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# $(L_p, L_q)$ boundedness of the fractional maximal operator on the dual of Laguerre hypergroup

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**Abstract.** *In this paper, we are interested in the Laguerre hypergroup $\mathbb{K} = [0,\infty) \times \mathbb{R}$ which is the fundamental manifold of the radial function space for the Heisenberg group. So, we consider the generalized shift operator, generated by the dual of the Laguerre hypergroup $\widehat{\mathbb{K}}$ which topologically can be identified with the so-called Heisenberg fan, the subset of $\mathbb{R}^2$:*

$$\Big( \bigcup_{m\in\mathbb{N}} \{(\lambda,\mu) \in \mathbb{R}^2 : \mu = |\lambda|(2m+\alpha+1), \lambda \neq 0\}\Big) \bigcup \{(0,\mu) \in \mathbb{R}^2 : \mu \geq 0\},$$

*by means of which fractional maximal function is investigated also the necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator on the dual of Laguerre hypergroup from the spaces $L_p(\widehat{\mathbb{K}})$ to the spaces $L_q(\widehat{\mathbb{K}})$ and from the spaces $L_1(\widehat{\mathbb{K}})$ to the weak spaces $WL_q(\widehat{\mathbb{K}})$ is obtained.*

**Keywords.** Dual of Laguerre hypergroup, Generalized translation operator, Fourier-Laguerre transform, Fractional integral.

**Mathematics Subject Classification (2010):** 42B20, 42B25, 42B35.

## 1 Introduction

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. The maximal function was firstly introduced by Hardy and Littlewood in 1930 (see [17]) for functions defined on the circle. It was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted measure spaces (see [9], [10], [21], [24], [26]). In the setting of hypergroups versions of Hardy–Littlewood maximal functions were given in [6] for the Jacobi hypergroups of compact type, in [7] for the Jacobi-type hypergroups, in [4] for the one-dimensional Chebli-Trimeche hypergroups, in [22] for the one-dimensional Bessel-Kingman hypergroups, in [11] (see also [12–14]) for the $n$-dimensional Bessel-Kingman hypergroups ($n \geq 1$), and in [15] for the dual of Laguerre hypergroups.

In the present work, we study fractional integral on the dual of Laguerre hypergroup [5, 18], so we fix $\alpha \geq 0$ and $\widehat{\mathbb{K}} \cong \mathbb{R} \times \mathbb{N}$ and we define fractional integral using the harmonic analysis on the Laguerre

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hypergroup and its dual which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2,20,23]).

The functional analysis and Fourier analysis on $\mathbb{K}$ and its dual have been extensively studied in [3] and [20], and hence, it is well known that the Fourier-Laguerre transform defined on $\mathbb{K}$ is a topological isomorphism from the Schwartz space $\mathcal{S}(\mathbb{K})$ onto $\mathcal{S}(\widehat{\mathbb{K}})$: the Schwartz space on $\widehat{\mathbb{K}}$ (see [[20], Proposition II.1]). Its inverse is given by

$$g^{\vee}(\xi) = \int_{\widehat{\mathbb{K}}} \varphi(\xi) g d\gamma_\alpha \tag{1.1}$$

where $d\gamma$ is the Plancherel measure on $\widehat{\mathbb{K}}$ given by $d\gamma(\lambda, m) = |\lambda|^{\alpha+1} d\lambda \otimes L_m^\alpha(0)\delta_m$. The topology on $\mathbb{K}$ is given by the norm $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$, while we assign to $\widehat{\mathbb{K}}$ the topology generated by the quasi-semi-norm $|(\lambda, m)|_{\widehat{\mathbb{K}}} = |\lambda|(m + \frac{\alpha+1}{2})$. For $r > 0$ we will denote by $\delta_r(\lambda, m) = (r\lambda, r^2 m)$ the dilation of $(\lambda, m) \in \widehat{\mathbb{K}}$.

The classical Riesz potential is an important technical tool in harmonic analysis, theory of functions and partial differential equations. In the present work, we study the fractional maximal function and fractional integral on the dual of Laguerre hypergroup. We define the fractional maximal function and the fractional integral using harmonic analysis on dual of Laguerre hypergroups which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see, for example [2,19,20, 23]). We obtain the necessary and sufficient conditions for the boundedness of the fractional maximal operator on the dual of Laguerre hypergroup from the spaces $L_p(\widehat{\mathbb{K}})$ to the spaces $L_q(\widehat{\mathbb{K}})$ and from the spaces $L_1(\widehat{\mathbb{K}})$ to the weak spaces $WL_q(\widehat{\mathbb{K}})$.

The paper organized as follows. In Section 2, we give the our main result on the boundness of the fractional maximal operator on the dual of Laguerre hypergroup. In Section 3, we present some definitions and auxiliary results. In section 4, we give polar coordinates in dual of Laguerre hypergroup and some lemmas needed to facilitate the proofs of our theorems. The main result of the paper is the inequality of Hardy-Littlewood-Sobolev type for the fractional integral, established in Section 5. We prove the boundedness of the fractional maximal operator from the spaces $L_p(\widehat{\mathbb{K}})$ to $L_q(\widehat{\mathbb{K}})$ and from the spaces $L_1(\widehat{\mathbb{K}})$ to the weak Lebesgue spaces $WL_q(\widehat{\mathbb{K}})$. We show that the conditions on the parameters ensuring the boundedness cannot be weakened.

Let $\alpha \geq 0$ be a fixed number, $\widehat{\mathbb{K}} = \mathbb{R} \times \mathbb{N}$ and where $d\gamma$ is the Plancherel measure on $\widehat{\mathbb{K}}$ given by $d\gamma(\lambda, m) = |\lambda|^{\alpha+1} d\lambda \otimes L_m^\alpha(0)\delta_m$.

For every $1 \leq p \leq \infty$, we denote by $L_p(\widehat{\mathbb{K}}) = L_p(\widehat{\mathbb{K}}; d\gamma_\alpha)$ the spaces of complex-valued functions $f$, measurable on $\widehat{\mathbb{K}}$ such that

$$\|f\|_{L_p(\widehat{\mathbb{K}})} = \left( \int_{\widehat{\mathbb{K}}} |f(\lambda, m)|^p \, d\gamma_\alpha(\lambda, m) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1, \infty),$$

and

$$\|f\|_{L_\infty(\widehat{\mathbb{K}})} = \underset{(\lambda,m)\in\widehat{\mathbb{K}}}{ess\sup} |f(\lambda, m)| \quad \text{if} \quad p = \infty.$$

We recall here that $d\gamma_\alpha$ is the positive measure defined on $\widehat{\mathbb{K}}$ by

$$\int_{\widehat{\mathbb{K}}} f(\lambda, m) d\gamma_\alpha(\lambda, m) = \sum_{m=0}^{\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} f(\lambda, m) |\lambda|^{\alpha+1} d\lambda.$$

For $1 \leq p < \infty$ we denote by $WL_p(\widehat{\mathbb{K}})$, the weak $L_p(\widehat{\mathbb{K}})$ spaces defined as the set of locally integrable functions $f(\lambda, m)$, $(\lambda, m) \in \widehat{\mathbb{K}}$ with the finite norm

$$\|f\|_{WL_p(\widehat{\mathbb{K}})} = \sup_{r>0} r \left( \gamma_\alpha \{(\lambda, m) \in \widehat{\mathbb{K}} : |f(\lambda, m)| > r\} \right)^{1/p}.$$

For $m \in \mathbb{N}$ we denote by $\widehat{\mathbb{K}}_m = \{(0,0)\} \cup \{R \setminus \{0\} \times \{0, 1, 2, \ldots, 2m\}\}$ and $\chi_{\widehat{\mathbb{K}}_m}$ the characteristic function of $\widehat{\mathbb{K}}_m$. In this section we introduce the ball in $\widehat{\mathbb{K}}$ with center $(\lambda, m)$ and radius $r > 0$ (for shortness $B_r$) to be the set

$$B_r(\lambda, m) = \{(\mu, n) \in \widehat{\mathbb{K}}_m, \ |(\lambda - \mu, \max(n - m, 0))|_{\widehat{\mathbb{K}}} < r\}.$$

We denote by

$$f_r(\lambda,m) = r^{-(\alpha+2)} f\left(\frac{\lambda}{r}, m\right)$$

the dilated of the function $f$ defined on $\widehat{\mathbb{K}}$ preserving the mean of $f$ with respect to the measure $d\gamma_\alpha$, in the sense that

$$\begin{aligned}
\int_{\widehat{\mathbb{K}}} f_r(\lambda,m)d\gamma_\alpha(\lambda,m) &= r^{-\alpha-2}\int_{\widehat{\mathbb{K}}} f\Big(\frac{\lambda}{r},m\Big)d\gamma_\alpha(\lambda,m)\\
&= r^{-\alpha-2}\sum_{m=0}^{\infty} L_m^{(\alpha)}(0)\int_{\mathbb{R}} f\Big(\frac{\lambda}{r},m\Big)|\lambda|^{\alpha+1}d\lambda\\
&= \sum_{m=0}^{\infty} L_m^{(\alpha)}(0)\int_{\mathbb{R}} f\Big(\lambda,m\Big)|\lambda|^{\alpha+1}d\lambda\\
&= \int_{\widehat{\mathbb{K}}} f(\lambda,m)d\gamma_\alpha(\lambda,m),\ \ \forall r>0 \text{ and } f\in L_1(\widehat{\mathbb{K}}).
\end{aligned}$$

The generalized translation operators $T^{(\alpha)}_{(\lambda,m)}$ on $\widehat{\mathbb{K}}$ are given for a suitable function $f$ by

$$T^{(\alpha)}_{(\lambda,m)} f(\mu,n) = \sum_{j\in\mathbb{N}_{m,n}} f(\lambda+\mu,j)C_j^{\alpha}((\lambda,m)(\mu,n))$$

where

$$C_j^{\alpha}((\lambda,m)(\mu,n)) = \frac{L_j^{\alpha}(0)}{\Gamma(\alpha+1)}\int_0^{\infty}\mathcal{L}_m^{\alpha}\Big(\Big|\frac{\lambda}{\lambda+\mu}\Big|x\Big)\mathcal{L}_n^{\alpha}\Big(\Big|\frac{\mu}{\lambda+\mu}\Big|x\Big)\mathcal{L}_j^{\alpha}(x)x^{\alpha}dx$$

and

$$\mathbb{N}_{m,n} = \begin{cases}\{0,1\dots,m+n\}, & \text{if } \ \lambda\mu>0,\\ \mathbb{N}, \ \ \text{if } \ \lambda\mu\le 0\end{cases}$$

with the assumption $C_j^{\alpha}((\lambda,m)(\mu,n)) = 0$ if $j\ge m+n+1$ and $\lambda\mu>0$.

Now on the dual of Laguerre hypergroup $\widehat{\mathbb{K}}$, we define the fractional maximal function by

$$M_\beta f(\lambda,m) = \sup_{r>0}\big(\gamma_\alpha B_r\big)^{\frac{\beta}{\alpha+2}-1}\int_{B_r} T^{(\alpha)}_{(\lambda,m)}|f(\mu,n)|\,d\gamma_\alpha(\mu,n), \quad 0\le\beta<\alpha+2$$

and the fractional integral by

$$I_\beta f(\lambda,m) = \int_{\widehat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)}|(\mu,n)|_{\widehat{\mathbb{K}}}^{\beta-\alpha-2}f(\mu,n)d\gamma_\alpha(\mu,n), \quad 0<\beta<\alpha+2.$$

The harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ (see [22]) is generated by the Laguerre singular differential operator

$$\mathcal{L}_\alpha = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}$$

and the norm $|(x,t)|_{\mathbb{K}} = (x^4+t^2)^{1/4}$, $(x,t)\in\mathbb{K}$, while its dual $\widehat{\mathbb{K}}$ is generated by the differential difference operator

$$\Lambda = \Lambda_1^2 - \Big(2\Lambda_2 + 2\frac{\partial}{\partial\lambda}\Big)^2,$$

where $\Lambda_1 = \frac{1}{|\lambda|}\Big(m\triangle_+\triangle_- + (\alpha+1)\triangle_+\Big)$ and $\Lambda_2 = \frac{-1}{2\lambda}\Big((\alpha+j+1)\triangle_+ + m\triangle_-\Big)$ and the quasinorm $|(\lambda,m)|_{\widehat{\mathbb{K}}} = |\lambda|(m+\frac{\alpha+1}{2})$, $(\lambda,m)\in\widehat{\mathbb{K}}$, where the difference operators $\triangle_\pm$ are given for a suitable function $\Phi:\mathbb{R}\times\mathbb{N}\to\mathbb{C}$ by:

$$\triangle_+\Phi(\lambda,m) = \Phi(\lambda,m+1)-\Phi(\lambda,m), \triangle_-\Phi(\lambda,m) = \Phi(\lambda,m)-\Phi(\lambda,m-1), \text{ if } m\ge 1$$

and $\triangle_-\Phi(\lambda,0) = \Phi(\lambda,0)$.

The inverse of the Laguerre-Fourier transform is given for suitable functions $\Psi$ defined on $\mathbb{R}\times\mathbb{N}$ by:

$$F^{-1}(\Psi)(x,t)=\int_{\mathbb{R}\times\mathbb{N}}\varphi_{\lambda,m}(x,t)\Psi d\gamma_\alpha,$$

where $d\gamma_\alpha$ is the measure defined for suitable functions $\Phi$ by

$$\int_{\mathbb{R}\times\mathbb{N}}\Phi(\lambda,m)d\gamma_\alpha(\lambda,m)=\sum_{m=0}^{\infty}L_m^\alpha(0)\int_{\mathbb{R}}\Phi(\lambda,m)|\lambda|^{\alpha+1}d\lambda.$$

The Fourier-Laguerre transform satisfies the following underlying properties $F^{-1}(\Lambda_1\Phi)=-x^2F^{-1}(\Phi)$ and $F^{-1}\left[\left(\Lambda_2+\frac{\partial}{\partial\lambda}\right)\Phi\right]=itF^{-1}(\Phi)$.

So, combining the above relations we obtain:

$$(1+\Lambda)\varphi_{\lambda,m}(x,t)=(1+x^4+4t^2)\varphi_{\lambda,m}(x,t).$$

Furthermore, we get the following fundamental relation

$$F^{-1}((1+\Lambda)\Phi)=(1+x^4+4t^2)F^{-1}(\Phi).$$

We remark that for appropriate functions $\Phi,\Psi:\mathbb{R}\times\mathbb{N}\to\mathbb{C}$ and $\varphi\in S_*(\mathbb{K})$, the operator $\Lambda$ and the Laguerre-Fourier transform satisfy respectivelly the following relations:

$$\int_{\mathbb{R}\times\mathbb{N}}\Lambda(\Phi)\Psi d\gamma_\alpha=\int_{\mathbb{R}\times\mathbb{N}}\Lambda(\Psi)\Phi d\gamma_\alpha,$$

$$\int_{\mathbb{K}}F^{-1}(\Phi)\varphi dm_\alpha=\int_{\mathbb{R}\times\mathbb{N}}F(\check{\varphi})\Phi d\gamma_\alpha,$$

where $\check{F}(\varphi)(\lambda,m)=F(\check{\varphi})(\lambda,m)=F(\varphi)(-\lambda,m)$.

These operators satisfy some basic properties which can be found in [2], [3] and [20], namely one has

$$\mathcal{L}_\alpha\varphi_{(\lambda,m)}=-|(\lambda,m)|_{\widehat{\mathbb{K}}}\varphi(\lambda,m)\text{ and }\Lambda\varphi_{(\lambda,m)}(x,t)=N^4(x,t)\varphi_{(\lambda,m)}. \tag{1.2}$$

Unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t)=e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right),\quad (x,t)\in\mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x)=e^{-x/2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree $m$ and order $\alpha$ (see [2]).

For $f\in L_1(\widehat{\mathbb{K}})$ the Fourier-Laguerre transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(\lambda,m)=\int\limits_{\widehat{\mathbb{K}}}\varphi_{-\lambda,m}(\lambda,m)f(\lambda,m)d\gamma_\alpha(\lambda,m),$$

such that

$$\|\mathcal{F}(f)\|_{L_\infty(\widehat{\mathbb{K}})}\le\|f\|_{L_1(\widehat{\mathbb{K}})}.$$

The generalized translation operators $T_{(\lambda,m)}^{(\alpha)}$ on the dual of Laguerre hypergroup satisfies the following properties

$$T_{(\lambda,m)}^{(\alpha)}f(\mu,n)=T_{(\mu,n)}^{(\alpha)}f(\lambda,m),\quad T_{(0,0)}^{(\alpha)}f(\mu,n)=f(\mu,n),$$

$$\|T_{(\lambda,m)}^{(\alpha)}f\|_{L_p(\widehat{\mathbb{K}})}\le\|f\|_{L_p(\widehat{\mathbb{K}})}\quad\text{for all}\quad f\in L_p(\widehat{\mathbb{K}}),\ 1\le p\le\infty, \tag{1.3}$$

$$\mathcal{F}\big(T_{(\lambda,m)}^{(\alpha)}f\big)(\lambda,m)=\mathcal{F}(f)(\lambda,m)\,\varphi_{\lambda,m}(\lambda,m).$$

The translation operator $T^{(\alpha)}_{(\lambda,m)}$ is defined by

$$T^{(\alpha)}_{(\lambda,m)} f(\mu,n) = \int_{\widehat{\mathbb{K}}} f(z,v) W_\alpha((\lambda,m),(\mu,n),(z,v)) z^{2\alpha+1} dz dv,$$

where $dzdv$ is the Lebesgue measure on $\widehat{\mathbb{K}}$, and $W_\alpha$ is an appropriate kernel satisfying

$$\int_{\widehat{\mathbb{K}}} W_\alpha((\lambda,m),(\mu,n),(z,v)) z^{2\alpha+1} dz dv = 1$$

(see [19]).

By using the generalized translation operators $T^{(\alpha)}_{(\lambda,m)}$, $(\lambda,m) \in \widehat{\mathbb{K}}$, we define a generalized convolution product $*$ on $\widehat{\mathbb{K}}$ by

$$\left(\delta_{(\lambda,m)} * \delta_{(\mu,n)}\right)(f) = T^{(\alpha)}_{(\lambda,m)} f(\mu,n),$$

where $\delta_{(\lambda,m)}$ is the Dirac measure at $(\lambda,m)$.

**Lemma 1.1** *[25] Let $\alpha \geq 0$. Then for all $(\lambda,m),(\mu,n)$ in $\widehat{\mathbb{K}}$ we have*

$$C^\alpha_j((\lambda,m)(\mu,n)) \geq 0, \textit{ for all } j \geq 0; \tag{1.4}$$

$$\sum_{j \in N_{m,n}} C^\alpha_j((\lambda,m)(\mu,n)) = 1; \tag{1.5}$$

$$C^\alpha_j((\lambda,m)(\mu,n)) = \frac{L^\alpha_j(0)}{L^\alpha_n(0)} \left|\frac{\lambda+\mu}{\mu}\right|^{\alpha+1} C^\alpha_n((-\lambda,m)(\mu+\lambda,j)). \tag{1.6}$$

We define the convolution product on the space $M_b(\widehat{\mathbb{K}})$ of bounded Radon measures on $\widehat{\mathbb{K}}$ by

$$(\mu * \nu)(f) = \int_{\widehat{\mathbb{K}} \times \widehat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} f(\mu,n)\, d\mu(\lambda,m)\, d\nu(\mu,n).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with } \check{g}(\mu,n) = g(y,-s),$$

where, $h$ and $g$ belong to the space $L_1(\widehat{\mathbb{K}})$ of the integrable functions on $\widehat{\mathbb{K}}$ with respect to the measure $d\gamma_\alpha(\lambda,m)$, and $h * g$ is the convolution product defined by

$$(h * g)(\lambda,m) = \int_{\widehat{\mathbb{K}}} T^{(\alpha)}_{(\lambda,m)} h(\mu,n)\, g(-\mu,n)\, dm_\alpha(\mu,n), \quad \text{for all} \quad (\lambda,m) \in \widehat{\mathbb{K}}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \leq p,\, r \leq q \leq \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\widehat{\mathbb{K}})$, and $g \in L_r(\widehat{\mathbb{K}})$, then $f * g \in L_q(\widehat{\mathbb{K}})$ and

$$\|f * g\|_{L_q(\widehat{\mathbb{K}})} \leq \|f\|_{L_p(\widehat{\mathbb{K}})} \|g\|_{L_r(\widehat{\mathbb{K}})}, \tag{1.7}$$

where $p' = p/(p-1)$.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2 Boundedness of the fractional maximal function on the dual of Laguerre hypergroup

In this section we introduce the ball in $\widehat{\mathbb{K}}$ with center $(\lambda, m)$ and radius $r > 0$ (for shortness $B_r$) to be the set

$$B_r(\lambda, m) = \{(\mu, n) \in \widehat{\mathbb{K}}_m : |(\lambda - \mu, \max(n - m, 0))|_{\widehat{\mathbb{K}}} < r\}.$$

**Lemma 2.1** *The measure of $B_r$ with respect to the Plancherel measure $d\gamma_\alpha$ is finite, in the sense that for all $r > 0$ and $(\lambda, m) \in \widehat{\mathbb{K}}$*

$$\gamma_\alpha(B_r) < \infty.$$

*Proof.* For fixed $(\lambda, m) \in \widehat{\mathbb{K}}$ and $r > 0$ one has

$$\begin{aligned}\gamma_\alpha(B_r) &= \int_{B_r} d\gamma_\alpha(\mu, n)\\ &= \sum_{n=0}^{\infty} L_n^\alpha(0) \int_{\frac{-r}{\max(n-m,0)+\frac{\alpha+1}{2}}+\lambda}^{\frac{r}{\max(n-m,0)+\frac{\alpha+1}{2}}+\lambda} |\mu|^{\alpha+1} d\mu\\ &= \sum_{n=m+1}^{\infty} L_n^\alpha(0) \int_{\frac{-r}{n-m+\frac{\alpha+1}{2}}+\lambda}^{\frac{r}{n-m+\frac{\alpha+1}{2}}+\lambda} |\mu|^{\alpha+1} d\mu + \sum_{n=0}^{m} L_n^\alpha(0) \int_{\frac{-r}{\frac{\alpha+1}{2}}+\lambda}^{\frac{r}{\frac{\alpha+1}{2}}+\lambda} |\mu|^{\alpha+1} d\mu\\ &= I_1 + I_2\end{aligned}$$

where

$$I_1 = \frac{1}{\alpha+2} \sum_{n=m+1}^{\infty} L_n^\alpha(0) \Big[\Big(\frac{r}{n-m+\frac{\alpha+1}{2}} + \lambda\Big)^{\alpha+2} - \Big(-\frac{r}{n-m+\frac{\alpha+1}{2}} + \lambda\Big)^{\alpha+2}\Big]$$

and

$$I_2 = \frac{\Big[\Big(\frac{r}{\frac{\alpha+1}{2}} + \lambda\Big)^{\alpha+2} - \Big(-\frac{r}{\frac{\alpha+1}{2}} + \lambda\Big)^{\alpha+2}\Big]}{\alpha+2} \sum_{n=0}^{m} L_n^\alpha(0).$$

Since $L_n^\alpha(0) \approx \frac{n^\alpha}{\Gamma(\alpha+1)}$ it follows $\frac{L_n^\alpha(0)}{\alpha+2}\Big[\Big(\frac{r}{n+\frac{\alpha+1}{2}} + \lambda\Big)^{\alpha+2} - \Big(\frac{r}{n+\frac{\alpha+1}{2}} - \lambda\Big)^{\alpha+2}\Big] \approx \frac{C_r}{n^2}$, and that $d\gamma_\alpha(B_r) < \infty$, where we have computed the result above for the case $\lambda > 0$. An analogous result follows for the complement case.

One can remark easily that the volume $\Omega_2$ of the ball $B_r$ depends not only on its radius $r$, but also it is largely close to its center $(\lambda, m)$ which means that the Plancherel measure $d\gamma_\alpha$ is not invariant under the standard translation over $\mathbb{R}^2$.

Now on the dual of Laguerre hypergroup $\widehat{\mathbb{K}}$, we define the fractional maximal function by

$$M_\beta f(\lambda, m) = \sup_{r>0} \big(\gamma_\alpha B_r\big)^{\frac{\beta}{\alpha+2}-1} \int_{B_r} T^{(\alpha)}_{(\lambda,m)} |f(\mu, n)|\, d\gamma_\alpha(\mu, n), \quad 0 \le \beta < \alpha + 2$$

If $\beta = 0$, then $M_0 = M$ is the Hardy-Littlewood maximal operator on the dual of Laguerre hypergroup (see [1]).

In the following theorem, we obtain the boundedness of the fractional maximal operator $M_\beta$ on the dual of the Laguerre hypergroup in the spaces $L_p(\widehat{\mathbb{K}})$ to $L_q(\widehat{\mathbb{K}})$, $1 < p < q < \infty$ and from the spaces $L_1(\widehat{\mathbb{K}})$ to the weak spaces $WL_q(\widehat{\mathbb{K}})$, $1 < q < \infty$.

**Theorem 2.1** *Let $0 \le \beta < \alpha + 2$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\alpha+2}$ and $1 \le p \le \frac{\alpha+2}{\beta}$.*

*1)If $p = 1$, $f \in L_1(\hat{K})$, then for all $\theta > 0$*

$$\int\limits_{\{(\lambda,m)\in\hat{K}: M_\beta f > \theta\}} d\gamma_\alpha(\lambda, m) \le \left(\frac{C}{\theta} \int_{\hat{K}} |f(\lambda, m)| d\gamma_\alpha(\lambda, m)\right)^{\frac{1}{q}}.$$

*2) If* $1 < p < \frac{\alpha+2}{\beta}$ *and* $f \in L_p(\hat{K})$*, then* $M_\beta f \in L_q(\hat{K})$ *and*

$$\left(\int_{\hat{K}} (M_\beta f(\lambda,m))^q d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{q}} \le C\left(\int_{\hat{K}} |f(\lambda,m)|^p dm_\alpha(\lambda,m)\right)^{\frac{1}{p}}.$$

*3) If* $p = \frac{\alpha+2}{\beta}$ *and* $f \in L_p(\hat{K})$*, then* $M_\beta f \in L_\infty(\hat{K})$ *and*

$$\sup_{(\lambda,m)\in\hat{K}} M_\beta f(\lambda,m) \le C\left(\int_{\hat{K}} |f(\lambda,m)|^p d\gamma_\alpha(\lambda,m)\right)^{\frac{1}{p}}.$$

*Proof.* The fractional maximal function $M_\beta f(\lambda,m)$ may be interpreted as a fractional maximal function defined on a space of homogeneous type. By this we mean a topological space $X$ equipped with a continuous pseudometric $\rho$ and a positive measure $\nu$ satisfying

$$\nu(E(\xi,2r)) \le \nu(E(\xi,r)), \tag{2.1}$$

with a constant $C$ independent of $\xi$ and $r > 0$.

Here $E(\xi,r) = \{\eta \in X \,:\, \rho(\xi,\eta) < r\},\ \rho(\xi,\eta) = |\xi-\eta|$.

Let $(X,\rho,\nu)$ be a space of homogeneous type. Define

$$M_{\nu,\beta} f(x) = \sup_{r>0} \nu(E(\xi,r))^{-1+\frac{\beta}{\alpha+2}} \int_{E(\xi,r)} |f(\eta)| d\mu(\eta), \qquad 0 < \beta < \alpha+2.$$

It is well known that the fractional maximal operator $M_{\nu,\beta}$ is of weak type $(1,q)$, $1-1/q = \beta/(\alpha+2)$ and is bounded from $L_p(X,\nu)$ to $L_q(X,\nu)$, $1/p - 1/q = \beta/(\alpha+2)$ for $1 < p < (\alpha+2)/\beta$ (see [8,16]).

We shall use this result in the case in which

$$X = \widehat{\mathbb{K}},\ \xi = (\lambda,m),\ \eta = (\mu,n) \in \widehat{\mathbb{K}},$$
$$\rho(\xi,\eta) = |(\lambda-\mu, \max\{|n-m|,0\})|_{\widehat{\mathbb{K}}},\ \ d\nu(\xi) = d\gamma_\alpha(\lambda,m).$$

It is clear that this measure satisfies the doubling condition (2.1). We will shall show that

$$M_\beta f(\lambda,m) \le C M_{\nu,\beta} f(2\lambda,m). \tag{2.2}$$

From the definition of the generalized shift operator it follows that

$$\begin{aligned}
&\int_{B_r(\lambda,m)} T_{(\lambda,m)} |f(\mu,n)| d\gamma_\alpha(\mu,n)\\
&= \int_{B_r(\lambda,m)} \sum_{j\in\mathbb{N}_{m,n}} C_j^\alpha((\lambda,m)(\mu,n)) |f(\mu+\lambda,j)| d\gamma_\alpha(\mu,n)\\
&= \sum_{n=0}^{\infty} L_n^\alpha(0) \int_{-r/\max(n-m,0)+(\alpha+1)/2+\lambda}^{r/\max(n-m,0)+(\alpha+1)/2+\lambda} \sum_{j\in\mathbb{N}_{m,n}} C_j^\alpha((\lambda,m)(\mu,n)) |f(\mu+\lambda,j)| |\mu|^{\alpha+1} d\mu.
\end{aligned}$$

From equalities (1.5) and (1.6) we get

$$\begin{aligned}
&\int_{B_r(\lambda,m)} T_{(\lambda,m)} |f(\mu,n)| d\gamma_\alpha(\mu,n) \le C \sum_{j\ge 0} L_n^\alpha(0) \int_{-r/\max(j-m,0)+(\alpha+1)/2+\lambda}^{r/\max(j-m,0)+(\alpha+1)/2+\lambda}\\
&\times \underbrace{\sum_{n\ge 0} C_n^\alpha((-\lambda,m)(\mu+\lambda,j))}_{=1} |\mu+\lambda|^{\alpha+1} |f(\mu+\lambda,j)| d\mu\\
&= C \sum_{j\ge 0} L_n^\alpha(0) \int_{-r/\max(j-m,0)+(\alpha+1)/2+\lambda}^{r/\max(j-m,0)+(\alpha+1)/2+\lambda} |\mu|^{\alpha+1} |f(\mu,j)| d\mu\\
&\le C \int_{B_r(2\lambda,m)} |f(\mu,j)| d\gamma_\alpha(\mu,j).
\end{aligned}$$

Since $\gamma_\alpha(B_r(2\lambda,m)) = 2^{\alpha+2}\gamma_\alpha(B_{r/2}(\lambda,m)) \le C\gamma_\alpha(B_r(\lambda,m))$, we obtain

$$M_\beta f(\lambda,m) \le CM_{\nu,\beta}f(2\lambda,m),$$

which completes the proof of parts 1) and 2).

3) Let $p = (\alpha+2)/\beta$, $f \in L_p(\widehat{\mathbb{K}})$ then applying Holder's inequality and the inequality (1.3) we have

$$\begin{aligned}
&(\gamma_\alpha(B_r))^{\frac{\beta}{\alpha+2}-1}\int_{B_r} T^\alpha_{(\lambda,m)}|f(\mu,n)|d\gamma_\alpha(\mu,n)\\
&\le (\gamma_\alpha(B_r))^{\frac{\beta}{\alpha+2}-1+\frac{1}{p'}}\Big(\int_{B_r}\Big(T^\alpha_{(\lambda,m)}|f(\mu,n)|\Big)^p d\gamma_\alpha(\mu,n)\Big)^{\frac{1}{p}}\\
&= \|T^{(\alpha)}_{(\lambda,m)}f\|_{L_p(\widehat{\mathbb{K}})} \le \|f\|_{L_p(\widehat{\mathbb{K}})}.
\end{aligned}$$

In the case $\beta = 0$ from Theorem 2.1 we get the following corollary.

**Corollary 2.1** *[1] 1) If $f \in L_p(\widehat{\mathbb{K}})$, $1 < p \le \infty$, then $Mf \in L_p(\widehat{\mathbb{K}})$ and*

$$\|Mf\|_{L_p(\widehat{\mathbb{K}})} \le C_p\|f\|_{L_p(\widehat{\mathbb{K}})},$$

*where $C_p > 0$ is independent of $f$.*

*2) If $f \in L_1(\widehat{\mathbb{K}})$, then for every $\tau > 0$*

$$\gamma_\alpha\{(\lambda,m) \in \hat{\mathbb{K}} : Mf(\lambda,m) > \tau\} \le \frac{C}{\tau}\int_{\hat{K}} |f(\lambda,m)|d\gamma_\alpha(\lambda,m),$$

*that is, we have*

$$\|Mf\|_{WL_1(\widehat{\mathbb{K}})} \le C\|f\|_{L_1(\widehat{\mathbb{K}})},$$

*where $C > 0$ is independent of $f$.*

In the following theorem, we obtain the necessary and sufficient conditions for the fractional maximal operator $M_\beta$ on the dual of the Laguerre hypergroup to be bounded from the spaces $L_p(\widehat{\mathbb{K}})$ to $L_q(\widehat{\mathbb{K}})$, $1 < p < q < \infty$ and from the spaces $L_1(\widehat{\mathbb{K}})$ to the weak spaces $WL_q(\widehat{\mathbb{K}})$, $1 < q < \infty$.

**Theorem 2.2** *Let $0 \le \beta < \alpha+2$ and $1 \le p \le \frac{\alpha+2}{\beta}$.*

*1) If $p = 1$, then the condition $1 - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_1(\widehat{\mathbb{K}})$ to $WL_q(\widehat{\mathbb{K}})$.*

*2) If $1 < p < \frac{\alpha+2}{\beta}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{\alpha+2}$ is necessary and sufficient for the boundedness of $M_\beta$ from $L_p(\widehat{\mathbb{K}})$ to $L_q(\widehat{\mathbb{K}})$.*

*3) If $p = \frac{\alpha+2}{\beta}$, then $M_\beta$ is bounded from $L_p(\widehat{\mathbb{K}})$ to $L_\infty(\widehat{\mathbb{K}})$.*

*Proof.* The sufficiency part of the proof follows from Theorem 2.1.

Let us prove the necessity:

1) Let $M_\beta$ bounded from $L_1(\widehat{\mathbb{K}})$ to $WL_q(\widehat{\mathbb{K}})$.

Define $f_r(\lambda,m) := f(r\lambda,m)$, then

$$\|f_r\|_{L_p(\widehat{\mathbb{K}})} = r^{-\frac{\alpha+2}{p}}\|f\|_{L_p(\widehat{\mathbb{K}})}$$

and

$$\|M_\beta f_r\|_{WL_q(\widehat{\mathbb{K}})} = r^{-\beta-\frac{\alpha+2}{q}}\|M_\beta f\|_{WL_q(\widehat{\mathbb{K}})}.$$

By the boundedness of $M_\beta$ from $L_1(\widehat{\mathbb{K}})$ to $WL_q(\widehat{\mathbb{K}})$, it follows that

$$\begin{aligned}
\|M_\beta f\|_{WL_q(\widehat{\mathbb{K}})} &= r^{\beta+\frac{\alpha+2}{q}}\|M_\beta f_r\|_{WL_q(\widehat{\mathbb{K}})}\\
&\le Cr^{\beta+\frac{\alpha+2}{q}}\|f_r\|_{L_1(\widehat{\mathbb{K}})}\\
&= Cr^{\beta+\frac{\alpha+2}{q}-(\alpha+2)}\|f\|_{L_1(\widehat{\mathbb{K}})}.
\end{aligned}$$

If $1 < \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_1(\widehat{\mathbb{K}})$ we have $\|M_\beta f\|_{WL_q(\widehat{\mathbb{K}})} = 0$ as $r \to 0$, which is impossible.

Similarly, if $1 > \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_1(\widehat{\mathbb{K}})$ we obtain $\|M_\beta f\|_{WL_q(\widehat{\mathbb{K}})} = 0$ as $r \to \infty$, which is also impossible.

Hence we get $1 = \frac{1}{q} + \frac{\beta}{\alpha+2}$.

2) Let $1 < p < \frac{\alpha+2}{\beta}$, $f \in L_p(\widehat{\mathbb{K}})$ and assume that the inequality

$$\|M_\beta f\|_{L_q(\widehat{\mathbb{K}})} \le C\|f\|_{L_p(\widehat{\mathbb{K}})}$$

then we have

$$\|M_\beta f_r\|_{L_q(\widehat{\mathbb{K}})} = r^{-\beta-\frac{2\alpha+4}{q}}\|M_\beta f\|_{L_q(\widehat{\mathbb{K}})}.$$

By the boundedness of $M_\beta$ from $L_p(\widehat{\mathbb{K}})$ to $L_q(\widehat{\mathbb{K}})$, we obtain

$$\begin{aligned}\|M_\beta f\|_{L_q(\widehat{\mathbb{K}})} &= r^{\beta+\frac{\alpha+2}{q}}\|M_\beta f_r\|_{L_q(\widehat{\mathbb{K}})}\\ &\le Cr^{\beta+\frac{\alpha+2}{q}}\|f_r\|_{L_p(\widehat{\mathbb{K}})}\\ &\le Cr^{\beta+\frac{\alpha+2}{q}-\frac{\alpha+2}{p}}\|f\|_{L_p(\widehat{\mathbb{K}})}.\end{aligned}$$

If $\frac{1}{p} > \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_p(\widehat{\mathbb{K}})$ we have $\|M_\beta f\|_{L_q(\widehat{\mathbb{K}})} = 0$ as $r \to 0$, which is impossible.

Similarly, if $\frac{1}{p} < \frac{1}{q} + \frac{\beta}{\alpha+2}$, then for all $f \in L_p(\widehat{\mathbb{K}})$ we obtain $\|M_\beta f\|_{L_q(\widehat{\mathbb{K}})} = 0$ as $r \to \infty$, which is also impossible.

Therefore $\frac{1}{p} = \frac{1}{q} + \frac{\beta}{\alpha+2}$.

The part 3) proved in Theorem 2.1.

Thus the proof of Theorem 2.2 is completed.

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