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Necessary Optimality Conditions In An Optimal Control Problem With Integro-Differential Equations Equality And Inequality Type Multipoint Functional Restraints

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Abstract. *A problem of optimal control of nonlinear integro-differential system of equations with multipoint quality test involving equality and inequality type multipoint functional restraints is considered. Necessary optimality conditions of first and second orders are proved.*

Keywords. integro-differential equation, necessary optimality condition of first and second orders, functional restraints.

1 Introduction

Theory of necessary optimality conditions of second order for optimal control problem involving different restraints (for example, functional restraints on trajectory) without the so called normality (see for example [1-13]) was developed a little.

In this direction show the works [1-17], ete.

In the present paper, we consider an optimal control problem described by the system of nonlinear integro-differential equations with multipoint test functional involving equality and inequality type multipoint functional restraints on the trajectory of the system.

Necessary optimality conditions of first and second orders without assumption of normality condition are obtained.

In particular, the case of degeneration of the analogue of the Legendre-Klebsh condition, is studied.

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2 Problem statement

Consider a problem on the minimum of a multipoint functional

$$S_0(u) = \varphi_0(x(T_1), x(T_2), ..., x(T_k)), \tag{2.1}$$

under the following restraints

$$S_\mu(u) = \varphi_\mu(x(T_1), x(T_2), ..., x(T_k)) \le 0, \mu = \overline{1, p}, \tag{2.2}$$

$$S_\mu(u) = \varphi_\mu(x(T_1), x(T_2), ..., x(T_k)) = 0, \mu = \overline{p+1, q}, \tag{2.3}$$

$$u(t) \in U \subset R^r, t \in T = [t_0, t_1], \tag{2.4}$$

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_{t_0}^{t} K(t, \tau, x(\tau), u(\tau))\, d\tau, \tag{2.5}$$

$$x(t_0) = x_0.$$

Here $U \subset R^r$ is the given nonempty, bounded and open set $f(t, x, u)$ $(K(t, \tau, x, u))$ is n -dimensional vector-function given and continuous in $T \times R^n \times R^r$ $(T \times T \times R^n \times R^r)$ together with partial derivatives with respect to (x, u) to second order inclusively, $\varphi_\mu(z_1, z_2, ..., z_k)$, $\mu = \overline{0, q}$ are the given twice continuously differentiable scalar functions in $R^{n \cdot k}$, $t_0,\ t_1,\ x_0$ are given, $T_i, i = \overline{1, k}$ $(t_0 < T_1 < T_2 < ... < T_k \le t_1) -$ is an $u(t)$ r-dimensional piecewise-continuous (with finite number of discontinuity points of first kind) vector of control actions.

We call any control function with above properties an available control. The available control $u(t)$ is said to be an admissible control if the solution $x(t)$ of system (2.5) corresponding to it satisfies restraints (2.2), (2.3).

An admissible control delivering minimum to functional (2.1) at restraints (2.2)-(2.5) is said to be an optimal control, the appropriate process an optimal process.

3 Basic results

Assume that $(u(t), x(t))$ is an optimal process in problem (2.1)-(2.5) and for all $\mu = \overline{1, p}$, $S_\mu(u) = 0$. This assumption is made for simplicity of the statement.

Introduce the Hamilton-Pontryagin function

$$H^{(\mu)}(t, x(t), u(t), \psi_\mu(t)) = \psi'_\mu(t)\ f(t, x(t), u(t))$$
$$+ \int_t^{t_1} \psi'_\mu(\tau)\ K(\tau, t, x(t), u(t))\, d\tau, \mu = \overline{0, q}$$

and make the denotation

$$H_x^{(\mu)}(t) \equiv H_x^{(\mu)}(t, x(t), u(t), \psi_\mu(t)),$$
$$H_{xx}^{(\mu)}(t) \equiv H_{xx}^{(\mu)}(t, x(t), u(t), \psi_\mu(t)),$$
$$H_{xu}^{(\mu)}(t) \equiv H_{xu}^{(\mu)}(t, x(t), u(t), \psi_\mu(t)),$$
$$H_{uu}^{(\mu)}(t) \equiv H_{uu}^{(\mu)}(t, x(t), u(t), \psi_\mu(t)),$$
$$f_x(t) \equiv f_x(t, x(t), u(t)), K_x(t, \tau) \equiv K_x(t, \tau, x(\tau), u(\tau)),$$
$$f_u(t) \equiv f_u(t, x(t), u(t)), K_u(t, \tau) \equiv K_u(t, \tau, x(\tau), u(\tau)).$$

Here $\psi_\mu(t) \in L_\infty(T, R^n)$ is the solution of Volterra type integral equation

$$\psi_\mu(t) = \int_t^{t_1} H_x^{(\mu)}(\tau)\ d\tau - \sum_{j=1}^{k} \alpha_j(t)\ \frac{\partial \varphi_\mu(x(T_1), ..., x(T_k))}{\partial z_j},$$

where $\alpha_j(t)$ is a characteristic function of the segment $[t_0, T_j]$.

By the scheme, for example from [18], one can show that the first and second (in the classical sense) variations of the functional $S_\mu(u)$, $\mu = \overline{0,\, q}$ at the "point" $u = u(t)$ have the following form

$$\delta^1 S_\mu(u;\ \delta u) = -\int_{t_0}^{t_1} H_u^{(\mu)'}(t)\ \delta u(t)\ dt, \mu = \overline{0,\, q}, \tag{3.1}$$

$$\begin{aligned}\delta^2 S_\mu(u;\ \delta u) = &\sum_{i,\, j=1}^{k} \delta x'(T_i)\ \frac{\partial^2 \varphi_\mu(x(T_1),\, ...,\, x(T_k))}{\partial z_i\, \partial z_j}\, \delta x\left(T_j\right) \\ &- \int_{t_0}^{t_1} \Big[\delta x'(t)\ H_{xx}^{(\mu)}(t)\ \delta x(t) + 2\, \delta x'(t)\ H_{xu}^{(\mu)}(t)\ \delta u(t) \\ &+ \delta u'(t)\ H_{uu}^{(\mu)}(t)\ \delta u(t)\Big]\, dt,\ \ \mu = \overline{0,\, q}.\end{aligned} \tag{3.2}$$

Here $\delta u(t)$ is an arbitrary piecewise-continuous r-dimensional vector-function with the values from R^r, i.e. $\delta u(t) \in K\, C(T,\ R^r)$ (admissible variation of control), and $\delta x(t)$ is the variation of trajectory being the solution of the equation in variations (see e.i. [18])

$$\delta \dot{x}(t) = f_x(t)\ \delta x(t) + \int_{t_0}^{t} (K_x(t,\, \tau)\ \delta x(\tau) + K_u(t,\, \tau)\ \delta u(\tau))\ d\tau + f_u(t)\, \delta u(t)\, , t \in T, \tag{3.3}$$

$$\delta x(t_0) = 0. \tag{3.4}$$

By the scheme for example of the papers [1-3, 13] and so on it is proved that for the optimality of the admissible control $u(t)$ in problem (2.1)-(2.5) it is necessary the existence of a vector $\lambda = (\lambda_0,\, \lambda_1,\, ...,\, \lambda_q) \in R^{q+1}$ such that

$$\lambda_\mu \geq 0, \mu = \overline{0,\, p}, \|\lambda\| = \sum_{\mu=0}^{q} |\lambda_\mu| = 1, \tag{3.5}$$

the relation

$$\sum_{\mu=0}^{q} \lambda_\mu\, \delta^1 S_\mu(u:\ \ \delta u) = 0 \tag{3.6}$$

is fulfilled for all $\delta u(t) \in K\, C(T\, ,\ R^r)$.

Assume

$$H^{(\lambda)} = \sum_{\mu=0}^{q} \lambda_\mu\, H^{(\mu)}, \psi_\lambda(t) = \sum_{\mu=0}^{q} \lambda_\mu\, \psi_\mu(t)\, .$$

By virtue of representation (3.1) we arrive at the following statement.

Theorem 3.1 *For optimality of the admissible control $u(t)$ in problem* (2.1)-(2.5) *it is necessary the existence of a vector $\lambda = (\lambda_0,\, \lambda_1,\ \ldots,\ \lambda_q) \in R^{q+1}$ such that $\lambda_\mu \geq 0$, $\mu = \overline{0,\, p}$, $\|\lambda\| = \sum\limits_{\mu=0}^{q} |\lambda_\mu| = 1$ for all $\theta \in [t_0,\, t_1)$*

$$H_u^{(\lambda)}(\theta) = 0. \tag{3.7}$$

Here $\theta \in [t_0,\, t_1)$ is an arbitrary continuity point of the control $u(t)$.

Necessary optimality condition (3.7) is the analogue of the Euler equation for the problem under consideration and is a necessary optimality condition of first order. Therefore, generally speaking, one can distinguish a great number of admissible controls suspicious for optimality.

In this connection, there arises a problem of the search of additional optimality criteria (necessary optimality conditions of second order) for eliminating nonoptimal controls satisfying the Euler equation.

Introduce

Definition 3.1 We call any admissible control satisfying the Euler equation a classic extremal.

Now derive necessary conditions for optimality of classic extremals.

Denote by $K\,(u\;:\;\delta u)$ the set of critical variations of the control $u\,(t)$ in problem (2.2)-(2.5).

$$K\,(u\;:\;\delta u) =$$

$$= \left\{\delta u\;:\;\delta^1 S_\mu\,(u;\;\delta u) \le 0,\;\mu = \overline{0,\,p}\,,\;\delta^1 S_\mu\,(u;\;\delta u) = 0,\;\mu = \overline{p+1,\,q}\right\}.$$

Note that the notion of the set of critical variations was derivedin [4, 5] (see also [6-10]).

Let $A\,(u)$ be a set of allvectors $\lambda \in R^{q+1}$ satisfying relations (3.5), (3.6).

Cite an implicit necessary optimality condition of second order for the problem under consideration

Theorem 3.2 *For optimality of the classic extremal $u\,(t)$ in problem* (2.1)-(2.5) *it is necessary that the relation*

$$\max_{\lambda\in A(u)} \sum_{\mu=0}^{q} \lambda_\mu\, \delta^2 S_\mu\,(u\;:\;\delta u) \ge 0 \tag{3.8}$$

to be fulfilled for all $\delta u \in K\,(u\;:\;\delta u)$.

Note that necessary optimality conditions of type (3.8) for the problems of mathematical simulation and optimal control of ordinary dynamic systems were first established in the papers [6, 7, 12, 13, 20] (see also [10, 11]), and in the case of the problem with vector quality index in the papers [1, 2, 12] and so on (see appropriate review [13, 20]).

The proof of inequality (3.8) is carried out by the scheme, for example from [1] without significant changes in reasonings.

Using the implicit necessary optimality condition of second order (3.8), we get constructively verifiable necessary optimality conditions.

Find the representation of the solution of the equation in variations (3.3)-(3.4).

Let $F\,(t,\,\tau)$ $(n\times n)$ be a matrix function being the solution of the problem

$$F_\tau\,(t,\,\tau) = -F\,(t,\,\tau)\;f_x\,(\tau) - \int_\tau^t F\,(t,\,s)\;K_x\,(s,\,\tau)\;d\tau,$$

$$F\,(t,\,t) = E,$$

$(E - (n\times n) -$ is a unit matrix).

Then the solution $\delta x\,(t)$ of the equation in variations (3.3)-(3.4) allows the representation (see e.i. [19])

$$\delta x\,(t) = \int_{t_0}^t Q\,(t,\,\tau)\;\delta u\,(\tau)\;d\tau, \tag{3.9}$$

where $Q\,(t,\,\tau)$ $(n\times n)$ is a matrix function defined by the formula

$$Q\,(t,\,\tau) = F\,(t,\,\tau)\;f_u\,(\tau) + \int_\tau^t F\,(t,\,s)\;K_u\,(s,\,\tau)\;ds.$$

Assume

$$M^{(\mu)}\,(\tau,\,s) = -\sum_{i,\,j=1}^{k} \alpha_i\,(\tau)\;\alpha_j\,(s)\;Q'\,(T_i,\,\tau)\;\frac{\partial^2\varphi_\mu\,(x\,(T_1)\,,\,...,\,x\,(T_k))}{\partial z_i\,\partial z_j}\,Q\left(T_j,\,s\right)$$

$$+\int_{\max\,(\tau,\,s)}^{t_1} Q'\,(T,\,\tau)\;H_{xx}^{(\mu)}\,(t)\;Q\,(T,\,s)\;dt\,,\quad \mu = \overline{0,\,q}\,, \tag{3.10}$$

$$M^{(\lambda)}\,(\tau,\,s) = \sum_{\mu=0}^{q} \lambda_\mu\, M^{(\mu)}\,(\tau,\,s)\,. \tag{3.11}$$

Bemeansofrepresentation (3.9) by analog with [19], the second variation (3.2) of the functional $S_v(u)$ is represented in the form

$$\delta^2 S_\mu(u:\ \delta u) = -\int_{t_0}^{t_1}\int_{t_0}^{t_1} \delta u'(\tau)\ M^{(\mu)}(\tau,\ s)\ \delta u(s)$$
$$-2\int_{t_0}^{t_1}\left[\int_t^{t_1} \delta u'(\tau)\ H_{ux}^{(\mu)}(\tau)\ Q(\tau,\ t)\ d\tau\right]\delta u(t)\ dt$$
$$-\int_{t_0}^{t_1} \delta u'(t)\ H_{uu}^{(\mu)}(t)\ \delta u(t)\ dt\,,\quad \mu = \overline{0,\ q}\,. \tag{3.12}$$

Taking into attention representation (3.12) in (3.8), we arrive at the following statement.

Theorem 3.3 *For the optimality of the classic extremal* $u(t)$ *in problem* (2.1)-(2.5) *it is necessary that for all* $\delta u(t) \in K\,C(T,\ R^r)$ *satisfying the conditions*

$$\int_{t_0}^{t_1} H_u^{(\mu)'}(t)\ \delta u(t)\ dt \ge 0\,,\quad \mu = \overline{0,\ p}\,,$$
$$\int_{t_0}^{t_1} H_u^{(\mu)'}(t)\ \delta u(t)\ dt = 0\,,\quad \mu = \overline{p+1,\ q}\,, \tag{3.13}$$

the inequality

$$\min_{\lambda\in A(u)}\left\{\int_{t_0}^{t_1}\int_{t_0}^{t_1} \delta u'(\tau)\ M^{(\lambda)}(\tau,\ s)\ \delta u(s)\ ds\,d\tau + \int_{t_0}^{t_1} \delta u'(t)\ H_{uu}^{(\lambda)}(t)\ \delta u(t)\ dt\right.$$
$$\left.+2\int_{t_0}^{t_1}\left[\int_t^{t_1} \delta u'(\tau)\ H_{ux}^{(\lambda)}(\tau)\ Q(\tau,\ t)\ d\tau\right]\delta u(t)\ dt\right\} \le 0 \tag{3.14}$$

to be fulfilled.

Necessary eptimality condition (3.14) is a constructively verifiable integral necessary optimality condition of second order.

Determining from it by this or other way the admissible variation $\delta u(t) \in KC(T,\ R^r)$ of the control $u(t)$, one can obtain more simple and convenient for practical use second order optimality conditions, in particular, the analogue of the Legendre-Klebsh condition follows.

Theorem 3.4 *For the optimality of the classic extremal* $u(t)$ *in problem* (2.1)-(2.5) *it is necessary that the inequality*

$$\min_{\lambda\in A(u)} v'\,H_{uu}^{(\lambda)}(\theta)\ v \le 0 \tag{3.15}$$

tobe fulfilled for all $v \in R^r$ *and* $\theta \in [t_0,\ t_1)$.

Inequality (3.15) is the analogue of the Legendre-Klebsh condition for the problem under consideration.

Introduce the denotation

$$P^{(\lambda)}(\theta,\ e) = \int_\theta^{t_1}\int_\theta^{t_1} e'(\tau)\ M^{(\lambda)}(\tau,\ s)\ e(s)\ ds\,d\tau + \int_\theta^{t_1} e'(\tau)\ H_{uu}^{(\lambda)}(t)\ e(t)\ dt$$
$$+2\int_\theta^{t_1}\left[\int_t^{t_1} e'(\tau)\ H_{ux}^{(\lambda)}(\tau)\ Q(\tau,\ t)\ d\tau\right] e(t)\ dt\,,$$
$$R^{(\lambda)}(\theta) = \int_\theta^{t_1}\int_\theta^{t_1} M^{(\lambda)}(\tau,\ s)\ ds\,d\tau + \int_\theta^{t_1} H_{uu}^{(\lambda)}(t)\ dt$$
$$+2\int_\theta^{t_1}\left[\int_t^{t_1} H_{ux}^{(\lambda)}(\tau)\ Q(\tau,\ t)\ d\tau\right] dt\,.$$

The following statements hold.

Theorem 3.5 *If $u\,(t)$ is an optimal control in problem* (2.1)-(2.5)*, then for all $\theta \in [t_0,\, t_1)$, and $e\,(t) \in K\,C\,(T,\, R^r)$ satisfying the condition*

$$\int_\theta^{t_1} H_u^{(\mu)'}\,(t)\; e\,(t)\; dt \geq 0\,, \mu = \overline{0,\, p}, \int_\theta^{t_1} H_u^{(\mu)'}\,(t)\; e\,(t)\; dt = 0\,, \mu = \overline{p+1,\, q}$$

the following inequality is fulfilled

$$\min_{\lambda \in A(u)}\; P^{(\lambda)}\,(\theta,\, e) \leq 0. \tag{3.16}$$

Theorem 3.6 *For optimality of the classic extremal $u\,(t)$ in problem* (2.1)-(2.5) *it is necessary that the inequality*

$$\min_{\lambda \in A(u)}\; v'\,R^{(\lambda)}\,(\theta)\; v \leq 0. \tag{3.17}$$

to be fulfilled for all $v \in R^r$, $\theta \in [t_0,\, t_1)$ such that

$$\int_\theta^{t_1} H_u^{(\mu)'}\,(t)\; v\,dt \geq 0\,, \mu = \overline{0,\, p}, \int_\theta^{t_1} H_u^{(\mu)'}\,(t)\; v\,dt = 0\,, \mu = \overline{p+1,\, q}.$$

For proving theorem 3.5 it sufficesin the necessary optimality condition (3.14) to determine the variation $\delta u\,(t)$ of the control $u\,(t)$ by the formula

$$\delta u\,(t) = \begin{cases} 0, & t \in [t_0,\, \theta)\,, \\ e\,(t)\,, & t \in [\theta,\, t_1]\;. \end{cases}$$

Here $\theta \in [t_0,\, t_1)$ is an arbitrary continuity point of $u\,(t)$, and $e\,(t) \in K\,C\,(T,\, R^r)$.

For $e\,(t) \equiv v$, $t \in [\theta,\, t_1]$theorem 3.5 yields theorem 3.6.

It is clear that optimality condition (3.17) is weaker than (3.16), but may be simply verified.

4 The case of degeneration of the analogue of the Legendre-Klebsh condition

In this item we study the case of degeneration of the analogue of the Legendre-Klebsh condition.

Definition 4.1 We call the classic extremal a singular control in the classic sense in problem (2.1)-(2.5) if for all $v \in R^r$ and $\theta \in [t_0,\, t_1)$

$$\min_{\lambda \in A(u)}\; v'\,H_{uu}^{(\lambda)}\,(\theta)\; v = 0. \tag{4.1}$$

Using the structure of the papers [14, 15] and necessary optimality condition (3.14), we study classic singular controls for optimality.

It holds

Theorem 4.1 *If $u\,(t)$ is a singular optimal control in the classic sense in problem* (2.1)-(2.5)*, then for any $\theta \in [t_0,\, t_1)$ and $v \in R^r$, satisfying the conditions*

$$H^{(\mu)'}\;[\theta]\; v \geq 0\,, \mu = \overline{0,\, p}, H_u^{(\mu)'}\;[\theta]\; v = 0\,, \mu = \overline{p+1,\, q}$$

the following inequality is fulfilled

$$\min_{\lambda \in A(u)}\; v'\;\left[M^{(\lambda)}\,(\theta,\, \theta) + H_{ux}^{(\lambda)}\,Q\,(\theta,\, \theta)\right]\; v \leq 0.$$

Theorem 3.7 is proved by means of relation (3.14) by using appropriate constructions from [14].

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