vehicle (UAV) equipped with four propellers driven by independent electric motors, which generate the thrust required for flight [1–5]. Owing to their maneuverability, mechanical simplicity, and relatively low cost, quadcopters have found widespread applications in aerial surveillance, environmental monitoring, inspection of engineering structures, precision agriculture, search-and-rescue operations, and many other fields.

Despite their extensive use, most practical applications still rely on manual or semi-autonomous remote control performed by a human operator. Consequently, the development of reliable algorithms for autonomous flight along prescribed trajectories remains an important research problem in modern control theory.

In [6], the motion of a quadcopter described by a system of linear ordinary differential equations on an infinite time interval was investigated. By applying the Linear Quadratic Regulator (LQR) theory, the authors proposed a computational algorithm for synthesizing optimal control inputs and trajectories that ensure the passage of the quadcopter through prescribed waypoints [7–13].

The present work extends this line of research by employing the methodology developed in [14, 15] to construct computational algorithms for the optimal control of the vertical motion and hovering of a quadcopter over a finite time interval. Unlike the problems considered in [14, 15], where only a one-dimensional motion was investigated, the present study addresses a three-dimensional control problem. Consequently, the desired trajectory is represented by a sequence of points in three-dimensional space, allowing the proposed algorithm to describe realistic quadcopter motion and hovering under equality constraints imposed on the control inputs.

The main contributions of this paper can be summarized as follows:

1. A finite-horizon linear quadratic optimal control problem for quadcopter motion subject to equality constraints on the control inputs is formulated
2. The original six-degree-of-freedom quadcopter model is decomposed into three independent lower-dimensional subsystems, thereby significantly reducing the computational complexity of the solution procedure
3. An analytical solution for each subsystem is derived using the Euler-Lagrange formalism in conjunction with the fundamental matrix approach
4. Three computational algorithms are proposed for constructing the optimal open-loop control inputs and the corresponding state trajectories over a finite time horizon
5. Numerical simulations are presented to validate the proposed methodology and demonstrate its effectiveness in generating smooth trajectories and stable hovering while satisfying the prescribed equality constraints on the control inputs

2 Problem formulation

As is well known [1, 2, 6], the motion of a controlled quadcopter is described by the following system of second-order ordinary differential equations:

$$m\,\ddot{x} = -u\,sin\theta,$$

$$m\,\ddot{y} = u\,cos\theta\,sin\varphi,$$

$$m\,\ddot{z} = u\,cos\theta\,cos\varphi - mg, \tag{2.1}$$

$$\ddot{\psi} = \tau_\psi,$$

$$\ddot{\theta} = \tau_\theta,$$

$$\ddot{\varphi} = \tau_\varphi,$$

where *m* denotes the mass of the quadcopter, and the control inputs are defined as follows:

$$u_f = f_1 + f_2 + f_3 + f_4,$$

$$\tau_\psi = \left[(f_2 + f_4) - (f_1 + f_2) \right] l, \tag{2.2}$$

$$\tau_\theta = (f_2 - f_4) l,$$

$$\tau_\varphi = (f_3 - f_1) l,$$

where l denotes the distance from the center of the quadcopter to the motors, and f_i represents the thrust generated by the (*i*)-th motor.

$$f_i = k_i \omega_i^2,$$

where ω_i is the angular velocity of the i -th rotor, and k_i is an experimentally determined positive constant. Furthermore, (x, y, z) represent the coordinates of the quadcopter's center of mass, whereas (ψ, θ, φ) are the Euler angles, representing the yaw, pitch, and roll angles, respectively.
As shown in [6], after introducing several simplifying assumptions and using the notation defined below,

$$x(t) = x_1(t), \quad \dot{x}(t) = x_2(t), \quad y(t) = y_1(t), \quad \dot{y}(t) = y_2(t),$$

$$\dot{z}(t) = z_1(t), \quad \dot{z}(t) = z_2(t), \quad \theta(t) = \theta_1(t), \quad \dot{\theta}(t) = \theta_2(t), \tag{2.3}$$

$$\varphi(t) = \varphi_1(t), \quad \dot{\varphi}(t) = \varphi_2(t), \quad \psi(t) = \psi_1(t), \quad \dot{\psi}(t) = \psi_2(t)$$

system (2.1) can be reduced to the following system:

$$\dot{x}_1 = x_2, \dot{x}_2 = -g\theta_1, \dot{y}_1 = y_2, \dot{y}_2 = g\,\varphi_1, \dot{z}_1 = z_2, \dot{z}_2 = u_f, \tag{2.4}$$

$$\dot{\theta}_1 = \theta_2, \theta_2 = \tau_\theta, \dot{\varphi}_1 = \varphi_2, \dot{\varphi}_2 = \tau_\varphi, \dot{\psi}_1 = \psi_2, \dot{\psi}_2 = \tau_\psi.$$

For system (2.4), a linear quadratic optimal control problem over the infinite time horizon $(0, \infty)$ was formulated in [6], where the quadcopter dynamics are described by a system of linear ordinary differential equations. The corresponding optimal control problem was solved using the linear quadratic regulator (LQR) framework.

In the present paper, we consider the problem of controlling the vertical motion and hovering of a quadcopter over a finite time interval $[T_0,\ T_1 + \triangle_1]$, subject to equality constraints imposed on the control inputs $[T_1,\ T_1 + \triangle_1]$. over a prescribed subinterval of the control horizon.

It is evident from the structure of system (2.4) that it can be decomposed into three independent lower-dimensional subsystems, each of which can be solved independently. Introducing the notation

$$X = \begin{bmatrix} x_2 \\ \theta_2 \\ x_1 \\ \theta_1 \end{bmatrix}, \; Y = \begin{bmatrix} y_2 \\ \varphi_2 \\ y_1 \\ \varphi_1 \end{bmatrix}, \; Z = \begin{bmatrix} z_2 \\ \psi_2 \\ z_1 \\ \psi_1 \end{bmatrix}, \tag{2.5}$$

the system (2.4) can be rewritten in the form

$$\begin{aligned} \dot{X} &= A_x X + B_x u_x, \\ \dot{Y} &= A_y Y + B_y u_y, \\ \dot{Z} &= A_z Z + B_z u_z, \end{aligned} \tag{2.6}$$

where the corresponding system matrices are defined by

$$A_x = \begin{bmatrix} 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B_x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_x = \tau_\theta, \tag{2.7}$$

$$A_y = \begin{bmatrix} 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad u_y = \tau_\varphi, \tag{2.8}$$

$$A_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad u_z = \begin{bmatrix} u_f \\ \tau_\psi \end{bmatrix}. \tag{2.9}$$

Since the primary objective of this study is the control of the quadcopter's vertical motion, we first address the altitude control problem.

3 Optimal control of the altitude and the yaw angle

Consider the motion of a quadcopter along the vertical axis over the time intervals $[T_0,\ T_1)$ and $(\ T_1,\ T_1 + \triangle_1]$ $(T_1 > T_0,\ \triangle_1 > 0)$, which is governed by the following linear differential equation [2,16]

$$\dot{Z} = A_z Z + B_z u_z + V_z \tag{3.1}$$

with initial conditions

$$H_z Z\,(T_0) = q_z \tag{3.2}$$

Assume that, the following condition holds at time T_1:

$$Z\,(T_1 + \varepsilon) = F_z Z\,(T_1 - \varepsilon) + B_z^1 u_z^1 (T_1 - \varepsilon), \tag{3.3}$$

where $H_z = \begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$, $u_z = \begin{bmatrix} u_z^1 \\ u_z^2 \end{bmatrix} = \begin{bmatrix} u_f \\ \tau_\psi \end{bmatrix}$, V is a four-dimensional external disturbance vector. Let $B_z = \begin{bmatrix} B_z^1 & B_z^2 \end{bmatrix}$, $F_z,\ B_z^1$ and the remaining matrices be known matrices with corresponding dimensions, ε be an arbitrarily small positive parameter, and $(\,T_1,\ T_1 + \triangle_1]$ let the disturbance vector remain bounded over the considered time interval.

The objective is to determine the control input $u_z\,(t)$ and the corresponding state trajectory satisfying the boundary conditions (3.2) and (3.3), while ensuring that the system dynamics (3.1) are fulfilled and the following quadratic performance index is minimized:

$$J_z = \frac{1}{2}\left(Z\left(T_1 - \varepsilon\right) - Z_d\right)' N_z \left(Z\left(T_1 - \varepsilon\right) - Z_d\right) + \left(u_z\left(T_1 - \varepsilon\right) - C_z\right)' \delta_z \left(u_z\left(T_1 - \varepsilon\right) - C_z\right)$$

$$+\frac{1}{2}\int_{T_0}^{T_1} \left[Z'\left(t\right) Q_z Z\left(t\right) + 2Z'\left(t\right) K_z u_z\left(t\right) + u_z'\left(t\right) R_z u_z\left(t\right)\right] dt$$

$$+\frac{1}{2}\int_{T_1}^{T_1+\triangle_1} \left(u_z\left(t\right) - C_z\right)'\ \gamma_z \left(u_z\left(t\right) - C_z\right) dt. \tag{3.4}$$

where $N_z \geq 0,\ \delta_z = \delta_z' > 0,\ Q_z = Q_z' \geq 0,\ \ R_z = R_z' > 0,\ \gamma_z = \gamma_z' > 0,\ \ K_z,\ Z_d,\ C_z$ the weighting matrices and vectors are assumed to be known and have compatible dimensions. It should be noted that the parameter $C_z = \begin{bmatrix} C_z^1 \\ 0 \end{bmatrix}$, C_z^1- defining the control constraint is a prescribed constant.

Applying the results obtained in [2,17], the Euler–Lagrange equations corresponding to system (3.1) and performance index (3.4) are derived for the intervals $[T_0,\ T_1)$ and $(\,T_1,\ T_1 + \triangle_1]$, respectively, in the following form [18,19]:

$$\begin{bmatrix} \dot{Z}\,(t) \\ \dot{\lambda}_z(t) \end{bmatrix} = \begin{bmatrix} A_z - B_z R_z^{-1} K_z' & -B_z R_z^{-1} B_z' \\ -Q_z' + K_z R_z^{-1} K_z' & -A_z' + K_z R_z^{-1} B_z' \end{bmatrix} \begin{bmatrix} Z\,(t) \\ \lambda_z(t) \end{bmatrix}$$

$$+\begin{bmatrix} V_z \\ 0 \end{bmatrix}, T_0 < t < T_1, \tag{3.5}$$

$$\begin{bmatrix} \dot{Z}\,(t) \\ \dot{\lambda}_z(t) \end{bmatrix} = \begin{bmatrix} A_z & -B_z \gamma_z^{-1} B_z' \\ 0 & -A_z' \end{bmatrix} \begin{bmatrix} Z\,(t) \\ \lambda_z(t) \end{bmatrix} + \begin{bmatrix} V_z + B_z C_z \\ 0 \end{bmatrix}, T_1 < t < T_1 + \triangle_1 \tag{3.6}$$

with boundary conditions [20,21]

$$N_z' Z\left(T_1 - \varepsilon\right) + N_z' Z_d - \lambda_z\left(T_1 - \varepsilon\right) + F_z' \lambda_z\left(T_1 + \varepsilon\right) = 0, \tag{3.7}$$

$$\delta_z u_z\left(T_1 - \varepsilon\right) - \delta_z C_z + (B_z^1)' \lambda_z\left(T_1 + \varepsilon\right) = 0, \tag{3.8}$$

$$\lambda_z\left(T_1 + \triangle_1\right) = 0, \tag{3.9}$$

$$H_z' \nu_z + \lambda_z\left(0\right) = 0, \tag{3.10}$$

where the vectors $\lambda_z\,(t)\,,\ \nu_z$ denote the corresponding Lagrange multipliers.

The optimal control $u_z\,(t)$ on the first interval $[T_0,\ T_1)$ is determined by

$$u_z\left(t\right) = -R_z^{-1} B_z' \lambda_z\left(t\right) - R_z^{-1} K_z' Z\left(t\right), \qquad [T_0,\ T_1) \tag{3.11}$$

whereas on the second interval it is given by

$$u_z\,(t) = -\gamma_z^{-1} B_z' \lambda_z\,(t) + C_z, \qquad\qquad (\,T_1,\ T_1 + \triangle_1]\,. \tag{3.12}$$

Substituting expression $u_z\,(T_1 - \varepsilon)$ (3.8) into (3.3) and $\lambda_z\,(T_1 + \varepsilon)$ using relation (3.7), we obtain

$$\begin{bmatrix} Z\,(T_1 + \varepsilon) \\ \lambda_z\,(T_1 + \varepsilon) \end{bmatrix} = \mathrm{F}_z^1 \mathrm{F}_z^2 \begin{bmatrix} Z\,(T_1 - \varepsilon) \\ \lambda_z\,(T_1 - \varepsilon) \end{bmatrix} + \mathrm{F}_z^1 \bar{q}_z, \tag{3.13}$$

where

$$\mathrm{F}_\mathrm{z}^1 = \begin{bmatrix} E & B_z' \delta_z^{-1} B_z' \\ 0 & E \end{bmatrix}^{-1}, \mathrm{F}_z^2 = \begin{bmatrix} F_z & 0 \\ (\mathrm{F}_z')^{-1} N'\ _z & (\mathrm{F}_z')^{\ -1} \end{bmatrix}, \bar{q}_z = \begin{bmatrix} B_z' C_z \\ (\mathrm{F}_z')^{-1} N'\ _z Z_d \end{bmatrix}. \tag{3.14}$$

and E denotes the identity matrix of appropriate dimension.
Combining the initial and boundary conditions (3.2), (3.9), and (3.10), we obtain

$$\begin{bmatrix} H_z & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & H_z' \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} Z(T_0) \\ \lambda_z(T_0) \\ Z(T_1 + \triangle_1) \\ \lambda_z(T_1 + \triangle_1) \\ \nu_z \end{bmatrix} = \begin{bmatrix} q_z \\ 0 \\ 0 \end{bmatrix}. \tag{3.15}$$

Let the fundamental matrix solutions of systems (3.5) and (3.6) be represented as

$$\begin{bmatrix} Z\,(T_1 - \varepsilon) \\ \lambda_z\,(T_1 - \varepsilon) \end{bmatrix} = H_z^1\,(\,T_1,\ T_0) \begin{bmatrix} Z\,(\,T_0) \\ \lambda_z\,(T_0) \end{bmatrix} + V_z^1 \tag{3.16}$$

and

$$\begin{bmatrix} Z(T_1 + \triangle_1) \\ \lambda_z\,(T_1 + \triangle_1) \end{bmatrix} = H_z^2\,(T_1 + \triangle_1, T_1)\ \begin{bmatrix} Z\,(T_1 + \varepsilon) \\ \lambda_z\,(T_1 + \varepsilon) \end{bmatrix} + V_z^2, \tag{3.17}$$

where $H_z^1\,(\,T_1,\ T_0)$ and $H_z^2\,(T_1 + \triangle_1, T_1)$ are the corresponding fundamental matrices and $V_z^1,\ V_z^2$ denote the associated forcing vectors.
Substituting expressions (3.13) and (3.16) into (3.17) yields

$$\begin{bmatrix} Z(T_1 + \triangle_1) \\ \lambda_z\,(T_1 + \triangle_1) \end{bmatrix} = H_z^2 F_z^1 F_z^2 H_z^1\ \begin{bmatrix} Z\,(T_0) \\ \lambda_z\,(T_0) \end{bmatrix} + H_z^2 F_z^1 F_z^2 V_z^1 + H_z^2 F_z^1 q_z + V_z^2. \tag{3.18}$$

Combining (3.15) and (3.18), we obtain the following system of linear algebraic equations:

$$\begin{bmatrix} -H_z^4 F_z^1 F_z^2 H_z^3 & E & 0 \\ H_z^3 & H_z^4 & H_z^5 \\ H_z^6 & H_z^7 & H_z^8 \end{bmatrix} \begin{bmatrix} Z(T_0) \\ \lambda_z(T_0) \\ Z(T_1 + \triangle_1) \\ \lambda_z(T_1 + \triangle_1) \\ \nu_z \end{bmatrix} = \begin{bmatrix} H_z^4 F_z^1 F_z^2 V_z^1 + H_z^4 F_z^1 q_z + V_z^2 \\ q_z \\ 0 \end{bmatrix}, \tag{3.19}$$

where

$$H_z^3 = \begin{bmatrix} H_z & 0 \\ 0 & E \end{bmatrix},\ H_z^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ \ H_z^5 = \begin{bmatrix} 0 \\ H_z' \end{bmatrix},\ \ H_z^6 = [0\ 0]\,,\ \ H_z^7 = [0\ \ E]\,,\ \ H_z^8 = 0. \tag{3.20}$$

After obtaining $Z(T_0)$, $\lambda_z(T_0)$ from the system (3.19), system (3.5) is solved with these initial conditions, yielding the state trajectory $Z(t)$over the interval $[T_0,\ T_1)$. The corresponding control input $u_z(t)$ on the interval $[T_0,\ T_1)$ is then determined from (3.11). Further, $Z(T_1+\varepsilon)$, $\lambda_z(T_1+\varepsilon)$ are computed using (3.13). These values are subsequently used as the initial conditions for solving system (3.6), which yields the state trajectory $Z(t)$ over the interval $(T_1,\ T_1+\triangle_1]$. Finally, the control input $u_z(t)$ on the interval $(T_1,\ T_1+\triangle_1]$ is determined from (3.12).

Thus, the following algorithm can be employed to determine the state trajectory $Z(t)$ and the corresponding control input $u_z(t)$:

Algorithm 1.

Step 1. Specify the parameters T_0, T_1, $\triangle_1$, ε, the matrices and the vectors A_z, B_z, V_z, q_z, H_z, F_z, Z_d, N_z, δ_z, C_z, Q_z, K_z, R_z, γ_z

Step 2. Compute the fundamental matrices $H_z^1(T_1,\ T_0)$ and $H_z^2(T_1+\triangle_1, T_1)$ corresponding to systems (3.5), (3.6), as well as the vectors V_z^1, V_z^2 defined by equations (3.16) , (3.17).

Step 3. Evaluate the matrices $H_z^3,\ H_z^4,\ H_z^5,\ H_z^6,\ H_z^7,\ H_z^8$ according equation (3.20).

Step 4. Solve the system of linear algebraic equations (3.19) to determine the initial vectors $Z(T_0)$, $\lambda_z(T_0)$.

Step 5. The state trajectory $Z(t)$ and the adjoint variable $\lambda_z(t)$ over the interval $[T_0,\ T_1)$ are obtained by solving system (3.5). Subsequently, the control input $u_z(t)$ is determined from (3.11).

Step 6. Determine the intermediate values $Z(T_1+\varepsilon)$, $\lambda_z(T_1+\varepsilon)$ from equation (3.13).

Step 7. The state trajectory $Z(t)$ and the adjoint variable $\lambda_z(t)$, over the interval $(T_1,\ T_1+\triangle_1]$. Subsequently, the control input $u_z(t)$ is determined from (3.12).

Example 1.

Let us solve the problem (3.1)-(3.4) using Algorithm 1. Assume that

$$A_z=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad B_z=\begin{bmatrix}1&0\\0&1\\0&0\\0&0\end{bmatrix},\quad u_z=\begin{bmatrix}u_f\\ \tau_\psi\end{bmatrix}.H_z=\begin{bmatrix}0&0&1&0\\0&0&0&1\end{bmatrix},V_z=[0\ -\mathrm{g}\ 0\ 0]'.$$

As a result of the MATLAB simulations, the following plots were obtained for the vertical motion and the yaw angle.

4 Optimal control of the longitudial motion y and the roll angle φ

Consider the longitudinal motion of the quadcopter along the y axis over the time intervals $[T_0,\ T_1)$ and $(T_1,\ T_1+\triangle_1]$ $(T_1>T_0,\ \triangle_1>0)$. The system dynamics are described by the following linear differential equation:

$$\dot{Y}=A_yY+B_yu_y+V_y \tag{4.1}$$

with initial conditions

$$H_yY(T_0)=q_y. \tag{4.2}$$

Assume that the following condition holds at time T_1:

$$Y(T_1+\varepsilon)=F_yY(T_1-\varepsilon)+B_yu_y(T_1-\varepsilon), \tag{4.3}$$

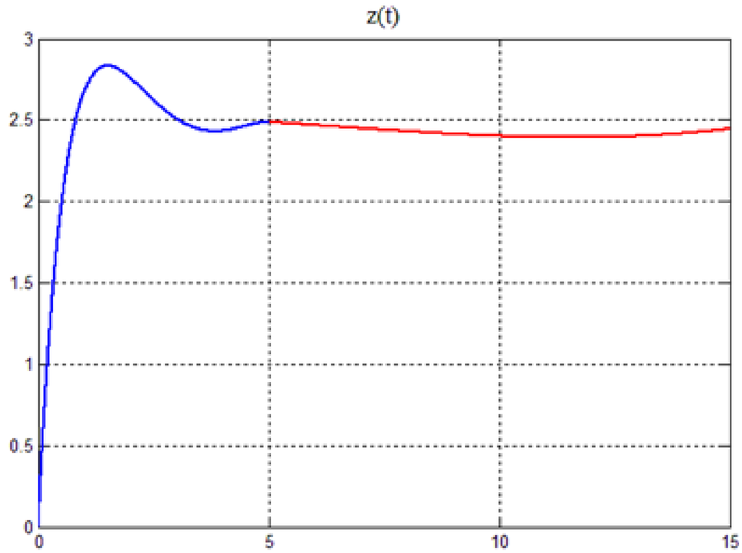

Figure 1. Vertical Position versus Time

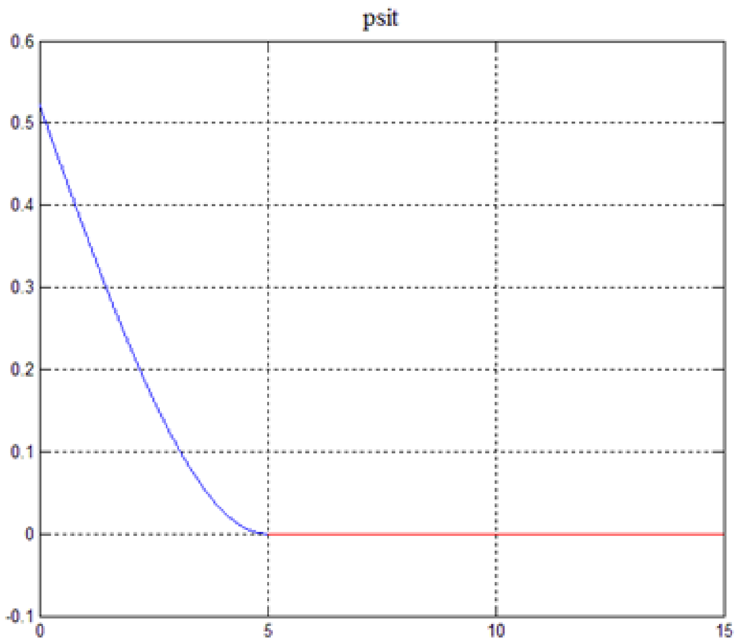

Figure 2.Yaw Angle versus Time

where $H_y = \begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$, V_y- is a four-dimensional external disturbance vector. Let all system $F_y,\ B_y$ matrices be known and of appropriate dimensions, ε be an arbitrarily small positive constant, and let the disturbance remain bounded over the entire control horizon.

The objective is to determine the control input $u_y\left(t\right)$ and the corresponding state trajectory $Y\left(t\right)$ satisfying the boundary conditions (4.2) and (4.3), such that system (4.1) is satisfied and the following performance index is minimized

$$J_y = \frac{1}{2}\left(Y\left(T_1-\varepsilon\right)-Y_d\right)' N_y \left(Y\left(T_1-\varepsilon\right)-Y_d\right)+$$

$$+(u_y\left(T_1-\varepsilon\right)-C_y)'\delta_y\left(u_y\left(T_1-\varepsilon\right)-C_y\right)+$$

$$+\frac{1}{2}\int_{T_1}^{T_1+\triangle_1}\left(u_y\left(t\right)-C_y\right)'\ \gamma_y\left(u_y\left(t\right)-C_y\right)dt, \tag{4.4}$$

where $N_y\geq 0,\ \delta_y=\delta_y'>0,\ Q_y=Q_y'\geq 0,\ \ R_y=R_y'>0,\ \gamma_y=\gamma_y'>0,\ \ K_y,\ Z_y,\ C_y$ the weighting matrices and vectors are assumed to be known and compatible with the dimensions of the system variables. The parameter C_y appearing in the equality constraint is regarded as a prescribed constant.
Following the approach developed in [2,17], the corresponding Euler-Lagrange equations on the intervals $[T_0,\ T_1)$ and $(\ T_1,\ T_1+\triangle_1]$ are obtained as

$$\begin{bmatrix} \dot{Y}\left(t\right) \\ \dot{\lambda}_y(t) \end{bmatrix} = \begin{bmatrix} A_y-B_yR_y^{-1}K_y' & -B_yR_y^{-1}B_y' \\ -Q_y'+K_yR_y^{-1}K_y' & -A_y'+K_yR_y^{-1}B_y' \end{bmatrix}\begin{bmatrix} Y\left(t\right) \\ \lambda_y(t) \end{bmatrix}+\begin{bmatrix} V_y \\ 0 \end{bmatrix}, T_0<t<T_1, \tag{4.5}$$

$$\begin{bmatrix} \dot{Y}\left(t\right) \\ \dot{\lambda}_y(t) \end{bmatrix} = \begin{bmatrix} A_y & -B_y\gamma_y^{-1}B_y' \\ 0 & -A_y' \end{bmatrix}\begin{bmatrix} Y\left(t\right) \\ \lambda_y(t) \end{bmatrix}+\begin{bmatrix} V_y+B_yC_y \\ 0 \end{bmatrix}, T_1<t<T_1+\triangle_1. \tag{4.6}$$

with boundary conditions [20,21]

$$N_y'Y\left(T_1-\varepsilon\right)+N_y'Y_d-\lambda_y\left(T_1-\varepsilon\right)+F_y'\lambda_y\left(T_1+\varepsilon\right)=0 \tag{4.7}$$

$$\delta_yu_y\left(T_1-\varepsilon\right)-\delta_yC_y+B_y'\lambda_y\left(T_1+\varepsilon\right)=0 \tag{4.8}$$

$$\lambda_y\left(T_1+\triangle_1\right)=0 \tag{4.9}$$

$$H_y'\nu_y+\lambda_y\left(0\right)=0 \tag{4.10}$$

where the vectors $\lambda_y\left(t\right),\ \nu_y$ denote the corresponding Lagrange multipliers. The $u_y\left(t\right)$ optimal control law on the first subinterval $[T_0,\ T_1)$ is given by

$$u_y\left(t\right)=-R_y^{-1}B_y'\lambda_y\left(t\right)-R_y^{-1}K_y'Y\left(t\right), \qquad [T_0,\ T_1) \tag{4.11}$$

whereas the optimal control on the second subinterval is determined by

$$u_y\left(t\right)=-\gamma_y^{-1}B_y'\lambda_y\left(t\right)+C_y, \qquad (\ T_1,\ T_1+\triangle_1]\,. \tag{4.12}$$

Substituting the expression for $u_y\left(T_1-\varepsilon\right)$ from (4.8) into (4.3) and using the expression for $\lambda_y\left(T_1+\varepsilon\right)$given by (4.7), we obtain

$$\begin{bmatrix} Y\left(T_1+\varepsilon\right) \\ \lambda_y\left(T_1+\varepsilon\right) \end{bmatrix} = \mathrm{F}_y^1 \mathrm{F}_y^2 \begin{bmatrix} Y\left(T_1-\varepsilon\right) \\ \lambda_y\left(T_1-\varepsilon\right) \end{bmatrix} + \mathrm{F}_y^1 q_y, \tag{4.13}$$

where

$$\mathrm{F}_y^1 = \begin{bmatrix} E & B'_y \delta_y^{-1} B'_y \\ 0 & E \end{bmatrix}^{-1}, \mathrm{F}_y^2 = \begin{bmatrix} F_y & 0 \\ \left(\mathrm{F}'_y\right)^{-1} N'_y & \left(\mathrm{F}'_y\right)^{-1} \end{bmatrix}, q_y = \begin{bmatrix} B'_y C_y \\ \left(\mathrm{F}'_y\right)^{-1} N'_y Y_d \end{bmatrix} \tag{4.14}$$

and E— is the identity matrix of appropriate dimension.
Combining conditions (4.2), (4.9), and (4.10) yields

$$\begin{bmatrix} H_y & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & H'_y \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} Y(T_0) \\ \lambda_y(T_0) \\ Y(T_1+\triangle_1) \\ \lambda_y(T_1+\triangle_1) \\ \nu_y \end{bmatrix} = \begin{bmatrix} q_y \\ 0 \end{bmatrix}. \tag{4.15}$$

Let the fundamental matrix solutions of systems (4.5) and (4.6) be represented as

$$\begin{bmatrix} Y\left(T_1-\varepsilon\right) \\ \lambda_y\left(T_1-\varepsilon\right) \end{bmatrix} = H_y^1\left(T_1,\ T_0\right) \begin{bmatrix} Y\left(T_0\right) \\ \lambda_y\left(T_0\right) \end{bmatrix} + V_y^1 \tag{4.16}$$

and

$$\begin{bmatrix} Y(T_1+\triangle_1) \\ \lambda_y\left(T_1+\triangle_1\right) \end{bmatrix} = H_y^2\left(T_1+\triangle_1, T_1\right) \begin{bmatrix} Y\left(T_1+\varepsilon\right) \\ \lambda_y\left(T_1+\varepsilon\right) \end{bmatrix} + V_y^2, \tag{4.17}$$

where $H_y^1\left(T_1,\ T_0\right)$ and $H_y^2\left(T_1+\triangle_1, T_1\right)$ denote the corresponding fundamental matrices, while V_y^1, and V_y^2 are the associated forcing vectors.
Substituting expressions (4.13) and (4.16) into (4.17) gives

$$\begin{bmatrix} Y(T_1+\triangle_1) \\ \lambda_y\left(T_1+\triangle_1\right) \end{bmatrix} = H_y^2 F_y^1 F_y^2 H_y^1 \begin{bmatrix} Y\left(T_0\right) \\ \lambda_y\left(T_0\right) \end{bmatrix} + H_y^2 F_y^1 F_y^2 V_y^1 + H_y^2 F_y^1 q_y + V_y^2. \tag{4.18}$$

Combining equations (4.15) and (4.18), we obtain the following system of linear algebraic equations:

$$\begin{bmatrix} -H_y^4 F_y^1 F_y^2 H_y^3 & E & 0 \\ H_y^3 & H_y^4 & H_y^5 \\ H_y^6 & H_y^7 & H_y^8 \end{bmatrix} \begin{bmatrix} Y(T_0) \\ \lambda_y(T_0) \\ Y(T_1+\triangle_1) \\ \lambda_y(T_1+\triangle_1) \\ \nu_y \end{bmatrix} = \begin{bmatrix} H_y^4 F_y^1 F_y^2 V_y^1 + H_y^4 F_y^1 q_y + V_y^2 \\ q_y \\ 0 \end{bmatrix}, \tag{4.19}$$

where

$$H_y^3 = \begin{bmatrix} H_y & 0 \\ 0 & E \end{bmatrix},\ H_y^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ H_y^5 = \begin{bmatrix} 0 \\ H'_y \end{bmatrix},\ H_y^6 = [0\ 0],\ H_y^7 = [0\ \ E],\ H_y^8 = 0. \tag{4.20}$$

After determining $Y\left(T_0\right),\ \lambda_y(T_0)$ from (4.19), we solve the system of differential equations with these initial conditions yielding the state trajectory $Y\left(t\right)$ on the interval $(T_0,\ T_1)$. The corresponding control input $u_y(t)$ on $[T_0,\ T_1)$ is then determined from (4.11). Further,

$Y\left(T_1+\varepsilon\right),\ \lambda_y\left(T_1+\varepsilon\right)$ are computed using (4.13). These values are subsequently used as the initial conditions for solving system (4.6), which yields the state trajectory $Y\left(t\right)$ over the interval $(\,T_1,\ T_1+\triangle_1]$. Consequently, the complete optimal state trajectory and control input for the longitudinal motion and the roll angle are obtained over the entire finite time horizon. Finally, the control input $u_y(t)$ on $(\,T_1,\ T_1+\triangle_1]$ is determined from (4.12).

Accordingly, the following algorithm is proposed for constructing the state trajectory $Y\left(t\right)$ and the corresponding control input $u_y\left(t\right)$:

Algorithm 2.

Step 1. Specify the parameters T_0, T_1, $\triangle_1$,ε, the matrices and vectors A_y, B_y, V_y, q_y, H_y, F_y, Y_d, N_y, δ_y, C_y, Q_y, K_y, R_y, γ_y.

Step 2. Compute the fundamental matrices $H_y^1\left(\,T_1,\ T_0\right)$ and $H_y^2\left(T_1+\triangle_1,T_1\right)$ corresponding to systems (4.5), (4.6), together with the vectors V_z^1, V_z^2 defined by equations (4.16), (4.17).

Step 3. Evaluate the matrices $H_y^3,\ H_y^4,\ \ H_y^5,\ \ H_y^6,\ \ H_y^7,\ \ H_y^8$ according equation (4.20).

Step 4. Solve the system of linear algebraic equations (4.19) to determine the initial vectors $Y\left(T_0\right),\ \lambda_y(T_0)$.

Step 5. The state trajectory $Y\left(t\right)$ and the adjoint variable $\lambda_y(t)$ over the interval $[T_0,\ T_1)$ are obtained by solving system (4.5). Subsequently, the control input $u_y\left(t\right)$ is determined from (4.11).

Step 6. Determine the intermediate values $Y\left(T_1+\varepsilon\right),\ \ \lambda_y\left(T_1+\varepsilon\right)$ from equation (4.13).

Step 7. The state trajectory $Y\left(t\right)$ and the adjoint variable $\lambda_y(t)$ over the interval $(\,T_1,\ T_1+\triangle_1]$ are obtained by solving system (4.6). Subsequently, the control input $u_y\left(t\right)$ is determined from (4.12).

Example 2.

Let us solve the problem (4.1)–(4.4) using Algorithm 2. Assume that

$$A_y=\begin{bmatrix}0&0&0&g\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix},\quad B_y=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad u_y=\tau_\varphi,\ \ H_y=\begin{bmatrix}0\,0\,1\,0\\0\,0\,0\,1\end{bmatrix},V_y=[0\;0\;0\;0]'.$$

In this case, the longitudinal motion is represented by the following graph.

As can be seen from Fig. 3, variations in the vertical coordinate z do not affect the longitudinal coordinate y.

5 Control of the lateral motion x of the flight and the pitch angle θ.

Consider the lateral motion of the quadcopter along the x -axis over the time intervals $[T_0,\ T_1)$ and $(\,T_1,\ T_1+\triangle_1]$ $(T_1>T_0,\ \triangle_1>0)$. The system dynamics are described by the following linear differential equation:

$$\dot{X}=A_xX+B_xu_x+V_x \tag{5.1}$$

with initial conditions

$$H_xX\left(T_0\right)=q_x \tag{5.2}$$

Assume that the following condition holds at time T_1:

$$X\left(T_1+\varepsilon\right)=F_xX\left(T_1-\varepsilon\right)+B_xu_x\left(T_1-\varepsilon\right), \tag{5.3}$$

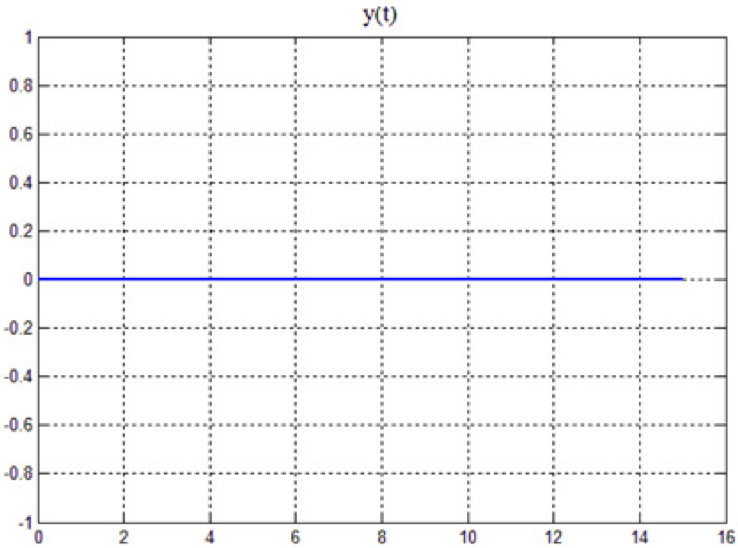

Figure 3. Longitudinal Position versus Time

where $H_x = \begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}$, V_x is a four-dimensional external disturbance vector. $F_x,\ B_x$ are known matrices with corresponding dimensions, ε is an arbitrarily small positive number, $(\,T_1,\ T_1 + \triangle_1]$ denotes the time interval during which, $u_x\,(t)$ remains bounded.

The problem is to find the control input$u_x\,(t)$ and the corresponding trajectory $X\,(t)$ subject to the boundary conditions (5.2) and (5.3), such that equation (5.1) is satisfied and the following cost functional is minimized

$$J_x = \frac{1}{2}\left(X\left(T_1 - \varepsilon\right) - X_d\right)' N_x \left(Y\left(T_1 - \varepsilon\right) - X_d\right) + \left(u_x\left(T_1 - \varepsilon\right) - C_x\right)' \delta_x \left(u_x\left(T_1 - \varepsilon\right) - C_x\right) +$$

$$+\frac{1}{2}\int_{T_0}^{T_1}\left[X'\left(t\right)Q_x X\left(t\right) + 2X'\left(t\right)K_x u_x\left(t\right) + u_x'\left(t\right)R_x u_x\left(t\right)\right]dt +$$

$$+\frac{1}{2}\int_{T_1}^{T_1+\triangle_1}\left(u_x\left(t\right) - C_x\right)'\ \gamma_x\left(u_x\left(t\right) - C_x\right)dt, \tag{5.4}$$

where $N_x \geq 0,\ \delta_x = \delta_x' > 0,\ \ Q_x = Q_x' \geq 0,\ \ R_x = R_x' > 0,\ \ \gamma_x = \gamma_x' > 0,\ \ K_x,\ Z_x$ are known matrices and vectors with corresponding dimension. C_x is known constant parameter.

Based on the results presented in [2, 17], the Euler-Lagrange equations corresponding to equation (5.1) and the functional (5.4) on the intervals $[T_0,\ T_1)$ and $(\,T_1,\ T_1 + \triangle_1])$, respectively, are obtained in the following form:

$$\begin{bmatrix} \dot{X}\left(t\right) \\ \dot{\lambda}_x(t) \end{bmatrix} = \begin{bmatrix} A_x - B_x R_x^{-1} K_x' & -B_x R_x^{-1} B_x' \\ -Q_x' + K_x R_x^{-1} K_x' & -A_x' + K_x R_x^{-1} B_x' \end{bmatrix} \begin{bmatrix} X\left(t\right) \\ \lambda_x(t) \end{bmatrix} + \begin{bmatrix} V_x \\ 0 \end{bmatrix}, T_0 < t < T_1, \tag{5.5}$$

$$\begin{bmatrix} \dot{X}\left(t\right) \\ \dot{\lambda}_x(t) \end{bmatrix} = \begin{bmatrix} A_x & -B_x\gamma_x^{-1}B'_x \\ 0 & -A'_x \end{bmatrix} \begin{bmatrix} X\left(t\right) \\ \lambda_x(t) \end{bmatrix} + \begin{bmatrix} V_x + B_xC_x \\ 0 \end{bmatrix}, T_1 < t < T_1 + \triangle_1 \quad (5.6)$$

with boundary conditions

$$N'_x X\left(T_1 - \varepsilon\right) + N'_x X_d - \lambda_x\left(T_1 - \varepsilon\right) + F'_x\lambda_x\left(T_1 + \varepsilon\right) = 0, \quad (5.7)$$

$$\delta_x u_x\left(T_1 - \varepsilon\right) - \delta_x C_x + B'_x\lambda_x\left(T_1 + \varepsilon\right) = 0, \quad (5.8)$$

$$\lambda_x\left(T_1 + \triangle_1\right) = 0, \quad (5.9)$$

$$H'_x\nu_x + \lambda_x\left(0\right) = 0, \quad (5.10)$$

where the vectors $\lambda_x\left(t\right),\ \nu_x$ denote the associated Lagrange multipliers with corresponding dimension.
The optimal control law over the first subinterval is determined by

$$u_x\left(t\right) = -R_x^{-1}B'_x\lambda_x\left(t\right) - R_x^{-1}K'_xX\left(t\right), \qquad [T_0,\ T_1] \quad (5.11)$$

while the optimal control over the second subinterval is given by

$$u_x\left(t\right) = -\gamma_x^{-1}B'_x\lambda_x\left(t\right) + C_x, \qquad (\ T_1,\ T_1 + \triangle_1]\,. \quad (5.12)$$

Substituting equation (5.8) into (5.4) and employing condition (5.7), we obtain

$$\begin{bmatrix} X\left(T_1 + \varepsilon\right) \\ \lambda_x\left(T_1 + \varepsilon\right) \end{bmatrix} = F_x^1F_x^2 \begin{bmatrix} X\left(T_1 - \varepsilon\right) \\ \lambda_x\left(T_1 - \varepsilon\right) \end{bmatrix} + F_x^1q_x, \quad (5.13)$$

where

$$F_x^1 = \begin{bmatrix} E & B'_x\delta_x^{-1}B'_x \\ 0 & E \end{bmatrix}^{-1}, F_x^2 = \begin{bmatrix} F_x & 0 \\ (\mathrm{F}'_x)^{-1}N'_{\ x} & (\mathrm{F}'_x)^{-1} \end{bmatrix}, q_x = \begin{bmatrix} B'_xC_x \\ (\mathrm{F}'_x)^{-1}N'_{\ x}X_d \end{bmatrix}, \quad (5.14)$$

and E denotes the identity matrix with corresponding dimension.
Combining equations (5.2), (5.9), and (5.10) yields

$$\begin{bmatrix} H_x & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & H'_x \\ 0 & 0 & 0 & E & 0 \end{bmatrix} \begin{bmatrix} X(T_0) \\ \lambda_x(T_0) \\ X(T_1 + \triangle_1) \\ \lambda_x(T_1 + \triangle_1) \\ \nu_x \end{bmatrix} = \begin{bmatrix} q_x \\ 0 \end{bmatrix}. \quad (5.15)$$

Let us the fundamental matrix solutions of systems (5.5) and (5.6) be represented by

$$\begin{bmatrix} X\left(T_1 - \varepsilon\right) \\ \lambda_x\left(T_1 - \varepsilon\right) \end{bmatrix} = H_x^1\left(\ T_1,\ T_0\right) \begin{bmatrix} X\left(\ T_0\right) \\ \lambda_x\left(T_0\right) \end{bmatrix} + V_x^1 \quad (5.16)$$

and

$$\begin{bmatrix} X(T_1 + \triangle_1) \\ \lambda_x\left(T_1 + \triangle_1\right) \end{bmatrix} = H_x^2\left(T_1 + \triangle_1, T_1\right) \begin{bmatrix} X\left(T_1 + \varepsilon\right) \\ \lambda_x\left(T_1 + \varepsilon\right) \end{bmatrix} + V_x^2, \quad (5.17)$$

where $H_x^1\,(\,T_1,\ T_0)$ and $H_x^2\,(T_1+\triangle_1, T_1)$ denote the fundamental matrices of systems (5.5) and (5.6), respectively, V_x^1, and V_x^2 are the corresponding forcing vectors of compatible dimensions.
Substituting equations (5.13) and (5.16) into (5.17) leads to

$$\begin{bmatrix} X(T_1+\triangle_1) \\ \lambda_x\,(T_1+\triangle_1) \end{bmatrix} = H_x^2F_x^1F_x^2H_x^1\ \begin{bmatrix} X\,(T_0) \\ \lambda_x\,(T_0) \end{bmatrix} + H_x^2F_x^1F_x^2V_x^1 + H_x^2F_x^1q_x + V_x^2. \quad (5.18)$$

Combining (5.15) and (5.18), we obtain the following system of linear algebraic equations:

$$\begin{bmatrix} -H_x^4F_x^1F_x^2H_x^3 & E & 0 \\ H_x^3 & H_x^4 & H_x^5 \\ H_x^6 & H_x^7 & H_x^8 \end{bmatrix} \begin{bmatrix} X(T_0) \\ \lambda_x(T_0) \\ X(T_1+\triangle_1) \\ \lambda_x(T_1+\triangle_1) \\ \nu_x \end{bmatrix} = \begin{bmatrix} H_x^4F_x^1F_x^2V_x^1 + H_x^4F_x^1q_x + V_x^2 \\ q_x \\ 0 \end{bmatrix}, \quad (5.19)$$

where

$$H_x^3 = \begin{bmatrix} H_y & 0 \\ 0 & E \end{bmatrix},\ H_x^4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ H_x^5 = \begin{bmatrix} 0 \\ H_x' \end{bmatrix},\ H_x^6 = [0\ 0]\,,\ H_x^7 = [0\ \ E]\,,\ H_x^8 = 0. \quad (5.20)$$

After determining $X(T_0),\ \lambda_x(T_0)$ from (5.19), we solve the system of differential equations (5.5) subject to these initial conditions. This yields the trajectory $X\,(t)$ on the interval $[T_0,\ T_1)$. The control input $u_x(t)$ on $[T_0,\ T_1)$ is determined from (5.11). Subsequently, $X\,(T_1+\varepsilon)\,,\ \ \lambda_x\,(T_1+\varepsilon)$ are obtained using formula (5.13). Next, the system of differential equations (5.6) is solved with these initial conditions, yielding the trajectory $X\,(t)$ on the interval $(\,T_1,\ T_1+\triangle_1]$. Finally, the control input $u_x(t)$ on the interval $(\,T_1,\ T_1+\triangle_1]$ is determined from (5.12).
Thus, the following algorithm can be proposed for determining $X\,(t)$ and $u_x\,(t)$:

Algorithm 3.

Step 1. Specify the parameters T_0, T_1, $\triangle_1$, ε, the matrices and vectors A_x, B_x, V_x, q_x, H_x, F_x, X_d, N_x, δ_x, C_x, Q_x, K_x, R_x, γ_x .
Step 2. Compute the fundamental matrices $H_x^1\,(\,T_1,\ T_0)$, $H_x^2\,(T_1+\triangle_1, T_1)$ corresponding to systems (5.5) and (5.6), together with the vectors V_x^1, V_x^2 defined by equations (5.16) and (5.17).
Step 3. Evaluate the matrices $H_x^3,\ H_x^4,\ \ H_x^5,\ \ H_x^6,\ \ H_x^7,\ \ H_x^8$ according equation (5.20).
Step 4. Solve the system of linear algebraic equations (5.19) to determine the initial vectors $(T_0)\,,\ \lambda_x(T_0)$.
Step 5. The system of differential equations (5.5) is solved to obtain $X\,(t)$ and $\lambda_x(t)$, and the control input $u_x\,(t)$on the interval $[T_0,\ T_1)$ is then determined from (5.11).
Step 6. Determine $X\,(T_1+\varepsilon)\,,\ \ \lambda_x\,(T_1+\varepsilon)$ from equation (5.13).
Step 7. The system of differential equations (5.6) is solved to obtain $X\,(t)$ and $\lambda_x(t)$, and the control input $u_x\,(t)$on the interval $(\,T_1,\ T_1+\triangle_1]$ is then determined from (5.17).

Example 3.

Let us solve the problem (5.1)–(5.4) using Algorithm 2. Assume that

$$A_x = \begin{bmatrix} 0 & 0 & 0 & -g \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B_x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, u_x = \tau_\theta,\ \ H_x = \begin{bmatrix} 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{bmatrix}, V_x = [0\,0\,0\,0]'.$$

The lateral motion is identical to that shown in Fig. 3. Combining all the above motion components yields the three-dimensional trajectory of the quadcopter, as shown in the following figure.

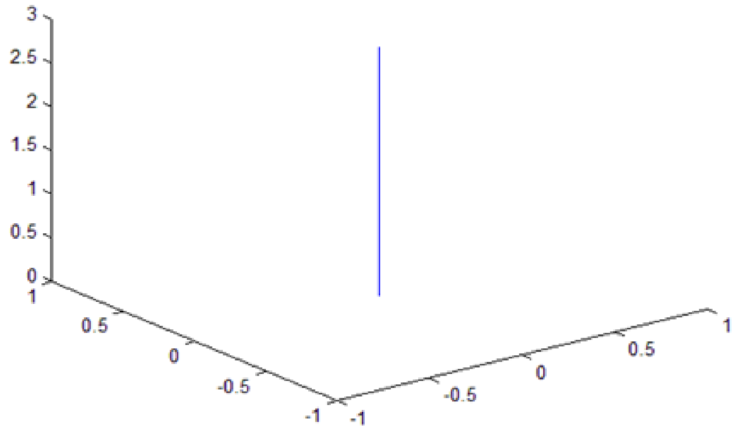

Figure 4. Three-dimensional optimal trajectory of the quadcopter during vertical motion and hovering

6 Conclusion

In this paper, a computational approach to the optimal control of the vertical motion and hovering of a quadcopter over a finite time interval has been developed. The considered problem incorporates equality constraints imposed on the control inputs over prescribed subintervals of the control horizon.The original quadcopter model was decomposed into three independent lower-dimensional subsystems describing the vertical, longitudinal, and lateral motions. For each subsystem, a linear quadratic optimal control problem was formulated and solved by applying the Euler–Lagrange approach. The resulting boundary-value problems were reduced to systems of linear algebraic equations through the use of fundamental matrix solutions. The proposed algorithms can serve as an efficient computational tool for the synthesis of optimal quadcopter trajectories in autonomous flight missions.

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