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On A Periodic Type Boundary Value Problem For A Third Order Equation In Hilbert Space

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Abstract. *In the paper we get sufficient conditions on the coefficients of a third order operator-differential equation in Hilbert space that provide the existence and uniqueness of the solution of a periodic type boundary value problem on a finite segment.*

Consider in a separable Hilbert space H the following boundary value problem:

$$\frac{d^3u}{dt^3} + A^3u + \sum_{j=0}^{4} A_{4-j}\frac{d^ju}{dt^j} = f\left(t\right), \quad t \in \left(0,T\right), \tag{1}$$

$$u^{(k)}\left(0\right) = e^{i\alpha}u^{(k)}\left(T\right), \quad k = 0,1,2, \tag{2}$$

where $\alpha \in R = (-\infty, \infty)$, $f\left(t\right)$, $u\left(t\right)$ are the functions with the values from H, and the operator coefficients satisfy the conditions:

1) A is a positive-definite self-adjoint operator in H;

2) the operators $B_j = A_jA^{-j}$, $j = \overline{0,4}$ are bounded in H.

Denote by H_γ a scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = D\left(A^\gamma\right)$, $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $x,y \in H_\gamma$, $\gamma \geq 0$. For $\gamma = 0$ we assume $H_0 = H$.

Let $L_2\left((0,T);H\right)$ be a Hilbert space of all functions $f\left(t\right)$ determined on the interval $(0,T)$ almost everywhere with the values in H for which

$$\|f\|_{L_2((0,T);H)} = \left(\int_0^T \|f\left(t\right)\|^2 dt\right)^{1/2} < \infty.$$

Following the monograph [1], we introduce the Hilbert space

$$W_2^3\left((0,T);H\right) = \left\{u, u''' \in L_2\left((0,T);H\right), A^3u \in L_2\left((0,T);H\right)\right\}$$

with the norm

$$\|u\|_{W_2^3((0,T);H)} = \left(\|u'''\|^2_{L_2((0,T);H)} + \left\|A^3u\right\|^2_{L_2((0,T);H)}\right)^{1/2}.$$

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From the trace theorem [1] it follows that

$$W_{2,\alpha}^{3}\left((0,T);H\right)=\left\{u:u\in W_{2}^{3}\left((0,T);H\right),u^{(k)}\left(0\right)=e^{i\alpha}u^{(k)}\left(T\right),k=0,1,2\right\}$$

is the complete subspace of the space $W_2^3\left((0,T);H\right)$.

The spaces $L_2\left(R;H\right)$ and $W_2^3\left(R;H\right)$ are similarly determined for $R=(-\infty,\infty)$.

Definition 1. *If for $f\in L_2\left((0,T);H\right)$ there exists a vector-function $u\left(t\right)\in W_2^3\left((0,T);H\right)$ that satisfies equation (1) almost everywhere in $(0,T)$, then $u\left(t\right)$ is said to be a regular solution of equation (1).*

Definition 2. I*f for any $f\left(t\right)\in L_2\left((0,T);H\right)$ there exists the regular solution $u\left(t\right)$ of equation (1) satisfying the boundary conditions (2) in the sense of convergence*

$$\lim_{t\to+0}\left\|u^{(k)}\left(t\right)-e^{i\alpha}u^{(k)}\left(T-t\right)\right\|_{3-k-1/2}=0,\quad k=0,1,2,$$

and it holds the estimation

$$\|u\|_{W_2^3((0,T);H)}\le const\,\|f\|_{L_2((0,T);H)},$$

the problem (1), (2) is said to be regular solvable.

In the present paper, we find conditions on the coefficients of the equation that provide regular solvability of problem (1), (2). Note that the boundary value problems on a final segment for second order elliptic operator differential equations were considered for example in the papers [2,8].

Denote by

$$P_0u=u'''+A^3u,\quad P_1u=\sum_{j=0}^{3}A_{3-j}u^{(j)},\quad u\in W_{2,\alpha}^{3}\left((0,T);H\right)$$

and

$$Pu=P_0u+P_1u,\quad u\in W_{2,\alpha}^{3}\left((0,T);H\right).$$

At first we investigate the solvability of the equation $P_0u=f$. It holds

Lemma 1. *For $u\in W_{2,\alpha}^{3}\left((0,T);H\right)$ it holds the equality*

$$\|P_0u\|_{L_2((0,T);H)}=\|u\|^2_{W_2^3((0,T);H)}. \tag{3}$$

Proof. Obviously, for $u\in W_{2,\alpha}^{3}\left((0,T);H\right)$

$$\|P_0u\|^2_{L_2((0,T);H)}=\left\|u'''+A^3u\right\|^2_{L_2((0,T);H)}$$

$$+\|u\|^2_{W_2^3((0,T);H)}+2\mathbf{Re}\left(u''',A^3u\right)_{L_2((0,T);H)}.$$

On the other hand, after integration by parts we get:

$$\left(u''',A^3u\right)_{L_2((0,T);H)}=\left(A^{1/2}u''\left(t\right),A^{5/2}u\left(t\right)\right)\Big|_0^T-\left(A^{3/2}u'\left(t\right),A^{3/2}u'\left(t\right)\right)\Big|_0^T$$

$$+\left(A^{5/2}u\left(t\right),A^{1/2}u''\left(t\right)\right)\Big|_0^T-\left(A^3u,u'''\right)_{L_2((0,T);H)}. \tag{4}$$

Since $u\in W_{2,\alpha}^{3}\left((0,T);H\right)$ $\left(u^{(k)}\left(0\right)=e^{i\alpha}u^{(k)}\left(T\right)\right)$, we get

$$2\mathbf{Re}\left(u''',A^3u\right)_{L_2((0,T);H)}=0.$$

Then the statement of the lemma follows from equality (4).

Corollary 1. $KerP_0 = \{u : P_0 u = 0\} = \{0\}$.

Theorem 1. *Let condition 1) be fulfilled. Then the equation $P_0 u = f$ is solvable for all $f \in L_2((0,T);H)$.*

Proof. Let $f_1(t) = f(t)$ for $t \in (0,T)$ and $f_1(t) = 0$ for $t \in R\setminus(0,T)$. Then the equation

$$P_0(d/dt)\,u(t) = u''' + A^3 u = f_1(t), \quad t \in R,$$

has the regular solution $u_1 \in W_2^3(R;H)$ in the form

$$u_1(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(i^3\xi^3 E + A^3\right)^{-1} \widehat{f}_1(\xi)\, e^{i\xi t} dt.$$

Indeed, from the Plancherel theorem it follows that

$$\|u_1\|^2_{W_2^3(R;H)} = \left\|A^3 \widehat{u}_1(\xi)\right\|^2_{L_2(R;H)} + \left\|\xi^3 \widehat{u}_1(\xi)\right\|^2_{L_2(R;H)}.$$

Since

$$\left\|A^3 u_1(\xi)\right\|_{L_2(R_+;H)} = \left\|A^3\left(i^3\xi^3 E + A^3\right)^{-1} \widehat{f}_1(\xi)\right\|_{L_2(R;H)}$$
$$\le \sup_{\xi} \left\|A^3\left(-i\xi^3 E + A^3\right)^{-1}\right\| \cdot \|f_1\|_{L_2(R;H)},$$

and $\|f_1\|_{L_2(R;H)} = \|f\|_{L_2((0,T);H)}$, then from the inequality

$$\left\|A^3\left(-i\xi^3 E + A^3\right)^{-1}\right\| = \sup_{u\in\sigma(A)} \left|\mu^3\left(-i\xi^3 + \mu^3\right)^{-1}\right| \le 1$$

it follows that $\left\|A^3 \widehat{u}_1(\xi)\right\|_{L_2(R;H)} \le \|f\|_{L_2((0,T);H)}$, consequently, $A^3 u \in L_2(R;H)$. We similarly get $\left\|u_1'''\right\|_{L_2(R;H)} = \left\|\xi^3 A_1 \widehat{u}(\xi)\right\|_{L_2(R;H)} \le \|f\|_{L_2((0,T);H)}$. Consequently, $u_1 \in W_2^3(R;H)$ and satisfies the equation $u'''(t) + A^3 u(t) = f_1(t)$ almost everywhere in R. Denote by $\omega(t)$ the contraction of $u_1(t)$ on $[0,T]$. Then $\omega(t) \in W_2^3((0,T);H)$, and by the trace theorem, $\omega^{(k)}(0) \in H_{3-k-1/2}$, $\omega^{(k)}(T) \in H_{3-k-1/2}$, $k = 0,1,2$. Then we'll find the solution of the equation $P_0 u = f$ in the form

$$u(t) = \omega(t) + e^{-tA}\varphi_1 + e^{\omega_1(t-T)A}\varphi_2 + e^{\omega_2(t-T)A}\varphi_3,$$

where $\omega_1 = \overline{\omega}_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, and φ_1, φ_2, φ_3 are still unknown vectors from $H_{5/2}$. Then from condition (2) it follows that the vectors φ_1, φ_2 and φ_3 satisfy the conditions:

$$\omega^{(k)}(0) + (-1)^k A^k \varphi_1 + \omega_1^k A^k e^{-\omega_1 AT}\varphi_1 + \omega_2^k A^k e^{-\omega_2 AT}\varphi_2$$
$$= \omega^{(k)}(T) + (-1)^k A^k e^{-AT}\varphi_1 + \omega_1^k A^k \varphi_1 + \varphi_2^k A^k \varphi_2, \;\; k = 0,1,2.$$

Hence we have

$$(-1)^k\left(E - e^{-AT}\right)\varphi_1 + \omega_1^k\left(e^{-\omega_1 AT} - E\right)\varphi_2 + \omega_2^k\left(e^{-\omega_2 AT} - E\right)\varphi_3$$
$$= A^{-k}\left(\omega^{(k)}(T) - \omega^{(k)}(0)\right), \quad k = 0,1,2,3.$$

Denoting by $\left(E - e^{-AT}\right)\varphi_1 = \psi_1$, $\left(e^{-\omega_1 AT} - E\right)\varphi_2 = \psi_2$, $\left(e^{-\omega_2 AT} - E\right)\varphi_3 = \psi_3$ with respect to ψ_1, ψ_2 and ψ_3 we get the system of equations

$$(-1)^k \psi_1 + \omega_1^k \psi_2 + \omega_2^k \psi_3 = A^{-k}\left(\omega^{(k)}(T) - \omega^{(k)}(0)\right), \;\; k = 0,1,2.$$

Since hence ψ_1, ψ_2 and ψ_3 are uniquely determined and the vectors $A^{-k}\left(\omega^{(k)}(T) - \omega^{(k)}(0)\right) \in H_{5/2}$, $k = 0,1,2$ then from the inversibility of the operators $E - e^{-AT}$, $e^{-\omega_1 AT} - E$ and $e^{-\omega_2 AT} - E$ in the space $H_{5/2}$ it follows that φ_1, φ_2 and $\varphi_3 \in H_{5/2}$. Thus, $u(t) \in W_2^3((0,T);H)$ is the solution of the equation $P_0 u = f$. Then from the Corollary 1 and lemma 1 it follows that $\|u\|_{W_2^3((0,T);H)} \le const\,\|f\|_{L_2((0,T);H)}$. The theorem is proved.

Now engage in regular solution of problem (1), (2). Prove the following theorem.

Theorem 2. *For all $u \in W_2^2\left((0,T);H\right)$ the following inequalities hold:*

$$\left\| A^{3-j}u^{(j)} \right\|_{L_2((0,T);H)} \leq c_j \left\| P_0 u \right\|_{L_2((0,H);H)}, \quad j=\overline{0,3}, \tag{5}$$

where $c_0 = c_3 = 1, \;\; c_1 = c_2 = 2^{1/3} \cdot 3^{-1/2}$.

Proof. The validity of inequality (5) for $j = 0$ and $j = 3$ follows from lemma 1. For $u \in W_{2,\alpha}^2\left((0,T);H\right)$, integrating by parts, we have:

$$\left\| Au'' \right\|_{L_2((0,T);H)}^2 = \left(Au'', Au''\right)_{L_2((0,T);H)} = \left(A^{1/2}u''(t), A^{5/2}u'(t)\right)\Big|_0^1$$

$$-\left(u''', A^2u'\right)_{L_2((0,T);H)} \leq \left\| u''' \right\|_{L_2((0,T);H)} \cdot \left\| A^2u' \right\|_{L_2((0,T);H)}. \tag{6}$$

Similarly we have

$$\left\| A^2u' \right\|_{L_2((0,T);H)}^2 \leq \left\| Au'' \right\|_{L_2((0,T);H)} \cdot \left\| A^3u \right\|_{L_2((0,T);H)}. \tag{7}$$

Taking into account inequality (6) in (7), we have

$$\left\| A^2u' \right\|_{L_2((0,T);H)}^2 \leq \left\| A^3u \right\|_{L_2((0,T);H)} \left\| u''' \right\|_{L_2((0,T);H)}^{1/2} \cdot \left\| A^2u' \right\|_{L_2((0,T);H)}^{1/2},$$

i.e.

$$\left\| A^2u' \right\|_{L_2((0,T);H)} \leq \left\| A^3u \right\|_{L_2((0,T);H)}^{2/3} \left\| u''' \right\|_{L_2((0,T);H)}^{1/3},$$

i.e. for any $\varepsilon > 0$ by the Young inequality we have:

$$\left\| A^2u' \right\|_{L_2((0,T);H)}^2 \leq \left(\varepsilon \cdot \left\| A^3u \right\|_{L_2((0,T);H)}^2\right)^{\frac{2}{3}} \left(\frac{1}{\varepsilon^2} \left\| u''' \right\|_{L_2((0,T);H)}^2\right)^{\frac{1}{3}}$$

$$\leq \frac{2}{3}\varepsilon \left\| A^3u \right\|_{L_2((0,T);H)}^2 + \frac{1}{3\varepsilon^2} \left\| u''' \right\|_{L_2((0,T);H)}^2.$$

Chosing $\frac{2}{3}\varepsilon = \frac{1}{3\varepsilon^2}$, i.e. $\varepsilon = \frac{1}{\sqrt[3]{2}}$, we get:

$$\left\| A^2u' \right\|_{L_2((0,T);H)}^2 \leq \frac{2^{\frac{2}{3}}}{3} \left(\left\| A^3u \right\|_{L_2((0,T);H)}^2 + \left\| u''' \right\|_{L_2((0,T);H)}^2\right)$$

$$= \frac{2}{3}^{\frac{2}{3}} \left\| u \right\|_{W_2^2((0,T);H)}^2 = \frac{2}{3}^{\frac{2}{3}} \left\| P_0 u \right\|_{L_2((0,T);H)},$$

i.e. inequality (5) is valid for $j = 1$ as well. For $j = 2$, taking into account inequality (7) in (6), inequality (5) is proved in the same way.

The theorem is proved. Now prove the basic theorem.

Theorem 3. *Let conditions 1), 2) be fulfilled, and it hold the inequality*

$$q = \sum_{j=0}^{4} c \left\| B_{4-j} \right\| < 1,$$

where the numbers c_j were determined from theorem 2. Then problem (1), (2) is regularly solvable.

Proof. Let us write problem (1), (2) in the form of the equation $Pu = f$, i.e. $P_0u + P_1u = f$, where $f \in L_2((0,T);H)$, $u \in W^3_{2,\alpha}((0,T);H)$. Then from theorem 1 it follows that P_0^{-1} exists and is bounded. Assuming $P_0u = \upsilon$, we get the equation $\upsilon + P_1P_0^{-1}\upsilon = f$ in $L_2((0,T);H)$. Hence we have that for $\upsilon \in L_2((0,T);H)$ it hold the inequalities:

$$\left\|P_1P_0^{-1}\upsilon\right\|_{L_2((0,T);H)} = \|P_1u\|_{L_2((0,T);H)} \leq \sum_{j=0}^{4}\left\|B_{3-j}\right\| \cdot \left\|A^{3-j}u^{(j)}\right\|_{L_2((0,T);H)}$$

$$\leq \sum_{j=0}^{4}\left\|B_{3-j}\right\| c_j \|P_0u\|_{L_2((0,T);H)} \leq q \leq \|P_0u\|_{L_2((0,T);H)} = q\,\|\upsilon\|_{L_2((0,T);H)}.$$

Since $q < 1$, then $\upsilon = \left(E + P_1P_0^{-1}\right)^{-1} f$, while $u = P_0^{-1}\left(E + P_1P_0^{-1}\right) f$ and $\|u\|_{W_2^2(R(0,T);H)} \leq const\,\|f\|_{L_2((0,T);H)}$. The theorem is proved.

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