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Riesz Potentials In Spaces Defined By Conditions On Local Oscillations Of Functions

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Abstract. *It is well known that potential type integrals play an important role in research of various problems of the harmonic analysis and have numerous applications in the theory of the partial differential equations. The present paper studies properties of Riesz potential in terms of local oscillation of functions.*

1. Preliminaries

Let R^n be an n- dimensional Euclidean space of the points $x = (x_1, x_2, ..., x_n)$, $B(a, r) := \{x \in R^n : |x - a| \le r\}$ is a closed ball in R^n of radius $r > 0$ with the center at the point $a \in R^n$. Denote the class of all local p-power summable functions defined on R^n by $L^p_{loc}(R^n)$, $(1 \le p < \infty)$, the class of all local bounded functions defined on R^n by $L^p_{loc}(R^n)$. By $L^p(R^n)$ we mean the usual Lebesgue space on R^n, and we denote by $\|\cdot\|_{L^p}$ the corresponding norm, that is

$$\|f\|_p = \|f\|_{L^p(R^n)} := \left(\int\limits_{R^n} |f(x)|^p \, dx\right)^{\frac{1}{p}} \text{ if } 1 \le p < \infty,$$

$$\text{and } \|f\|_\infty = \|f\|_{L^\infty(R^n)} := ess\sup\left\{|f(x)| := x \in R^n\right\}.$$

Denote by P_k the totality of all polynomials on R^n whose degrees are equal to or less than k.

Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, $1 \le q \le \infty$, $k \in N$ (N is a set of natural numbers). Define the following functions

$$\mu^k_f(x; r)_p := \inf_{\pi \in P_{k-1}} \|f - \pi\|_{L^p(B(x,r))}, \quad r > 0, \quad x \in R^n,$$

$$\mu^k_f(r)_{pq} := \begin{cases} \left\|\mu^k_f(\cdot; r)_p\right\|_{L^q(R^n)} & if \quad 1 \le q < \infty \\ \sup\limits_{x \in R^n} \mu^k_f(x; r)_p & if \quad q = \infty. \end{cases}$$

From definitions easily follows, that $\mu^{k+1}_f(x; r)_p \le \mu^k_f(x; r)_p$ and $\mu^{k+1}_f(r)_{pq} \le \mu^k_f(r)_{pq}$ $(x \in R^n, r > 0)$. We can show that if $B(x_1, r_1) \subset B(x_2, r_2)$ then $\mu^k_f(x_1; r_1)_p \le \mu^k_f(x_2; r_2)_p$.

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It follows that $\mu_f^k(r)_{pq}$ increases monotonically.

Proposition 1. Let $f \in L_{loc}^{q_2}(R^n)$, $1 \le q_1 < q_2 \le \infty$, $k \in N$. Then the following inequality is true

$$\mu_f^k(x;r)_{q_1} \le |B(0,1)|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot r^{n\left(\frac{1}{q_1}-\frac{1}{q_2}\right)} \cdot \mu_f^k(x;r)_{q_2}, \quad \left(x \in R^n, r>0\right). \tag{1}$$

Proof. Let $1 \le q_1 < q_2 \le \infty$. If we denote $q := \frac{q_2}{q_1}$, $\frac{1}{q}+\frac{1}{q'}=1$, then for any polynomial $\pi \in P_{k-1}$ we obtain by using Holder's inequality

$$\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_1}\,dt \le \left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_1\cdot q}\,dt\right)^{\frac{1}{q}} \cdot \left(\int\limits_{B(x,r)} dt\right)^{\frac{1}{q'}}$$

$$= |B(x,r)|^{\frac{1}{q'}} \cdot \left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_2}\,dt\right)^{\frac{q_1}{q_2}}$$

$$= |B(0,1)|^{\frac{1}{q'}} \cdot r^{n\cdot\frac{1}{q'}} \left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_2}\,dt\right)^{\frac{q_1}{q_2}}.$$

It follows that

$$\left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_1}\,dt\right)^{\frac{1}{q_1}}$$

$$\le |B(0,1)|^{\frac{1}{q'\cdot q_1}} \cdot r^{n\cdot\frac{1}{q'\cdot q_1}} \left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_2}\,dt\right)^{\frac{1}{q_2}}.$$

Since

$$\frac{1}{q'\cdot q_1} = \frac{1}{q_1}\left(1-\frac{1}{q}\right) = \frac{1}{q_1}\left(1-\frac{q_1}{q_2}\right) = \frac{1}{q_1}-\frac{1}{q_2},$$

we have

$$\left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_1}\,dt\right)^{\frac{1}{q_1}}$$

$$\le |B(0,1)|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot r^{n\cdot\left(\frac{1}{q_1}-\frac{1}{q_2}\right)} \cdot \left(\int\limits_{B(x,r)} |f(t)-\pi(t)|^{q_2}\,dt\right)^{\frac{1}{q_2}}.$$

From this it is easy to obtain the inequality (1) in the case $1 \le q_1 < q_2 < \infty$. The case $1 \le q_1 < q_2 = \infty$ is easily verified. The proposition is proved.

Corollary 1. *Let $f \in L_{loc}^{q_2}(R^n)$, $1 \le q_1 < q_2 \le \infty$, $k \in N$, $1 \le p \le \infty$.*

Then the next inequality holds true

$$\mu_f^k(r)_{q_1 p} \le |B(0,1)|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot r^{n\cdot\left(\frac{1}{q_1}-\frac{1}{q_2}\right)} \mu_f^k(r)_{q_2 p}, \quad r>0. \tag{2}$$

Let $x = (x_1, x_2, ..., x_n) \in R^n$, $v = (v_1, v_2, ..., v_n)$, $v_j\,(i = 1, 2, ..., n)$ be non-negative integers, $|v| = v_1 + v_2 + ... + v_n$, $x^v = x_1^{v_1} \cdot x_2^{v_2} \cdots x_n^{v_n}$. Apply the orthogonalization process by the scalar product

$$(f, g) = \frac{1}{|B(0,1)|} \int\limits_{B(0,1)} f(t)\, g(t)\, dt$$

to the system of the power functions $\{x^v\}$, $|v| \leq k$, $(k \in N \cup \{0\})$ arranged by partially lexicographic order[1] [1], where$|E|$ is the Lebesque measure of the set $E \subset R^n$. Denote by $\{\varphi_v\}$, $|v| \leq k$ the obtained orthogonal normed system.

Let $L^1_{loc}(R^n)$. Suppose that ([2], [3]):

$$P_{k,B(a,r)} f(x) = \sum_{|v| \leq k} \left(\frac{1}{|B(a,r)|} \int\limits_{B(a,r)} f(t)\, \varphi_v \left(\frac{t-a}{r} \right) dt \right) \varphi_v \left(\frac{x-a}{r} \right).$$

It is obvious that $P_{k,B(a,r)} f$ is a polynomial degree of which is equal or less than k.

Denote

$$O_k(f, B(a,r))_p := \left\| f - P_{k-1,B(a,r)} f \right\|_{L^p(B(a,r))}$$

for $f \in L^p_{loc}(R^n)$ $(1 \leq p \leq \infty)$. Let us call $O_k(f, B(a,r))$ local oscillation of k-th order of the function f on the ball $B(a,r)$ in the metric L^p.

Note that if $k = 0$ then

$$P_{k,B(a,r)} f(x) \equiv \frac{1}{|B(a,r)|} \int\limits_{B(a,r)} f(t)\, dt =: f_{B(a,r)},$$

and therefore

$$O_1(f, B(a,r))_1 = \int\limits_{B(a,r)} \left| f(t) - f_{B(a,r)} \right| dt.$$

It is known that [4] for each polynomial $\pi \in P_{k-1}$ and each ball $B(x,r) \subset R^n$ the inequality

$$\left\| f - P_{k-1,B(x,r)} \right\|_{L^p(B(x,r))} \leq C \left\| f - \pi \right\|_{L^p(B(x,r))}$$

is true, where the positive constant C does not depend on p, f, B and π. Hence it follows that

$\exists C > 0$, $\forall x \in R^n$, $\forall r > 0$:

$$\mu^k_f(x; r)_p \leq O_k(f, B(x,r))_p \leq C \cdot \mu^k_f(x; r)_p\,.$$

It should be mentioned that the theory of spaces defined by local oscillation has been developed by several authors, for instance F.John and L.Nirenberg [5], S.Campanato [6], N.G.Meyers [7], S.Spanne [8], J.Peetre [9], D.Sarason [10] etc. (see also [11], [12]).

2. Definition and some properties of spaces $L^{k,\varphi}_{p,q,\theta}$

Denote by Φ the class of all the positive functions $\varphi(t)$ monotonically increasing in $(0, +\infty)$ such that $\varphi(+0) = 0$.

Let ν is a positive number. We denote by Φ_ν a set of all $\varphi \in \Phi$ such that $\varphi(t) \cdot t^{-\nu}$ almost decreases[2] on $(0, +\infty)$.

Let $k \in N$, $1 \leq p, q, \theta \leq \infty$, $\varphi \in \Phi_{k+\frac{n}{p}}$. We denote by $L^{k,\varphi}_{p,q,\theta}$ the class of all the functions $f \in L^p_{loc}(R^n)$ such that $\|f\|_{L^{k,\varphi}_{p,q,\theta}} < +\infty$ where for $1 \leq \theta < \infty$

$$\|f\|_{L^{k,\varphi}_{p,q,\theta}} := \left(\int\limits_0^\infty \left(\frac{\mu^k_f(t)_{pq}}{\varphi(t)} \right)^\theta \frac{dt}{t} \right)^{\frac{1}{\theta}},$$

[1] It means that x^ν precedes x^μ if either $|\nu| < |\mu|$, or $|\nu| = |\mu|$ but the first nonzero difference $\nu_i - \mu_i$ is negative.

[2] A nonnegative function $h(t), t \in (0, +\infty$, is said to be almost decreasing when there exists a constant $c > 0$ such that $h(t_1) \geq ch(t_2)$ is satisfied for every $t_1, t_2 \in (0, +\infty)$ with $t_1 < t_2$.

and for $\theta = \infty$

$$\|f\|_{L^{k,\varphi}_{p,q,\theta}} := \sup\left\{\frac{\mu^k_f(t)_{pq}}{\varphi(t)} : t > 0\right\}.$$

If we consider the class $L^{k,\varphi}_{p,q,\theta}$ as a subset in the quotient space $L^p_{loc}(R^n)/P_{k-1}$, then $\|\cdot\|_{L^{k,\varphi}_{p,q,\theta}}$ is the norm on $L^{k,\varphi}_{p,q,\theta}$. We can show that $L^{k,\varphi}_{p,q,\theta}$ with the norm $\|\cdot\|_{L^{k,\varphi}_{p,q,\theta}}$ is a complete normed space.

Theorem 1. *Suppose* $1 \le q_1 < q_2 \le \infty$, $1 \le p, \theta \le \infty$, $\varphi \in \Phi_{k+\frac{n}{q_2}}$.

Then

$$L^{k,\varphi}_{q_2,p,\theta} \subset L^{k,\Psi}_{q_1,p,\theta}$$

and

$$\exists c > 0,\ \forall f \in L^{k,\varphi}_{q_2,p,\theta} : \|f\|_{L^{k,\Psi}_{q_1,p,\theta}} \le c \cdot \|f\|_{L^{k,\varphi}_{q_2,p,\theta}},$$

where $\psi(r) = \varphi(r) \cdot r^{n\cdot\left(\frac{1}{q_1}-\frac{1}{q_2}\right)}$, $r > 0$.

Proof. If $f \in L^{k,\varphi}_{q_2,p,\theta}$, then using inequality (2) in case $1 \le \theta < \infty$ we have

$$\|f\|_{L^{k,\Psi}_{q_1,p,\theta}} = \left(\int_0^\infty \left(\frac{\mu^k_f(t)_{q_1 p}}{\psi(t)}\right)^\theta \frac{dt}{t}\right)^{\frac{1}{\theta}}$$

$$\le |B(0,1)|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot \left(\int_0^\infty \left(\frac{\mu^k_f(t)_{q_2 p}}{\varphi(t)}\right)^\theta \frac{dt}{t}\right)^{\frac{1}{\theta}} = |B(0,1)|^{\frac{1}{q_1}-\frac{1}{q_2}} \cdot \|f\|_{L^{k,\varphi}_{q_2,p,\theta}}.$$

Analogously we consider the case $\theta = \infty$. The theorem is proved.

Proposition 2. Suppose that $1 \le p, q, \theta \le \infty$, $\varphi, \psi \in \Phi_{k+\frac{n}{p}}$ and

$$c > 0,\ \forall t \in (0, +\infty) : \varphi(t) \le c \cdot \psi(t).$$

Then $L^{k,\varphi}_{p,q,\theta} \subset L^{k,\psi}_{p,q,\theta}$ and

$$\exists C > 0,\ \forall f \in L^{k,\varphi}_{p,q,\theta} : \|f\|_{L^{k,\psi}_{p,q,\theta}} \le C \cdot \|f\|_{L^{k,\varphi}_{p,q,\theta}},$$

that is space $L^{k,\varphi}_{p,q,\theta}$ is continuously embedded in space $L^{k,\psi}_{p,q,\theta}$.

Proof. Let $1 \le p, q \le \infty$, $1 \le \theta < \infty$. Then we have

$$\|f\|_{L^{k,\varphi}_{p,q,\theta}} = \left(\int_0^\infty \left(\frac{\mu^k_f(t)_{pq}}{\varphi(t)}\right)^\theta \frac{dt}{t}\right)^{\frac{1}{\theta}} \ge c^{-1} \left(\int_0^\infty \left(\frac{\mu^k_f(t)_{pq}}{\psi(t)}\right)^\theta \frac{dt}{t}\right)^{\frac{1}{\theta}} = c^{-1} \|f\|_{L^{k,\psi}_{p,q,\theta}}.$$

The case $1 \le p, q \le \infty$, $\theta = \infty$ is similar. The proposition is proved.

Corollary 2. *Let* $1 \le p, q, \theta \le \infty$, $\varphi, \psi \in \Phi_{k+\frac{n}{p}}$ *and*

$$\exists c_1 > 0,\ c_2 > 0,\ \ \forall r \in (0, +\infty) : c_1\varphi(r) \le \psi(r) \le c_2\varphi(r).$$

Then $L^{k,\varphi}_{p,q,\theta} = L^{k,\psi}_{p,q,\theta}$ *and their norms are equivalent.*

From inequality $\mu^{k+1}_f(r)_{pq} \le \mu^k_f(r)_{pq}$ $(r > 0)$ it is easy to obtain that $L^{k,\varphi}_{p,q,\theta}$ is continuously embedded in space $L^{k+1,\varphi}_{p,q,\theta}$.

We denote the set of all the positive functions $\psi(\delta)$ monotonically increasing in $(0,+\infty)$ by Ψ. If ν is a positive number, then we denote by Ψ_ν a set of all $\psi \in \Psi$ such that $\psi(t)\cdot t^{-\nu}$ almost decreases on $(0,+\infty)$.

If $\psi \in \Psi_k$, $k \in N$, then we will denote by BMO_ψ^k a class of all the functions $f \in L_{loc}^1(\mathbf{R}^n)$ for which the following relation

$\exists C>0,\ \forall a \in \mathbf{R}^n, \forall r>0$:

$$\Omega_k(f, B(a,r))_1 := \frac{1}{|B(a,r)|} \int\limits_{B(a,r)} \left| f(t) - P_{k-1,B(a,r)} f(t) \right| dt \le C\psi(r)$$

is valid.

We define the norm on BMO_ψ^k by the equality

$$\|f\|_{BMO_\psi^k} := \sup\left\{ \frac{\Omega_k(f, B(a,r))_1}{\psi(r)} : r>0,\ \ a \in \mathbf{R}^n \right\}.$$

In particular, if $k=1$, $\psi(\delta) \equiv 1$ then $BMO_\psi^k = BMO$, where BMO is the space of all local summable functions of bounded mean oscillation. The class BMO for the first time was introduced in [5].

It is easy to see that if $p=1$, $q=\infty$, $\theta=\infty$, $\varphi(\delta) = \delta^n \psi(\delta)$ then $L_{p,q,\theta}^{k,\varphi} = BMO_\psi^k$ and their norms are equivalent.

Consider also a class VMO which was introduced in [10]: VMO is the class of all $f \in BMO$ for which the relation

$$\lim_{r\to 0} \sup_{a\in R^n} \frac{1}{|B(a,r)|} \int\limits_{B(a,r)} \left| f(t) - f_{B(a,r)} \right| dt = 0$$

is valid. For $f \in VMO$ we define $\|f\|_{VMO} := \|f\|_{BMO}$.

For $f \in L_{loc}^1(R^n)$, $k \in N$ we assume that

$$M_f^k(\delta) = \sup_{\substack{0<r\le\delta \\ x\in R^n}} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \left| f(t) - P_{k-1,B(x,r)} f(t) \right| dt, \quad \delta>0.$$

Let $\psi \in \Psi_k$, $k \in N$. By VMO_ψ^k we denote the class of all functions $f \in BMO_\psi^k$, for which $M_f^k(\delta) = o(\psi(\delta))$, $\delta \to 0$. In class VMO_ψ^k we introduce the norm by the equality

$$\|f\|_{VMO_\psi^k} := \|f\|_{BMO_\psi^k}.$$

In the case of $k=1$, $\psi(\delta) \equiv 1$ equality $VMO_\psi^k = VMO$ holds true.

3. Operator $R_{\alpha,k}f$ in spaces defined by conditions on local oscillations of functions

Consider the following potential type integral operator

$$R_{\alpha,k} f(x) = \int\limits_{\mathbf{R}^n} \left\{ K_\alpha(x-y) - \left(\sum_{|\nu|\le k-1} \frac{x^\nu}{\nu!} D^\nu K_\alpha(-y) \right) X_{\{|t|>1\}}(y) \right\} f(y)\, dy,$$

where $K_\alpha(x) = |x|^{\alpha-n}$, $0<\alpha<n$, $\nu=(\nu_1,\nu_2,...,\nu_n)$, ν_i $(i=1,2,...,n)$ are non-negative integers $x^\nu = x_1^{\nu_1}\cdot x_2^{\nu_2}\cdots x_n^{\nu_n}$, $\nu! = \nu_1!\cdot\nu_2!\cdots\nu_n!$, $|\nu| = \nu_1+\nu_2+...+\nu_n$, $k\in N$,

$$D^\nu g := \frac{\partial^{|\nu|} g}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}},$$

$X_{\{|t|>1\}}$ is the characteristic function of the set $\{t \in R^n : |t|>1\}$.

Operator $R_{\alpha,k}f$ is certain modification of the Riesz potential

$$I_\alpha f(x) = \int_{R^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

It should be noted that if $f \in L^p(R^n)$ and $1 \le p < \frac{n}{\alpha}$, then the integral $R_{\alpha,k}f$ differs from integral $I_\alpha f$ by a polynomial power of which is equal or less than $k-1$. If $p \ge \frac{n}{\alpha}$, then the potential $I_\alpha f$ no defined for all functions $f \in L^p(R^n)$.

Moreover, for example, if $1 \le p \le \infty$ and $k + \frac{n}{p} > \alpha$, then for $f \in L^p(\mathbf{R}^n)$ integral $R_{\alpha,k}f(x)$ absolutely converges almost everywhere.

Note that modified Riesz potential similar to the $R_{\alpha,k}f$ was considered, for example, in T.Kurokawa [13], T.Shimomura and Y.Mizuta [14] etc. (see also [15]).

The following assertion holds.

Theorem 2 [16]. *Let $f \in L^p_{loc}(R^n)$, $1 \le p \le \infty$, k, $l \in N$, $k \ge [\alpha] + l$ ($[\alpha]$ designates the entire part of number α), $0 < \alpha < n$, $x \in R^n$ and*

$$\int_1^\infty t^{-k-\frac{n}{p}+\alpha-1} \cdot \mu^l_f(x;t)_p \, dt < +\infty.$$

Then the inequality

$$\mu^k_{\overline{f}}(x;\delta)_p \le C \cdot \delta^{k+\frac{n}{p}} \int_\delta^\infty t^{-k-\frac{n}{p}+\alpha-1} \mu^l_f(x;t)_p \, dt, \quad \delta > 0, \tag{3}$$

is valid, where $\overline{f} := R_{\alpha,k}f$, and the constant $C > 0$ is independent of f, δ and x.

Theorem 3. *Let $f \in L^p_{loc}(R^n)$, $1 \le p$, $q \le \infty$, $0 < \alpha < n$, k, $l \in N$, $k \ge [\alpha] + l$ and*

$$\int_1^\infty t^{-k-\frac{n}{p}+\alpha-1} \cdot \mu^l_f(t)_{pq} \, dt < +\infty.$$

Then the inequality

$$\mu^k_{\overline{f}}(\delta)_{pq} \le C \cdot \delta^{k+\frac{n}{p}} \int_\delta^\infty t^{-k-\frac{n}{p}+\alpha-1} \mu^l_f(t)_{pq} \, dt, \quad \delta > 0, \tag{4}$$

is valid, where $\overline{f} := R_{\alpha,k}f$, and the constant $C > 0$ is independent of f and δ.

By virtue of this theorem we prove the following theorem on the action of the operator $f \mapsto R_{\alpha,k}f$ in spaces $L^{k,\varphi}_{p,q,\theta}$.

Theorem 4. *Let $1 \le p$, q, $\theta \le \infty$, $l \in N$, $0 < \alpha < n$, $k \ge [\alpha] + l$, $\varphi \in \Phi_{l+\frac{n}{p}}$, $\varphi_1(\delta) = \delta^\alpha \varphi(\delta)$, $(\delta > 0)$ and*

$$\delta^{k+\frac{n}{p}-\alpha} \int_\delta^\infty t^{-k-\frac{n}{p}+\alpha-1} \varphi(t) \, dt = O(\varphi(\delta)) \quad (\delta > 0).$$

If $f \in L^{l,\varphi}_{p,q,\theta}$, then $R_{\alpha,k}f \in L^{k,\varphi_1}_{p,q,\theta}$ and it holds the inequality

$$\left\| R_{\alpha,k}f \right\|_{L^{k,\varphi_1}_{p,q,\theta}} \le C \left\| f \right\|_{L^{l,\varphi}_{p,q,\theta}},$$

where the constant $C > 0$ is independent of f.

Corollary 3 [4]. *Let $0 < \alpha < n$, $l \in N$, $k \ge [\alpha] + l$, $\psi \in \Psi_l$ and*

$$\delta^{k-\alpha} \int_\delta^\infty t^{-k+\alpha-1} \psi(t) \, dt = O(\psi(\delta)). \tag{5}$$

Then $f \mapsto R_{\alpha,k} f$ is a *bounded operator from BMO_{ψ}^{l} into $BMO_{\psi_1}^{k}$, where $\psi_1(\delta) = \delta^{\alpha}\psi(\delta)$, $(\delta > 0)$.*

Since the function $\psi(\delta) \equiv 1$ satisfies the condition (5), we obtain the following assertion.

Corollary 4. *Let $0 < \alpha < n$, $k = [\alpha] + l$. Then $f \mapsto R_{\alpha,k} f$ is a bounded operator from space BMO into the space $BMO_{\psi_1}^{k}$, where $\psi_1(\delta) = \delta^{\alpha}$ $(\delta > 0)$.*

Lemma 1. Let $\varphi, \psi \in \Psi$, $\mu > 0$, $\varphi \in Z_\mu$ i.e.

$$\delta^{\mu}\int_{\delta}^{\infty} \frac{\varphi(t)}{t^{\mu+1}}\,dt = O(\varphi(\delta)), \quad \delta > 0,$$

and let $\psi(\delta) = o(\varphi(\delta))$, $\delta \to 0$, $\psi(\delta) = O(\varphi(\delta))$, $\delta > 0$. Then

$$\delta^{\mu}\int_{\delta}^{\infty} \frac{\psi(t)}{t^{\mu+1}}\,dt = o(\varphi(\delta)), \quad \delta \to 0.$$

Proof. If $\varphi \in Z_\mu$, then there is a number $\beta \in (0, \mu)$ such that $\frac{\varphi(t)}{t^{\beta}}$ is almost decreasing, i.e.

$$\exists c > 0, \ \ \forall t_1, t_2 \in (0, +\infty) : \left(t_1 < t_2 \Rightarrow \frac{\varphi(t_2)}{t_2^{\beta}} \le c \cdot \frac{\varphi(t_1)}{t_1^{\beta}} \right).$$

If $\psi(r) = o(\varphi(r))$, $r \to 0$, and $\psi(r) = O(\varphi(r))$, $r > 0$, then there is a bounded function $\overline{\alpha}(t) \ge 0$ such that $\lim\limits_{t \to 0} \overline{\alpha}(t) = 0$ and for $r \in (0, 1]$ holds the equality

$$\psi(r) = \overline{\alpha}(r) \cdot \varphi(r).$$

We introduce the function

$$\alpha(r) = \begin{cases} \sup\limits_{0 < t \le r} \overline{\alpha}(t), & \text{if } r \in (0, 1], \\ c_0 = \|\overline{\alpha}\|_{L^{\infty}(0, +\infty)}, & \text{if } r > 1. \end{cases}$$

Then $\psi(r) \le \alpha(r) \cdot \varphi(r)$, $r \in (0, +\infty)$, $\alpha(r)$ is monotonically increasing on $(0, +\infty)$ and $\lim\limits_{r \to 0} \alpha(r) = 0$. Considering it, further we have

$$\delta^{\mu}\int_{\delta}^{\infty} \frac{\psi(t)}{t^{\mu+1}}\,dt \le \delta^{\mu}\int_{\delta}^{\infty} \frac{\alpha(t) \cdot \varphi(t)}{t^{\mu+1}}\,dt$$

$$= \delta^{\mu}\int_{\delta}^{\infty} \frac{\alpha(t)}{t^{\mu-\beta+1}} \cdot \frac{\varphi(t)}{t^{\beta}}\,dt \le c \cdot \frac{\varphi(\delta)}{\delta^{\beta}} \cdot \delta^{\mu}\int_{\delta}^{\infty} \frac{\alpha(t)}{t^{\mu-\beta+1}}\,dt = c \cdot \varphi(\delta) \cdot \delta^{\mu-\beta} \cdot \int_{\delta}^{\infty} \frac{\alpha(t)}{t^{\mu-\beta+1}}\,dt. \tag{6}$$

We show that if $\nu > 0$, then

$$J(\delta) = \delta^{\nu}\int_{\delta}^{\infty} \frac{\alpha(t)}{t^{\nu+1}}\,dt = o(1), \quad \delta \to 0.$$

Let $0 < \delta \le 1$. Then we have

$$J(\delta) = \delta^{\nu}\int_{\delta}^{\infty} \frac{\alpha(t)}{t^{\nu+1}}\,dt = \delta^{\nu}\int_{\delta}^{\sqrt{\delta}} \frac{\alpha(t)}{t^{\nu+1}}\,dt + \delta^{\nu}\int_{\sqrt{\delta}}^{1} \frac{\alpha(t)}{t^{\nu+1}}\,dt + \delta^{\nu}\int_{1}^{\infty} \frac{\alpha(t)}{t^{\nu+1}}\,dt$$

$$\leq \alpha\left(\sqrt{\delta}\right)\cdot\delta^{\nu}\int\limits_{\delta}^{\infty}\frac{1}{t^{\nu+1}}dt+\left(\sqrt{\delta}\right)^{\nu}\cdot\alpha\left(1\right)\cdot\left(\sqrt{\delta}\right)^{\nu}\cdot\int\limits_{\sqrt{\delta}}^{\infty}\frac{1}{t^{\nu+1}}dt+c_{0}\cdot\delta^{\nu}\int\limits_{1}^{\infty}\frac{1}{t^{\nu+1}}dt.$$

All terms on the right side of this inequality tend to zero when $\delta\rightarrow 0$. It follows that $\lim\limits_{\delta\rightarrow 0}J\left(\delta\right)=0$. Therefore, due to the relation (6) we have

$$\delta^{\mu}\int\limits_{\delta}^{\infty}\frac{\psi\left(t\right)}{t^{\mu+1}}dt=o\left(\varphi\left(\delta\right)\right),\ \ \delta\rightarrow 0.$$

Theorem 5. *Let $0<\alpha<n$, $k\geq[\alpha]+l$, $k,\ l\in N$, $\varphi\in Z_{k-\alpha}$. Then the operator $\overline{f}=R_{\alpha,k}f$ is a bounded map from VMO_{φ}^{l} to $VMO_{\varphi_1}^{k}$, where $\varphi_1\left(\delta\right)=\delta^{\alpha}\cdot\varphi\left(\delta\right)$, $\delta\in(0,+\infty)$.*

Proof. From Theorem 2 we receive the following estimate in terms of characteristics $M_{f}^{k}\left(\delta\right)$:

$$M\frac{k}{f}\left(\delta\right)\leq c\cdot\delta^{k}\int\limits_{\delta}^{\infty}\frac{M_{f}^{l}\left(t\right)}{t^{k-\alpha+1}}dt,\quad\delta>0. \tag{7}$$

If $f\in VMO_{\varphi}^{k}$, then from estimate (7) it follows that

$$M\frac{k}{f}\left(\delta\right)\leq c\cdot\delta^{k}\int\limits_{\delta}^{\infty}\frac{\|f\|_{VMO_{\varphi}^{l}}\cdot\varphi\left(t\right)}{t^{k-\alpha+1}}dt\leq c\cdot\|f\|_{VMO_{\varphi}^{l}}\cdot\delta^{\alpha}\cdot\delta^{k-\alpha}\int\limits_{\delta}^{\infty}\frac{\varphi\left(t\right)}{t^{k-\alpha+1}}$$

$$\leq c_{1}\cdot\|f\|_{VMO_{\varphi}^{l}}\cdot\delta^{\alpha}\cdot\varphi\left(\delta\right)=c_{1}\cdot\|f\|_{VMO_{\varphi}^{l}}\cdot\varphi_{1}\left(\delta\right),\quad\delta>0.$$

Hence we have

$$\left\|\overline{f}\right\|_{BMO_{\varphi_1}^{k}}=\sup_{\delta>0}\frac{M\frac{k}{f}\left(\delta\right)}{\varphi_{1}\left(\delta\right)}\leq c\cdot\|f\|_{VMO_{\varphi}^{l}}\,. \tag{8}$$

It is necessary to show, that $\overline{f}\in VMO_{\varphi_1}^{k}$.
Let $0<\delta\leq 1$. Then from inequality (7) we obtain

$$M\frac{k}{f}\left(\delta\right)\leq c\cdot\delta^{\alpha}\cdot\delta^{k-\alpha}\int\limits_{\delta}^{\infty}\frac{M_{f}^{l}\left(t\right)}{t^{k-\alpha+1}}dt,\quad\delta>0. \tag{9}$$

Since $M_{f}^{l}\left(r\right)=o\left(\varphi\left(r\right)\right)$, $r\rightarrow 0$, $M_{f}^{l}\left(r\right)=O\left(\varphi\left(r\right)\right)$, $r>0$, and $\varphi\in Z_{k-\alpha}$, then from the lemma 1 we obtain that

$$\delta^{k-\alpha}\int\limits_{\delta}^{\infty}\frac{M_{f}^{l}\left(t\right)}{t^{k-\alpha+1}}dt=o\left(\varphi\left(\delta\right)\right),\quad\delta\rightarrow 0.$$

Then from (9) it follows that $M\frac{k}{f}\left(\delta\right)=o\left(\delta^{\alpha}\cdot\varphi\left(\delta\right)\right),\quad\delta\rightarrow 0.$

Corollary 5. *Let $0<\alpha<n$, $k=[\alpha]+1$. If $f\in VMO$, then $\overline{f}=R_{\alpha,k}f\in VMO_{\psi_1}^{k}$, where $\psi_1\left(\delta\right)=\delta^{\alpha}$ $(\delta>0)$.*

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