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Quality Properties Of Solutions Of The Basic Equation Of Perturbation Theory For One-Dimensional Magnetic Schrodinger Operator

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Abstract. *By means of the quadratic forms method under certain conditions on magnetic and electric potential a self-adjoint and one-dimensional Schrodinger operator was constructed. Smoothness and behavior at infinity of the solutions of the basic equation of perturbation theory for one-dimensional magnetic Schrodinger operator was studied.*

Keywords. magnetic Schrodinger operator · quadratic form · perturbation theory · magnetic potential.

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1. Self-adjointness

In this item, in the space $L_1(R_1)(R_1 = (-\infty, +\infty)$ we study the self-adjointness of one-dimensional magnetic Schrodinger operator generated by the differential expression

$$\Delta_{a,V} = \left(\frac{1}{i}\frac{d}{dx} + a(x)\right)^2 + V(x), \tag{1}$$

where $a(x)$ and $V(x)$ are magnetic and electric potentials, respectively, and these potentials are real functions satisfying the following conditions:

a) $\Phi(x) \equiv a^2(x) + V(x) + ia'(x) \in L_1(R_1)$;

b) $a(x) \in L_1(R_1)$.

Subject to conditions a) and b) differential expression (1) may be written in the form

$$\Delta_{a,V} = -\frac{d^2}{dx^2} + W,$$

where

$$W = -2i\frac{d}{dx}a(x) + \Phi(x). \tag{2}$$

It is known that if $a(x)$ and $V(x)$ are sufficiently smooth bounded functions, then minimal (in this case they are also maximal) operators H_0 and $H = H_0 + W$ that correspond to differential expressions $-\dfrac{d^2}{dx^2}$ and $\Delta_{a,V}$, respectively, are self-adjoint operators in $L_2(R_1)$ with the same domains of definition $W_2^2(R_1)$ (second order Sobolev space). Generally speaking, under conditions a) and b), the differential expression

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$\varDelta_{a,V}$ doesn't determine the minimal operator on the linear manifold $C_0^\infty(R_1)$. For constructing a self-adjoint operator by means of this expression, we will use the method of quadratic forms. To this end, recall some denotation and notation (see for detailed information the books [1, p. 303], [2, p. 185], [3, p. 386]).

Let E be Hilbert space and the linear manifold $Q(q)$ be dense in E. Denote by $q(\varphi,\psi)$ a complex-valued one-and-a half linear form with domain of definition $Q(q)$, and by $q(\varphi) = q(\varphi,\varphi)$ a quadratic form associated with $q(\varphi,\psi)$. If the one-and-a half form $q(\varphi,\psi)$ is generated by some linear operator A, i.e.

$$\forall\varphi \in Q(q),\ \forall\psi \in D(A) \Longrightarrow q(\varphi,\psi) = (\varphi, A\psi),$$

then domain of its definition is denoted by $Q(q) = Q(A)$.

Definition 1. *Let the operator A be selfadjoint and lower bounded. The symmetric operator B is said to be A -bounded in the sense of forms if*

i) $Q(A) \subseteq Q(B)$,

ii) $\exists a, b > 0, \forall\varphi \in Q(A) \Longrightarrow |(\varphi, B\varphi)| \le a\,(\varphi, A\varphi) + b(\varphi,\varphi)$.

Then the lower bound of all such a is called A -bound of the operator B in the sense of forms.

Let us consider in $L_2(R_1)$ the quadratic forms

$$h_0(\varphi) = \int_{-\infty}^{+\infty} \left|\varphi'\right|^2 dx,$$

$$h_{a,V}(\varphi) = h_0(\varphi) + (W\varphi,\varphi),$$

where W is an operator acting by formula (2). Obviously, $h_0(\varphi)$ corresponds to the self-adjoint operator $H_0 := -\dfrac{d^2}{dx^2}$ with domain of definition $W_2^2(R_1)$. It is known that $Q(h_0) = W_2^1(R_1) = D(H_0^{1/2})$ (first order Sobolev space), and $\forall\varphi\in Q(h_0), h_0(\varphi) = (H_0^{1/2}\varphi, H_0^{1/2}\varphi)$.

Theorem 1. *Let conditions a) and b) be fulfilled. Then there exists a unique lower bounded self-adjont operator $H = H_0 + W$ responding to the form $h_{a,V}(\varphi) = h_0(\varphi) + (W\varphi,\varphi)$ with $Q(H_0) = Q(H)$ such that any essential domain of the operator H_0 is also an essential domain for the operator H. In particular, the space of basic functions $C_0^\infty(R_1)$ is an essential domain of the operator H.*

Before we pass to the proof of theorem 1, note that the sum $H_0 + W$ is understood in the sense of forms and it may differ from the operator sum.

Proof of theorem 1. Obviously, the operator W acting by formula (2) is symmetric. Show $Q(H_0) \subseteq Q(W)$. Take an arbitrary element φ from $Q(H_0)$. From the equality $Q(H_0) = W_2^1(R_1)$ it follows that $\varphi \in AC_{loc}(R_1) \cap L_\infty(R_1)$, i.e. φ is a locally absolute function, and $\varphi(\pm\infty) = 0$. Taking into attention conditions a) and b), from the equality

$$a(x) = a(x_0) + \int_{x_0}^{x} a'(t)dt$$

it follows that $a(x) \in AC_{loc}(R_1) \cap L_\infty(R_1)$.

From condition a) and $\varphi \in AC_{loc}(R_1) \cap L_\infty(R_1)$ we have:

$$\begin{aligned} \left|\int_{-\infty}^{+\infty} \varPhi(x)\overline{\varphi(x)}dx\right| &\le \max_{-\infty<x<+\infty} |\varphi(x)| \int_{-\infty}^{+\infty} |\varPhi(x)|\,dx \\ &= \|\varphi\|_{L_\infty(R_1)} \|\varPhi(x)\|_{L_1(R_1)} < +\infty. \end{aligned} \tag{3}$$

From the equality

$$\int_{-\infty}^{+\infty} (a(x)\varphi(x))'\overline{\varphi(x)}dx = -\int_{-\infty}^{+\infty} (a(x)\varphi(x))\overline{\varphi'(x)}dx$$

and the Schwatz inequality we get:

$$\left|\int_{-\infty}^{+\infty} -2i(a(x)\varphi(x))'\overline{\varphi(x)}dx\right| \leq 2\int_{-\infty}^{+\infty}\left|a(x)\left\|\varphi(x)\right\|\overline{\varphi'(x)}\right|dx$$
$$\leq 2\left\|a\right\|_{L_\infty(R_1)}\left\|\varphi(x)\right\|_{L_2(R_1)}\left\|\varphi'(x)\right\|_{L_2(R_1)} < +\infty. \tag{4}$$

Then from inequalities (3) and (4) it follows that $\forall\varphi\in Q(H_0)$ the expression

$$(W\varphi,\varphi) = \int_{-\infty}^{+\infty}(W\varphi(x))\overline{\varphi(x)}dx$$

$$= -\int_{-\infty}^{+\infty}\left\{-2i(a(x)\varphi(x))' + \left[a^2(x)+V(x)+ia'(x)\right]\varphi(x)\right\}\overline{\varphi(x)}dx$$

has a meaning. This means that $\varphi\in Q(W)$, whence it follows that $Q(H_0)\subseteq Q(W)$.

Conditions a) and b) yield that the operator

$$W = -2i\frac{d}{dx}a(x) + \varPhi(x) = -2ia(x)\frac{d}{dx} + \overline{\varPhi(x)}$$

belongs to the Kato class. From the Schechter theorem [4, theorem 7.3] we get that the relative H_0-bound of the operator W equals zero. If we take into account that the space of the basic functions $C_0^\infty(R_1)$ is the essential domain of the operator H_0, we can be convinced that all the statements of the theorem follow from KLMN theorem (see e.i. [(5 p. 11)]. The theorem is proved.

2. Investigation of the equation $f + K(\lambda)f = 0$ on the half-plane $C_+ = \{\lambda\in C : \mathrm{Im}\lambda > 0\}$

Let $h(x)\in C_0^\infty(R_1)$ and $z=\lambda^2, \mathrm{Im}\lambda>0$. Assume $u_0(\lambda)\equiv u_0(x,\lambda) = R_0(\lambda^2)h(x)$, $u(\lambda)\equiv u(x,\lambda) = R(\lambda^2)h(x)$, where $R_0(\lambda^2) = (H_0-\lambda^2)^{-1}$ and $R(\lambda^2) = (H-\lambda^2)^{-1}$ are the resolvents of the operators H_0 and H, respectively. Taking into account that the operators $-i\dfrac{d}{dx}$ and $R_0(\lambda^2)$ are permutational, $R_0(\lambda^2)$ is an integral operator with the kernel

$$G_0(x,y,\lambda) = -\frac{e^{i\lambda|x-y|}}{2i\lambda}$$

and the space of all basic functions $C_0^\infty(R_1)$, according to theorem 1, is the essential domain of both operators H_0 and H, for $u(\lambda)$ we get the inhomogeneous equation

$$u(\lambda) + K(\lambda)u(\lambda) = u_0(\lambda), \tag{5}$$

where $K(\lambda)$ is an integral operator with the kernel

$$K(x,y,\lambda) = -\frac{e^{i\lambda|x-y|}}{2i\lambda}\left[\varPhi(y) + 2\lambda sgn(x-y)a(y)\right].$$

Denote by $C(R_1)$ the Banach space of functions continuous and bounded on R_1 and with the norm $\sup\limits_{-\infty<x<+\infty}|f(x)| = \|f\|_{C(R_1)} < +\infty$.

In the paper [6, theorem 1] it is proved that the operator $K(\lambda)$ is analytic with respect to λ in the upper part of the complex plane $C_+ = \{\lambda\in C : \mathrm{Im}\lambda>0\}$ in the uniform operator topology and for all λ from $\overline{C}_+\setminus\{0\} = \{\lambda\in C : \mathrm{Im}\lambda\geq 0,\ \lambda\neq 0\}$ is compact in $C(R_1)$, and continuous in the uniform operator topology. These results allow as to apply to the equation

$$f + K(\lambda)f = 0 \tag{6}$$

the Fredholm analytic theorem [1, p. 224, theorem VI.14]. According to Fredholm's theory, inhomogeneous equation (5) for $\mathrm{Im}\lambda > 0$ has a unique solution in $C(R_1)$, if the corresponding homogeneous equation (6) has only a zero solution. Denote by $\mathcal{L}_+$ the set of those points from the half-plane C_+ for which homogeneous equation (6) has a nontrivial solution in $C(R_1)$. Preliminarily we prove the following two lemmas.

Lemma 1. *If conditions a) and b) are fulfilled, then the operator $M(\lambda)$ with the kernel*

$$M(x,y,\lambda) = K(x,y,\lambda)e^{-i\lambda(|x|-|y|)}$$

is analytic with respect to λ in the half-plane C_+ in uniform operator topology and for all λ from $\overline{C}_+\setminus\{0\}$ is compact in $C(R_1)$, and continuous in uniform operator topology.

Proof. Let $\mathrm{Im}\lambda \geq 0$. Then from the equality

$$e^{i\lambda|x-y|}e^{-i\lambda(|x|-|y|)} = e^{i\lambda(|x-y|+|y|-|x|)}$$

and inequality

$$|x-y| \geq |x|-|y|$$

we have

$$\left|e^{i\lambda(|x-y|+|y|-|x|)}\right| = e^{-\mathrm{Im}\lambda(|x-y|+|y|-|x|)} \leq 1.$$

Hence we get

$$|M(x,y,\lambda)| \leq |T(x,y,\lambda|\,.$$

The proof of the lemma follows from this inequality and theorem 1 of [6].

Lemma 2. *Let τ be an arbitrary positive number. Then for any real numbers z and y the following equality is valid*

$$\int\limits_{-\infty}^{+\infty} e^{-\tau(|x-y|+|x-z|)}dx = \frac{1}{\tau}e^{-\tau|y-x|}(1+\tau\,|y-z|). \tag{7}$$

Proof. At first consider the case $y<z$. Represent the integral

$$J = \int\limits_{-\infty}^{+\infty} e^{-\tau|x-y|}e^{-\tau|x-z|}dx$$

in the sum of three integrals

$$\begin{aligned} J = \int\limits_{-\infty}^{y} e^{-\tau|x-y|}e^{-\tau|x-z|}dx + \int\limits_{y}^{z} e^{-\tau|x-y|}e^{-\tau|x-z|}dx \\ + \int\limits_{z}^{+\infty} e^{-\tau|x-y|}e^{-\tau|x-z|}dx = J_1+J_2+J_3. \end{aligned} \tag{8}$$

Since in the integral J_1 $x<y<z$, we have

$$J_1 = \int\limits_{-\infty}^{y} e^{-\tau|x-y|}e^{-\tau|x-z|}dx = \int\limits_{-\infty}^{y} e^{-\tau[(y-x)+(z-x)]}dx.$$

Making a change $s=(y-x)+(z-x)$, in the last integral we get

$$J_1 = \int\limits_{+\infty}^{z-y} e^{-\tau s}\frac{ds}{-2} = \frac{1}{2\tau}e^{-\tau s}\Big|_{+\infty}^{z-y} = \frac{1}{2\tau}e^{-\tau(z-y)}. \tag{9}$$

Taking into account that in the integral J_2 $y < x < z$, we have

$$J_2 = \int\limits_y^z e^{-\tau(x-y)} e^{-\tau(z-x)} dx = \int\limits_y^z e^{-\tau(z-y)} dx = (z-y)e^{-\tau(z-y)}. \quad (10)$$

From the inequalities $y < z < x$ it follows

$$J_3 = \int\limits_z^{+\infty} e^{-\tau|x-y|} e^{-\tau|x-z|} dx = \int\limits_z^{+\infty} e^{-\tau[(x-y)+(x-z)]} dx.$$

If in the last integral we make the change $s = (x-y) + (x-z)$, we get

$$J_3 = \int\limits_z^{+\infty} e^{-\tau[(x-y)+(x-z)]} dx = \int\limits_{z-y}^{+\infty} e^{-\tau s} \frac{ds}{2} = -\frac{1}{2\tau} e^{-\tau s} \Big|_{z-y}^{+\infty} = \frac{1}{2\tau} e^{-\tau(z-y)}. \quad (11)$$

From equalities (8)-(11) for $y < z$ we get

$$J = \int\limits_{-\infty}^{+\infty} e^{-\tau|x-y|} e^{-\tau|x-z|} dx = \frac{1}{\tau} e^{-\tau(z-y)} (1 + \tau(z-y)). \quad (12)$$

Taking into account the symmetry of the integral J with respect to variables z and y, from (12) we get equality (7). The lemma is proved.

Theorem 2. *Let $\sigma + i\tau = \lambda \in \mathcal{L}_+$ and $f(x)$ be a nontrivial solution of homogeneous equation (6) from $C(R_1)$. Then, if the conditions a) and b) are fulfilled, then*

$$\sup_{-\infty < x < +\infty} e^{\tau|x|} |f(x)| < +\infty. \quad (13)$$

Proof. Let $\sigma + i\tau = \lambda \in \mathcal{L}_+,\ f(x)$ be the solution of equation (6), χ_n be an operator of multiplication by the characteristic function of the section $[-n, n],\ K^{(n)}(\lambda) = K(\lambda)\chi_n$. It is clear that

$$\lim_{n\to\infty} \left\| K^{(n)}(\lambda) - K(\lambda) \right\|_{C(R_1)\to C(R_1)} = 0.$$

Then according to general theory of compact operators [see [7, p. 41] or [8]) there exists a sequence of numbers $\{\gamma_n\}$ and a sequence of functions $\{f_n(x)\} \subset C(R_1)$ such that for any n

$$f_n(x) + \gamma_n K^{(n)}(\lambda) f_n(x) = 0,$$

moreover, $\lim\limits_{n\to\infty} \gamma_n = 1,\ \lim\limits_{n\to\infty} \|f_n(x) - f(x)\|_{C(R_1)=0}$. It is clear that for any n the function

$$g_n(x) = e^{-i\lambda|x|} f_n(x)$$

is the solution of the equation

$$g_n(x) + \gamma_n M^{(n)}(\lambda) g_n(x) = 0,$$

where $M^{(n)}(\lambda)$ is an integral operator with the kernel

$$M^{(n)}(x, y, \lambda) = K^{(n)}(x, y, \lambda) e^{-i\lambda(|x|-|y|)} = K(x, y, \lambda)\chi_n(x) e^{-i\lambda(|x|-|y|)}.$$

It is easy to show that if $y \in [-n, n]$, $|x| \geq n+1$, then there exists $c > 0$ such that

$$|K(x, y, \lambda)| \leq \frac{c}{2|\lambda|} e^{-\mathrm{Im}\lambda|x|} \left(|\Phi(y)| + |a(y)| \right).$$

Hence and from the equality

$$f_n(x) = -\gamma_n K^{(n)}(\lambda) f_n(x)$$

it follows that if the conditions a) and b) are fulfilled, then the function $g_n(x) = e^{-i\lambda|x|} f_n(x)$ belongs to the space $C(R_1)$. Show that

$$\sup_n \|g_n\|_{C(R_1)} = \sup_n \left\{ \sup_{-\infty<x<+\infty} |g_n(x)| \right\} < +\infty. \tag{14}$$

It inequality (14) is not valid, then there will be found the subsequence $\{g_{n_l}(x)\} \subset \{g_n(x)\}$ such that $\lim \|g_{n_l}\|_{C(R_1)} = +\infty$. For the normalized sequence $\widetilde{g}_{n_l} = \dfrac{g_{n_l}(x)}{\|g_{n_l}\|_{C(R_1)}}$ we have

$$\widetilde{g}_{n_l}(x) = -\gamma_{n_l} M^{(n_l)}(\lambda) \widetilde{g}_{n_l}(x). \tag{15}$$

According to lemma 1, the operator $M^{(n_l)}(\lambda)$ is compact. Therefore, from equality (15) is follows that there exist $\widetilde{g}(x) \in C(R_1)$ and the subsequence $\left\{\widetilde{g}_{n_{l_k}}(x)\right\} \subset \{g_{n_l}(x)\}$ such that the sequence $\widetilde{g}_{n_{l_k}}(x)$ uniformly converges to $\widetilde{g}(x)$ as $k \to \infty$. Passing to limit we find

$$f(x) = \lim_{k\to\infty} \widetilde{f}_{n_{l_k}}(x) = \lim_{k\to\infty} e^{i\lambda|x|} \widetilde{g}_{n_{l_k}}(x) = e^{i\lambda|x|} \widetilde{g}(x).$$

Hence it follows that the function $\widetilde{g}(x)$ may not identically equal to zero since by the supposition the function $f(x)$ is a nontrivial solution of equation (6). On the other hand, by $\lim\limits_{k\to\infty} \left\|g_{n_{l_k}}\right\|_{C(R_1)} = +\infty$ we get $\widetilde{g}(x) = 0$. The obtained contradiction shows the validity of inequality (14). Inequality (13) follows from inequality (14) and equalities $g_n(x) = e^{-i\lambda|x|} f_n(x)$, $\lim\limits_{n\to\infty} \|f_n(x) - f(x)\|_{C(R_1)} = 0$. The theorem is proved.

Theorem 3. *Let $\sigma + i\tau = \lambda \in \mathcal{L}_+$ and $f(x)$ be a nontrivial solution of homogeneous equation (6) from $C(R_1)$. Then if the conditions a) and b) are fulfilled, then $f(x) \in W_2^1(R_1)$.*

Proof. From theorem 2 it follows that $f(x) \in C(R_1) \cap L_2(R_1)$. Show that the generalized derivative of the function $f(x)$ also belongs to $L_2(R_1)$. To this end, we calculate the first order derivative of the function

$$K(x, y, \lambda) = -\frac{e^{i\lambda|x-y|}}{2i\lambda} \left[\Phi(y) + 2\lambda sgn(x-y) a(y)\right]$$

with respect to the variable x:

$$\begin{aligned} \frac{\partial K(x,y,\lambda)}{\partial x} &= -\frac{e^{i\lambda|x-y|}}{2} sgn(x-y) \left[\Phi(y) + 2\lambda sgn(x-y) a(y)\right] \\ &-2\frac{e^{i\lambda|x-y|}}{i} \delta(x-y) a(y) = -\frac{e^{i\lambda|x-y|}}{2} sgn(x-y) \Phi(y) \\ &-\lambda e^{i\lambda|x-y|} a(y) - 2\frac{e^{i\lambda|x-y|}}{i} \delta(x-y) a(y), \end{aligned} \tag{16}$$

where $\delta(x-y)$ is the Dirac function. Taking into account the equality $e^{i\lambda|x-y|}\delta(x-y) = \delta(x-y)$ in (16), we get

$$\frac{\partial K(x,y,\lambda)}{\partial x} = -\frac{e^{i\lambda|x-y|}}{2} sgn(x-y) \Phi(y) - \lambda e^{i\lambda|x-y|} a(y) + 2i\delta(x-y) a(y). \tag{17}$$

Note that the function $a(y)$ is continuous, the order $\delta(x-y)$ of the function equals zero, and therefore the derivative $\delta(x-y)a(y)$ has a meaning. Using the equalities

$$f(x) = -K(\lambda) f(x) = -\int_{-\infty}^{+\infty} K(x, y, \lambda) f(y) dy,$$

from (17) we get:

$$f'(x) = -\int\limits_{-\infty}^{+\infty} \frac{\partial K(x,y,\lambda)}{\partial x} f(y)dy = \psi_1(x,\lambda) + \psi_2(x,\lambda) + \psi_3(x,\lambda),$$

where

$$\psi_1(x,\lambda) = \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda|x-y|}}{2} sgn(x-y)\varPhi(y)f(y)dy,$$

$$\psi_2(x,\lambda) = \int\limits_{-\infty}^{+\infty} \lambda e^{i\lambda|x-y|} a(y)f(y)dy,$$

$$\psi_3(x,\lambda) = -2i\int\limits_{-\infty}^{+\infty} \delta(x-y)a(y)f(y)dy.$$

Show that all three functions $\psi_1(x,\lambda),\ \psi_2(x,\lambda),\ \psi_3(x,\lambda)$ belong to the space $L_2(R_1)$. From the equalities

$$\psi_3(x,\lambda) = -2i\int\limits_{-\infty}^{+\infty} \delta(x-y)a(y)f(y)dy = -2ia(x)f(x)$$

and inequality (14) and also from the conditions on the function $a(x)$ it follows that $\psi_3(x,\lambda) \in L_2(R_1)$. Now prove that $\psi_1(x,\lambda) \in L_2(R_1)$. To this end, we calculate the integral

$$\int\limits_{-\infty}^{+\infty} |\psi_1(x,\lambda)|^2 dx = \int\limits_{-\infty}^{+\infty} \left| \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda|x-y|}}{2} sgn(x-y)\varPhi(y)f(y)dy \right|^2 dx$$

$$= \int\limits_{-\infty}^{+\infty} \left\{ \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda|x-y|}}{2} sgn(x-y)\varPhi(y)f(y)dy \right\}$$

$$\times \left\{ \int\limits_{-\infty}^{+\infty} \frac{e^{-i\overline{\lambda}|x-z|}}{2} sgn(x-z)\overline{\varPhi(z)f(z)}dz \right\} dx$$

$$= \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \left\{ \int\limits_{-\infty}^{+\infty} \frac{e^{i\lambda|x-y|}}{2} sgn(x-y) \frac{e^{-i\overline{\lambda}|x-z|}}{2} sgn(x-z)dx \right\}$$

$$\times \varPhi(y)f(y)\overline{\varPhi(z)f(z)}dydz$$

$$= \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \left\{ \frac{1}{4}\int\limits_{-\infty}^{+\infty} e^{i\sigma(|x-y|-|x-z|)-\tau(|x-y|+|x-z|)} sgn(x-y)sgn\,(x-z)\,dx \right\}$$

$$\times \varPhi(y)f(y)\overline{\varPhi(z)f(z)}dydz,$$

whence, according to lemma 2, we have the estimation

$$\int\limits_{-\infty}^{+\infty} |\psi_1(x,\lambda)|^2 dx \le \frac{1}{4\tau}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} e^{-\tau|y-z|}\left(1+\tau\,|y-z|\right)|\varPhi(y)f(y)|\left|\overline{\varPhi(y)f(y)}\right| dydz.$$

From inequalities (13) and conditions on the function $\varPhi(x)$ it follows that $\psi_1(x,\lambda) \in L_2(R_1)$. Quite similarly we get $\psi_2(x,\lambda) \in L_2(R_1)$. The theorem is proved.

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