Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.
Mathematics, **47** (4), 1-7 (2027).

Investigation of resolvent of operator generated by differential expressions with operator coefficients in Hilbert space

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Received: 02.01.2026 / Revised: 12.09.2026 / Accepted: 16.11.2026

Abstract. *One of the main problems of spectral theory of differential operators is the research of completeness of the system of eigen-and adjoint functions of given operator. The completeness of eigen-and adjoint functions for polynomial operator bundles was researched for the first time by M.V.Keldysh [5]. Later on, many authors dealt with these problems and got fundamental results. In this paper such problems were also researched. The theorems on the boundedness of resolvent of operators generated by high order differential expressions with unbounded operator coefficient in Hilbert space were proved. The proved theorems play an important role in research of completeness of the system of eigen-and adjoint elements. In this direction we should note the works of J.E.Allahverdiev [2], M.G.Gasymov [4], S.S.Mirzoev [6], A.A.Shkalikov [7], Sh. Agmon [1], F. Browder [3] and others. The stady of the solvability of partial operator-differential equations in consider in works Aslanov H.I.[3-5], Aslanov H.I., Hatamova R.F. [6], Mirzoev S.S. [10], Mirzoev S.S., Ismailova M.F. [11], Mirzoev S.S., Suleymanov N.M. [12], Ismailova M.F. [15], Jafarov I.Y.[16] and others. In the presented paper, the steady of the properties of the resolvent corresponding to a given equation plays a particularly important role i solving operator-differential equations and investigating the properties of their spectrum. Some important theorems concerning the boundedness of the resolvent are proved in this work.*

Keywords. Hilbert space, differential equation, self-adjoint operator, compact operator, resolvent, eigenvalue, eigen-functions, completeness, boundedness

Mathematics Subject Classification (2010): 35J25, 47A10, 58J40

1 Introduction

Let $H_0, H_1, ..., H_{2m}$ be Hilbert spaces, $H_{i+1} \subset H_i$, $i = 0, 1, ...,$ 2m-1, and also all inclusions are compact.

Let's consider the differential expressions

$$L(x, D)u = \sum_{|\alpha| \leq 2m} A_\alpha(x) D^\alpha u, \quad x \in R^n \tag{1.1}$$

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where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n,$

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}, \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$$

The function $u(x) \in H_{2m}$ is such that $D^\alpha u \in H_{2m-|\alpha|}$ it assumes that for every $x \in R^n$operator $A_\alpha(x), \alpha \neq 0$ is bounded: $H_{2m-|\alpha|} \to H_0, A_0(x) = A_0 + \gamma(x)$ where A_0 is a positive defined selfadjoint operator $H_0 \to H$ such that A_0^{-1}completely continuous.

Complex –valued function $\gamma(x)$ is assuming to be measurable and local bounded. The theorem an boundedness of resolvent of the operator $L(x, D)$ generated by the expression (1.1) is proved.

2 Problem statement

Before the going over the finding the estimate of resolvent of the operator $L(x, D)$ which is main for our purpose, we will formulate the conditions the coefficients of operator $L(x, D)$ must satisfy later.

We will denote

$$R_0(x, \xi) = \left[\sum_{|\alpha|=2m} A_\alpha(x)(i\xi)^\alpha\right]^{-1}, \xi = (\xi_1, \xi_2, ..., \xi_n),$$
$$\xi^\alpha = \xi_1^{\alpha_1}, \xi_2^{\alpha_2}, ..., \xi_n^{\alpha_n}.$$

Let's assume that

1. $R_0(x, \xi)$ for all $\xi \in R^n/0$ is a bounded operator $H_0 \to H_{2m}$, and also

$$\sum_{i=0}^{2m} |\xi|^i \|R_0\|_{H_0 \to H_{2m-i}} \leq \delta_1$$

where $\delta_1 = const$, doesn't depend on x, ξ.

2. There is a ray $l = \{\lambda : \arg \lambda = \beta\}$ of complex plane λ such that the operator $R(x, \xi, \lambda) = \left[\sum_{|\alpha|\leq 2m} A_\alpha(i\xi)^\alpha - \lambda E\right]^{-1}$ is a bounded operator $H_0 \to H_{2m}$, for $\lambda \in l, \xi \in R^n, |x| > c$ and

$$(|\gamma(x)| + |\lambda|) \|R\|_{H_0 \to H_0} + \sum_{i<2m} |\xi|^{2m-i} \|R\|_{H_0 \to H_{2m-i}} \leq \delta_2.$$

3. The values $\sup\limits_{|x-x_0|\leq h} \|A_\alpha(x) - A_\alpha(x_0)\|$, $\sup\limits_{|x-x_0|\leq h} \left|\frac{\gamma(x)-\gamma(x_0)}{\partial(x_0)}\right|$ approach zero for $h \to 0$ evenly by $x_0, |\alpha| = 2m$.

4. $\sum\limits_{0<|\alpha|\leq 2m} \|A_\alpha(x)\| \leq \delta_3$.

Let's set H_i the space with the scalar product:

$$(f, g)_{H_i} = \int_{R^n} (f(x), g(x)_{H_i} dx, \quad f, g \in H_i, i = 0, 1, ..., 2m$$

3 Main Results

The following theorem holds.

Theorem 3.1 *Let the conditions 1-4 fulfilled and*

$$\sum_{|\alpha|\leq 2m}\int_{R^n}\|D^\alpha u\|^2_{H_0}\,dx+\int_{R^n}\left|\gamma^2(x)\right|^2\|u\|^2_{H_0}\,dx<\infty$$

for $\lambda\in l, |\lambda|\geq\lambda_0$.

Then

$$\sum_{|\alpha|\leq 2m}\int_{R^n}\|D^\alpha u\|^2_{H_0}\,dx\leq c\,\|(L-\lambda E)u\|^2_{H_0}$$

where$c=const$, doesn't depend on λ.

Proof. Let's set $(L-\lambda E)u=f$. We will decompose R^n to the system of cubes with edge h such that the union of their interiors concurs with $R^n$and wery point is covered by the finite number of cubes. Let Π_i be some of this systems cubes. P_i is the center of Π_i. Let's consider the partition of unity

$$\sum_{i=1}^{\infty}\theta_i(x)\equiv 1,\quad \theta_i(x)\in\overset{\circ}{C}{}^\infty(\Pi_i)$$

It's easy to see that

$$(L-\lambda E)\theta_i(x)u=f_i \tag{3.1}$$

where

$$f_i=\sum_{\substack{0<|\alpha|\leq 2m\\ |\alpha'|>0,|\beta'|>0,\,\alpha'+\beta'=\alpha}}A_\alpha(x)C_{\alpha'\beta'}D^{\alpha'}\theta_i(x)D^{\beta'}u+f\theta_i(x), C_{\alpha'\beta'}=const \tag{3.2}$$

From (3.2) it follows that

$$A_0(P_i)\theta_i u+\sum_{|\alpha|=2m}A_\alpha(P_i)D^\alpha\theta_i u-\lambda\theta_i u=f_i-\sum_{|\alpha|=2m}A_\alpha(P_i)D^\alpha\theta_i u$$

$$-\sum_{|\alpha|=2m}(A_\alpha(x)-A_\alpha(P_i))D^\alpha\theta_i u=F_i(x) \tag{3.3}$$

Let's set $\theta_i u=v_i$. We will apply the Fourier transform by x to both sides of the equality (3.3). It is possible because $v_i(x)$ have compact support. As a result we get:

$$\left[A_p(P_i)+\sum_{|\alpha|=2m}A_\alpha(P_i)(i\xi)^\alpha-\lambda\right]\tilde{v}_i(\lambda)=\tilde{F}_i(\lambda) \tag{3.4}$$

From the assumption 2 for $\lambda\in l$ we get:

$$(|\gamma(P_i)|+|\lambda|)\,\|\tilde{v}_i(\lambda)\|_{H_0}+\|\tilde{v}_i(\lambda)\|_{H_{2m}}\leq c_2\left\|\tilde{F}_i(\lambda)\right\|_{H_0}$$

$$\leq c_3\left\|\tilde{f}_i(\lambda)\right\|_{H_0}+c_3\,\|v_i\|_{H_{2m-1}}+\varepsilon(h)\,\|v_i\|_{H_{2m}} \tag{3.5}$$

where $\varepsilon(h) \to 0$ for $h \to 0$.

Since the inclusions $H_i \subset H_{i-1}$ are compact

$$\|w\|_{H_{2m-1}} \leq \varepsilon \|w\|_{H_{2m}} + c(\varepsilon) \|w\|_{H_{01}}, \ \forall \varepsilon > 0. \tag{3.6}$$

Choosing $\varepsilon = \frac{1}{2}c_3$ from (3.5) for guide small h we will get:

$$(|\gamma(P_i)| + |\lambda|) \|\tilde{v}_i(\lambda)\|_{H_0} + \|\tilde{v}_i(\lambda)\|_{H_{2m}} \leq c_4 \left\|\tilde{f}(\lambda)\right\|_{H_0}$$

if $|\lambda|$ is guide large.

Hence and from Parseval's equality we get

$$(|\gamma(P_i)|^2 + |\lambda^2|) \int_{\Pi_i} \|v_i\|^2_{H_0} \, dx + \int_{\Pi_i} \|v_i\|^2_{H_{2m}} \, dx \leq c \int_{\Pi_i} \|f_i\|^2_{H_0} \, dx.$$

From this inequality it follows:

$$\int_{\Pi_i} (|\gamma(x)|^2 + |\lambda^2|) \|v_i\|^2_{H_0} \, dx + \int_{\Pi_i} \|v_i\|^2_{H_{2m}} \, dx$$

$$\leq c(h) \int_{\Pi_i} \|u\|^2_{H_{2m-1}} \, dx + c \int_{\Pi_i} \|f\|^2_{H_0} \, dx. \tag{3.7}$$

Let's sum up inequalities (3.7) by all $i = 1, 2, ...$ As a result we will get:

$$\int_{R^n} (|\gamma(x)|^2 + |\lambda^2|) \|u\|^2_{H_0} \, dx + \int_{R^n} \|u(x)\|^2_{H_{2m}} \, dx$$

$$\leq c(h) \int_{R^n} \|u\|^2_{H_{2m-1}} \, dx + c \|f\|^2_{H_0} \, dx. \tag{3.8}$$

For estimation of the first term on the right side of the inequality (3.8) we use (3.6). As a result choosing $\varepsilon > 0$ in (3.6) such that $\varepsilon c(h) = \frac{1}{2}$ we will get:

$$\int_{R^n} (|\gamma(x)|^2 + |\lambda^2|) \|u\|^2_{H_0} \, dx + \int_{R^n} \|u\|^2_{H_{2m}} \, dx \leq$$

$$\leq c \|f\|^2_{H_0} \, dx. + c(\varepsilon, h) \int_{R^n} \|u\|^2_{H_0} \, dx. \tag{3.9}$$

If $|\lambda|^2 > 2c(\varepsilon, h)$, then from (3.9) required inequality (1.1) follows. The theorem 1 is proven.

Theorem 3.2 *If the conditions of Theorem 3.1 are fulfilled then the operator* $(L - \lambda E)^{-1}$ *is a bounded operator* $H_0 \to H_0$ *for every* $\lambda \in l, |\lambda| \geq \lambda_0$.

Proof. It is enough to prove only the existence of the solution of equation

$$(L(x, D))u - \lambda u = f(x) \tag{3.10}$$

for every $f(x) \in H_0$.

Let $\theta_i(x)$ be the same partition of unity which was used in the proof of Theorem 3.1, $f_i = \theta_i f$.

Consider the equation:

$$\sum_{|\alpha|=2m} A_\alpha(P_i) D^\alpha w_i + A_0(P_i) w_i = f_i(x)$$

It's solution exits and can be defined by formula

$$\tilde{w}_i = \left[\sum_{|\alpha|=2m} A_\alpha(P_i)(i\xi)^\alpha + A_0(P_i) \right]^{-1} \tilde{f}_i(\lambda)$$

Let $\sigma_i(x)$ be some partition of unity such that $\sigma_i(x)\psi_i \equiv \psi_i$. Such partition of unity exists, it is used a lot in the theory of elliptic boundary-value problems (see [8])

Let's define a function

$$w(x) = \sum_{i=1}^{\infty} \sigma_i(x) w_i = L(f)$$

It is easy to see that

$$(L - \lambda E)w = \sum_{i=1}^{\infty} \left[\sum_{|\alpha|=2m} \sigma_i(x) A_\alpha(x) D^\alpha w_i \right.$$

$$\left. + \sum_{\substack{\alpha'+\beta'=\alpha \\ |\alpha'|>0, |\beta'|>0, |\alpha|<2m}} d_{\alpha'\beta'}(x) A_\alpha(x) D^{\beta'} w_i + A_0(x)\sigma_i w_i - \lambda \sigma_i(x) w_i \right]$$

$$= \sum_{i=1}^{\infty} \left[\sigma_i f_i + \sum_{|\alpha|=2m} \sigma_i [A_\alpha(x) - A_\alpha(x_i)]\, D^\alpha w_i \right.$$

$$\left. + \sum_{\substack{\alpha'+\beta'=\alpha \\ |\alpha|>0, |\beta'|>0, |\alpha|<2m}} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i + (A_0(x) + A_0(x_i))\sigma_i w_i \right]$$

$$= f + \sum_{i=1}^{\infty} \left[\sum_{|\alpha|=2m} \sigma_i(x) [A_\alpha(x) - A_\alpha(x_0) D^\alpha w_i \right] D^\alpha w_i$$

$$+ \sum_{\substack{\alpha'+\beta'=\alpha \\ |\alpha'|>0, |\beta'|>0, |\alpha|<2m}} A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i + (A_0(x) - A_0(x_i))\sigma_i w_i = f + Tf. \quad (3.11)$$

Here $d_{\alpha'\beta'}(x)$ are infinitely differentiable functions with a support in Π_i, T is operator $H_0 \to H_0$.

Let's estimate $\|T\|$. For $|\alpha| = 2m$ we have:

$$\|\sigma_i(x)(A_0(x) - A_\alpha(x_0)) D^\alpha w_i\|_{H_0} \le c \int_{\Pi_i} \|D^\alpha w_i\|^2_{H_0}\, dx$$

It $h$is quite small then from III it follows:

$$(|\gamma(x_i)| + |\lambda|)\, \|w_i\|_{H_0} + \|w_i\|_{H_{2m}} \le c\, \|f\|_{H_0} \quad (3.12)$$

It means that

$$\|\sigma_i(x)[A_0(x) - A_\alpha(x_0)] D^\alpha w_i\|_{H_0} \le c\varepsilon\, \|f_i\|_{H_0} \quad (3.13)$$

Besides

$$\left\| A_\alpha(x) d_{\alpha'\beta'}(x) D^{\beta'} w_i \right\|_{H_0} \le c\varepsilon \left\| w_i \right\|_{H_{2m-1}} \le$$

$$\le \varepsilon \left\| w_i \right\|_{H_{2m}} + c_\varepsilon \left\| w_i \right\|_{H_0} \le \varepsilon \left\| f \right\|_{H_0} + (1+|\lambda|)^{-1} c_\varepsilon \left\| f_i \right\|_{H_0}. \tag{3.14}$$

Finally

$$\left\| A_0(x) - A_0(x_i)\sigma_i w_i \right\|_{H_0} \le c\varepsilon \left\| w_i \right\|_{H_0} \tag{3.15}$$

Note that the sum by *i* in (3.11) is local finite, i.e. for wery *x* the sum has a finite number of non-zero terms.

From (3.13)-(3.15) it follows that $\|T\| \le \frac{1}{2}$ if *h* is quite small. Thus, $(L - \lambda E)w = f + Tf$.

Let $T\varphi + \varphi = f$. Such φexist by virtue of Banach theorem. It's obvious that $L(\varphi)$ is a solution of the equation (3.10). Indeed, $(L - \lambda E)\varphi = T\varphi + \varphi = f$. So the solvability of the equation (3.10) is proved.

The theorem is proved.

the authors express their deep qratitude to professors M.Bayramoqlu and S.S. Mirzoyev for useful discussions of the results obtained in the paper.

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