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The local density and the local weak density in the space of idempotent probability measures

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Abstract. *In this work, the local weak density and the local density of the space of idempotent probability measures of an infinite compact space are studied. It is proved that the local weak density of the space of idempotent probability measures with a finite support $I_n(X)$ of an infinite compact space X does not exceed from the local weak density of an infinite compact space X. It was also proved that the local density of the space of idempotent probability measures with a finite support $I_n(X)$ of an infinite compact space X does not exceed from the local density of an infinite compact space X.*

Keywords. Local density, local weak density, idempotent probability measures, idempotent probability measures with a finite support.

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1 Introduction

Let X be a compact Hausdorf space. As usual, let $\mathcal{C}(X)$ be the Banach space of continuous functions on X equipped with the sup-norm. For an arbitrary $c \in \mathbb{R}$, we denote by c_X the constant function on X taking the value c. We also denote by $w(X)$ the weight of a topological space X.

In what follows, by a functor we mean a covariant functor. The probability measure functor on the category **Comp** is denoted by *P* (for its properties, see, for example, [12]).

We denote by $\exp$ the hyperspace functor acting on the category **Comp**. For every compact Hausdorff space X, the space $\exp X$ consists of all non-empty closed subsets of X and is equipped with the Vietoris topology. A base of this topology is formed by the sets

$$\langle U_1, ..., U_n \rangle = \{A \in \exp X \mid A \subset \bigcup_{i=1}^{n} U_i,\ A \cap U_i \neq \emptyset,\ i = 1, ..., n\},$$

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where the sets $U_1, ..., U_n$ are open in X, $n \in \mathbb{N}$. If $f : X \to Y$ is a continuous map, then the map $\exp f : \exp X \to \exp Y$ is defined by the formula $\exp f(A) = f(A)$, $A \in \exp X$.

We equip the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with the metric ϱ defined by the formula $\varrho(x, y) = |e^x - e^y|$ (with the convention that $e^{-\infty} = 0$). We also assume that $\mathbb{R}^n_{\max} = (\mathbb{R}_{\max})^n$.

A functor on the category **Comp** is said to be *open* if it preserves the class of open surjective maps. We recall that a map of topological spaces is said to be *open* if the image of every open set is open. A surjective map $f : X \to Y$ between compact Hausdorff spaces is open if and only if the map $y \mapsto f^{-1}(y) : Y \to \exp X$ is continuous. Moreover, if X and Y are metrizable, then f is open if and only if for an arbitrary sequence $(y_i)_{i=1}^{\infty}$ convergent to $y \in Y$ and any $x \in X$ such that $f(x) = y$ there is a sequence $(y_i)_{i=1}^{\infty}$ convergent to x and such that $f(x_i) = y_i$ for every i.

Following the style of notation adopted in idempotent mathematics (see, for example, [6], [7], [8]), we denote by $\odot : \mathbb{R} \times \mathcal{C}(X) \to \mathcal{C}(X)$ the map acting by the formula $(\lambda, \varphi) \mapsto \lambda_X + \varphi$, and $\oplus : \mathcal{C}(X) \times \mathcal{C}(X) \to \mathcal{C}(X)$ the map of pointwise maximum,$(\varphi, \psi) \mapsto \max\{\varphi, \psi\}$.

Definition 1.1 *A functional $\mu : \mathcal{C}(X) \to \mathbb{R}$ is said to be an idempotent measure (or Maslov measure) if the following conditions hold for any $\varphi, \psi \in \mathcal{C}(X)$:*

1) $\mu(c_X) = c$;

2) $\mu(c \odot \varphi) = c \odot \mu(\varphi)$;

3) $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$.

The number $\mu(\varphi)$ (which is also denoted by $\int_X^{\oplus} \varphi d\mu$) is referred to as the *Maslov integral* of the function $\varphi \in \mathcal{C}(X)$ with respect to the measure μ.

We denote by $I(X)$ the set of all idempotent measures on X. The set $I(X)$ is equipped with the weak topology. A base of this topology is formed by the sets

$$\langle \mu : \varphi_1, ..., \varphi_n; \varepsilon \rangle = \{\nu \in I(X) \mid |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, ..., n\},$$

where $\mu \in I(X)$, $\varphi_i \in \mathcal{C}(X)$, $i = 1, ..., n$, and $\varepsilon > 0$.

It is worth giving an example of an idempotent measure. Let $x_1, ..., x_n$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}_{\max}$ have the property $\max\{\lambda_1, ..., \lambda_n\} = 0$. We define $\mu : \mathcal{C}(X) \to \mathbb{R}$ by putting $\mu(\varphi) = \max\{\varphi(x_i) + \lambda_i \mid i = 1, ..., n\}$. As usual, for every $x \in X$ we denote by δ_x (or$\delta(x)$) the functional on $\mathcal{C}(X)$ defined by the formula $\delta_x(\varphi) = \varphi(x)$, $\varphi \in \mathcal{C}(X)$ (the Dirac probability measure concentrated at the point x). Then we can write $\mu = \bigoplus_{i=1}^{n} \lambda_i \odot \delta_{x_i}$.

A support $\operatorname{supp}(\mu)$ of a measure $\mu \in I(X)$ is the smallest closed subset $F \subset X$ such that $\mu(F) = \mu(X)$. For a compact X and for a natural number n we put [5]

$$I_n(X) = \{\mu \in I(X) : \ |\operatorname{supp}(\mu)| \leq n\}, \quad I_\omega(X) = \cup\{I_n(X) : \ n = 1, 2, \ldots\}.$$

It is clear that the space $I_n(X)$ is close in the space $I(X)$ for a compact X and for a $I(f)(I_n(X)) \subset I_n(Y)$, where $f : X \to Y$ is a continuous function.

The weak density of a topological space X is defined as the least cardinal $\tau \geq \aleph_0$ such that X has a $\pi-$base B decomposed into τ systems of open sets with the finite intersection property; i.e., $B = \cup\{B_\alpha : \alpha \in A\}$, where B_α is a system of open sets with the finite intersection property for each $\alpha \in A$ and $|A| = \tau$. The weak density of a topological space X is denoted by $wd(X)$. If $wd(X) = \aleph_0$, then X is said to be weakly separable [2].

Proposition 1.1 *[1]. If $wd(X) \leq \tau$ and $f : X \to Y$ is a continuous mapping of X onto Y, then $wd(Y) \leq \tau$.*

Proposition 1.2 *[3]. If $wd(X) \leq \tau$ and G is an arbitrary non-empty open subset of the space X, then $wd(G) \leq \tau$.*

A topological space X is said to be locally weakly τ-dense at the point $x \in X$, if τ is the smallest cardinal number such that x has a neighborhood of weak density τ in X [13]. Local weak density at the point x is denoted by $lwd(x)$. The local weak density of the space X is a supremum of all cardinal numbers $lwd(x)$ for $x \in X$ and denoted by $lwd(X) = \sup\{lwd(x) : x \in X\}$. If $lwd(X) = \aleph_0$, then X is said to be locally weakly separable. It is known that $lwd(X) \leq wd(X)$ for any topological space.

In [11] shown that the following statements hold.

Theorem 1.1 *The Cartesian product $X = \prod_{i=1}^{n} X_i$ is a locally weakly τ-dense if and only if X_i is a locally weakly τ-dense for all $i = 1, 2, \ldots, n$.*

Proposition 1.3 *Let X_α is a locally weakly τ-dense for any $\alpha \in A$. Then $X = \oplus\{X_\alpha : \alpha \in A\}$ is a locally weakly τ-dense.*

A subset D of a topological space X is called a dense set in X, if $[D] = X$. Define the density $d(X)$ of X by $d(X) = \min\{|D| : D \text{ is a dense subset of } X\}$ [4]. If $d(X) = \aleph_0$, then X is said to be separable.

For a density of topological space following statements are hold [4].

Theorem 1.2 *If $d(X) \leq \tau$ and $f : X \to Y$ is a continuous mapping of X onto Y, then $d(Y) \leq \tau$.*

Proposition 1.4 *If $d(X) \leq \tau$ and G is an arbitrary non-empty open subset of the space X, then $d(G) \leq \tau$.*

We say that the local density of a topological space X is τ at the point x, if τ is the smallest cardinal number such that x has a neighborhood of density τ in X. The local density at the point x is denoted by $ld(x)$. The local density of the topological space X is defined as the supremum of all numbers $ld(x)$ for $x \in X$: $ld(X) = \sup\{ld(x) : x \in X\}$ [14]. If $ld(X) = \aleph_0$, then X is said to be locally separable. It is known that $ld(X) \leq d(X)$ for any topological space.

For an infinity T_1-space following statements are hold.

Theorem 1.3 *[10]. The Cartesian product $X = \prod_{i=1}^{n} X_i$ is a locally τ-dense, if X_i is a locally τ-dense for each $i = 1, 2, \ldots, n$.*

Proposition 1.5 *[9]. Let X_α is a locally τ-dense for any $\alpha \in A$. Then $X = \oplus\{X_\alpha : \alpha \in A\}$ is a locally τ-dense.*

The results obtained in this paper for the functors I_n and I_ω acting on the category **Comp** help us study the cardinal properties of the extensions of these functors to the category **Tych** of Tychonoff spaces.

2 Main result

Theorem 2.1 *Let X be an infinite compact space. Then $ld(I_n(X)) \leq ld(X)$.*

Proof. Let X be an infinite compact space and let $ld(X) = \tau \geq \aleph_0$. Then by the Theorem 1.3 we obtain $ld(X^n) = \tau$. In [15], M.M. Zarichnyi shows that the space $I_n(X)$ can be represented as a continuous image of the space $X \times \sigma^{n-1}$, where σ^{n-1} is the $(n-1)$-dimensional simplex. The mapping $\pi : X \times \sigma^{n-1} \to I_n(X)$ is defined by the formula $\pi(x_1, \ldots, x_n, m_1, \ldots, m_n) = \sum_{i=1}^{n} m_i \delta_{x_i}$, where $(m_1, \ldots, m_n) \in \sigma^{n-1}$, $\sum_{i=1}^{n} m_i = 1$ and $m_i \geq 0$ for all $i \in N$, δ_{x_i} are Dirac's measures at the points x_i, respectively. The mapping π is perfect, since π is continuous mapping defined on compact $X \times \sigma^{n-1}$. Take an arbitrary point $\mu \in I_n(X)$. Then the set $\pi^{-1}(\mu) \subset X \times \sigma^{n-1}$ is compact in $X \times \sigma^{n-1}$. For every

point $\xi \in \pi^{-1}(\mu)$ there exists a neighborhood $O\xi$ of ξ such that $d(O\xi) \leq \tau$. The family of all these neighborhoods covers the set $\pi^{-1}(\mu)$. Since $\pi^{-1}(\mu)$ is compact, there is a finite sequence $O\xi_1, O\xi_2, \ldots, O\xi_n$ of open sets such that $\pi^{-1}(\mu) \subset \bigcup_{i=1}^{n} O\xi_i$ and $d(O\xi_i) \leq \tau$ for all $i = 1, 2, \ldots, n$. Assume $G = \bigcup_{i=1}^{n} O\xi_i$. Then we have $d(G) \leq \tau$ and $X \times \sigma^{n-1} \backslash G \subset X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)$. This implies $\pi\left(X \times \sigma^{n-1} \backslash G\right) \subset \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. Let's show that $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. We suppose that it does not hold, i.e. $\mu \in \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. In this case, there is a point $\xi \in X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)$ such that $\pi(\xi) = \mu$. But it is impossible, since $\xi \notin \pi^{-1}(\mu)$. Therefore $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$ and, a fortiori, $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash G\right)$. The set $\pi\left(X \times \sigma^{n-1} \backslash G\right)$ is closed as the image of a closed set under the closed mapping. Hence, the set $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)$ is an open set containing μ. We shall show $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right) \subset \pi(G)$. Indeed, since $X \times \sigma^{n-1} \backslash G \subset \pi^{-1}\left(\pi\left(X \times \sigma^{n-1} \backslash G\right)\right)$, we obtain

$$\pi^{-1}\left(I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)\right) = X \times \sigma^{n-1} \backslash \pi^{-1}\left(\pi\left(X \times \sigma^{n-1} \backslash G\right)\right) \subset$$

$$\subset X \times \sigma^{n-1} \backslash \left(X \times \sigma^{n-1} \backslash G\right) = G.$$

Therefore $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right) \subset \pi(G)$. Hence, $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)$ is an open subset of $\pi(G)$. Since $d(G) \leq \tau$, by theorem 1.2 we obtain $d(\pi(G)) \leq \tau$. Then by proposition 1.4 we have $d\left(I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)\right) \leq \tau$. In this way, we found the neighborhood $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)$ of density $\leq \tau$ for an arbitrarily chosen point $\mu \in I_n(X)$. Therefore $ld(I_n(X)) \leq \tau$. The theorem 2.1 is proved.

Proposition 2.1 *Let X be an infinite compact space. Then $ld(I_\omega(X)) \leq ld(X)$.*

Proof. Let X be an infinite compact space and let $ld(X) = \tau \geq \aleph_0$. By the theorem 2.1 we obtain $ld(I_n(X)) \leq ld(X) = \tau$. Since $I_n(X)$ is a locally τ-dense for any $n \in \mathbf{N}$. By the proposition 1.5 we obtain the space $I_\omega(X)$ is a locally τ-dense. Hence $ld(I_\omega(X)) \leq \tau$. The proposition 2.1 is proved.

Theorem 2.2 *Let X be an infinite compact space. Then $lwd(I_n(X)) \leq lwd(X)$.*

Proof. Let X be an infinite compact space and let $lwd(X) = \tau \geq \aleph_0$. Then by the theorem 1.1 we have $lwd(X^n) = \tau$. In [15], M.M. Zarichnyi shows that the space $I_n(X)$ can be represented as a continuous image of the space $X \times \sigma^{n-1}$, where σ^{n-1} is the $(n-1)$-dimensional simplex. The mapping $\pi : X \times \sigma^{n-1} \to I_n(X)$ is defined by the formula $\pi(x_1, \ldots, x_n, m_1, \ldots, m_n) = \sum_{i=1}^{n} m_i \delta_{x_i}$, where $(m_1, \ldots, m_n) \in \sigma^{n-1}$, $\sum_{i=1}^{n} m_i = 1$ and $m_i \geq 0$ for all $i \in N$, δ_{x_i} are Dirac's measures at points x_i, respectively. The mapping π is perfect, insofar as π is continuous mapping defined on compact $X \times \sigma^{n-1}$. Let us take an arbitrary point $\mu \in I_n(X)$. Then the set $\pi^{-1}(\mu) \subset X \times \sigma^{n-1}$ is compact in $X \times \sigma^{n-1}$. For any point $\xi \in \pi^{-1}(\mu)$ there exists a neighborhood $O\xi$ of ξ such that $wd(O\xi) \leq \tau$. The family of all these neighborhoods covers the set $\pi^{-1}(\mu)$. Seeing $\pi^{-1}(\mu)$ is compact, there is a finite sequence $O\xi_1, O\xi_2, \ldots, O\xi_n$ of open sets such that $\pi^{-1}(\mu) \subset \bigcup_{i=1}^{n} O\xi_i$ and $wd(O\xi_i) \leq \tau$ for all $i = 1, 2, \ldots, n$. Suppose $G = \bigcup_{i=1}^{n} O\xi_i$. Then we have $wd(G) \leq \tau$ and $X \times \sigma^{n-1} \backslash G \subset X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)$. Therefore $\pi\left(X \times \sigma^{n-1} \backslash G\right) \subset \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. Let's show that $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. Assume, it does not hold, i.e. $\mu \in \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$. In this case, there is a point $\xi \in X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)$ such that $\pi(\xi) = \mu$. But it is impossible, insofar as $\xi \notin \pi^{-1}(\mu)$.

Hence, $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash \pi^{-1}(\mu)\right)$ and, all the more so, $\mu \notin \pi\left(X \times \sigma^{n-1} \backslash G\right)$. The set $\pi\left(X \times \sigma^{n-1} \backslash G\right)$ is closed as the image of a closed set under a closed mapping. Therefore, the set $I_n(X) \backslash \pi\left(X \times \sigma^{n-1} \backslash G\right)$ is an open set containing μ. We must show that

$I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)\subset\pi(G)$. Indeed, since $X\times\sigma^{n-1}\setminus G\subset\pi^{-1}\left(\pi\left(X\times\sigma^{n-1}\setminus G\right)\right)$, we have

$$\pi^{-1}\left(I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)\right)=X\times\sigma^{n-1}\setminus\pi^{-1}\left(\pi\left(X\times\sigma^{n-1}\setminus G\right)\right)\subset$$

$$\subset X\times\sigma^{n-1}\setminus\left(X\times\sigma^{n-1}\setminus G\right)=G.$$

This implies that $I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)\subset\pi(G)$. Hence, $I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)$ is an open subset of $\pi(G)$. Insofar as $wd(G)\leq\tau$, by proposition 1.1 we have $wd(\pi(G))\leq\tau$. Then by proposition 1.2 we obtain $wd\left(I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)\right)\leq\tau$. In this way, we found the neighborhood $I_n(X)\setminus\pi\left(X\times\sigma^{n-1}\setminus G\right)$ of weak density $\leq\tau$ for arbitrarily chosen point $\mu\in I_n(X)$. Hence, $lwd(I_n(X))\leq\tau$. The inequality $lwd(I_n(X))\leq lwd(X)$ is proved.

Proposition 2.2 *Let X be an infinite compact space. Then $lwd(I_\omega(X))\leq lwd(X)$.*

Proof. Let X be an infinite compact space and let $lwd(X)=\tau\geq\aleph_0$. By the theorem 2.2 we have $lwd(I_n(X))\leq lwd(X)=\tau$. Since $I_n(X)$ is a locally weakly τ-dense for any $n\in\mathbf{N}$, by the proposition 1.3 we obtain the space $I_\omega(X)$, which is a locally weakly τ-dense. Hence, $lwd(I_\omega(X))\leq\tau$. The proposition 2.2 is proved.

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