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Spectroidal Operators In Banach Space

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Abstract. *In the present paper we introduce spectroidal operators in Banach space, define their place in hierarcy of a class of operators, give constructions of spectroidal operators, prove criteria of closure of the numerical range and the Weyl theorem for Brauder's essential spectrum. We also introduce a subclass of spectroidal operators (σ-unitary) being the Banach analogue of unitary operators in Hilbert space. The structure of the spectrum, the Weyl essential spectrum were described and the criterion of closure of the numerical range was proved. These results show both similarity and some distinctions with the operators having the same name in Hilbert space.*

Preliminaries.

Let X be a Banach space, H a Hilbert space (both over the field of complex numbers $\mathbf{C}$); $S(X)$ a unit sphere in X; * a Banach conjugation. Denote by $\overline{M}$ a closure, by coM a convex shell, by $\overline{co}M$ a convex closure of the set M. Denote algebra of linear operators in X and H by $B(X)$ and $B(H)$, respectively. The space X is said to be uniformly rounded if for any sequences $x_n, y_n \in S(X)$ from $\|x_n + y_n\| \to 2$ it follows $\|x_n - y_n\| \to 0$ as $n \to \infty$.

For $T \in B(X)$ we use the following denotation for the parts of the spectrum $\sigma(T)$: $\alpha_p(T) = \{\lambda \in \mathbf{C} : Ker(T-\lambda) \neq 0\}$ is a point spectrum or a set of eigen values T; $\sigma_\pi(T) = \{\lambda \in \mathbf{C}$: there exists a sequence $x_n \in S(X)$ such that $\|(T-\lambda)x_n\| \to 0$ as $n \to \infty\}$ is an approximately point spectrum; $\sigma_\delta(T) = \{\lambda \in \mathbf{C} : Ran(T-\lambda) \neq X\}$ is an approximately defective spectrum; $\sigma_j(T) = \left\{\lambda \in \mathbf{C} : \overline{Ran}(T-\lambda) \neq X\right\}$ is a contractive spectrum or a contraction spectrum; $\sigma_r(T) = \{\lambda \in \mathbf{C} : Ker(T-\lambda) = 0, \overline{Ran}(T-\lambda) \neq X\big\}$ is a residual spectrum. Here Ker, Ran, $\overline{Ran}$ denote a kernel, range, closure of the range, operator, respectively.

The set $V(T) = \{\lambda \in \mathbf{C} : \lambda = f(Tx), x \in S(X), f \in D(x,X)\}$, where $D(x,X) = \{f \in S(X^*) : f(x) = 1\}$ for $x \in S(X)$ is called Bauerian or spatial numerical range of the operator $T \in B(X)$.

In any Banach space X one can give a mapping $s : X \times X \to C$ such that $s[x,y]$ linearly with respect to the first argument, $[x,x] = \|x\|^2$ and the Cauchy inequality is fulfilled: $|[x,y]| \leq \|x\| \cdot \|y\|$. The mapping [] is called a semi-inner product in X, generating a norm in X. In the general Banach space X such a semi-scalar product s is not unique and their set is denoted by $\sum$. The set $W_s(T) = \{\lambda \in \mathbf{C} : \lambda = s[Tx,x], x \in S(X)\}$ is the Lumerian numerical range for $T \in B(X)$ responding to $s \in \sum$. The set $\mathcal{V}(T) = \left\{\lambda \in \mathbf{C} : \lambda = F(T), F \in B(X)^*, \|F\| = F(T) = 1\right\}$, where I is a unique operator in X, is called the algebraic numerical range of the operator $T \in B(X)$. If $X = H$ with a inner product $<,>$ then for any $T \in B(H)$ there hold the relations $W_s(T) = V(T) = W(T)$ and $\mathcal{V}(T) = \overline{W}(T)$ $s \in \sum$, where $W(T) = \{\lambda \in \mathbf{C} : \lambda =< Tx, x >, x \in S(H)\}$ is a Hausdorff numerical range. For $T \in B(X)$ for

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any $s \in \sum$ the relations $W_s(T) \subset V(T) \subset \mathcal{V}(T)$ and $\overline{co}W_s(T) = \overline{co}V(T) = \mathcal{V}(T)$ are valid. See for numerical ranges of the operator in [4,5].

The operator $T \in B(X)$ is Hermitian if some of $W_s(T)$, and therefore any numerical range of the operator T is real. If $T = H + iK$, where H, K are commuting Hermitian operators, then T is called a normal operator. A spectral (numerical) radius of the operator $T \in B(X)$ $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ $(\nu(T) = Sup\{|\lambda| : \lambda \in V(T)\})$. For $T \in B(X)$ the intermediate inequality $r(T) \le \nu(T) \le \|T\|$, is fulfilled and both inequalities are strong in the general case . The operator $T \in B(X)$ is called radioloid if $r(T) = \|T\|$ and normaloid $\nu(T) = \|T\|$. For $T \in B(H)$ these two notion are equivalent, and for $T \in B(X)$ in the arbitrary X the normaloidness follows from radioloidness, but not vice-versa. $T \in B(X)$ is called transloidal if $r(T - \lambda) = \|T - \lambda\|$ for any $\lambda \in \mathbf{C}$ and convexoidal if $co\sigma(T) = \overline{V}(T)$.

If for $T \in B(X)$ the equality $\left\|(T-\lambda)^{-1}\right\| = d(\lambda, \sigma(T))$ is fulfilled for any $\lambda \notin \sigma(T)$, where $d(\lambda, \sigma(T)) = \inf\{|z - \lambda| : z \in \sigma(T)\}$, then it is said that T is an operator of the class $G_1(X)$.

The operator $T \in B(X)$ is called norm-unitary if T is inversible and $\|T\| = \left\|T^{-1}\right\| = 1$. The point $x \in K$ s called an extremal point of the convex set $K \subset \mathbf{C}$ if $K \setminus \{x\}$ is a convex set. We denote the set of such points by $extK$. If the point $x \in K$ is such that there exists rectilinear l support to K with the property $K \cap l = \{x\}$, then x is said an exponed point of the convex set K. We denote the set of such points by $\exp K$.

§ 0. **Motivation and content of the paper.**

The results from [7], [9]; of von Neumann [2, p. 459-469], the classic results on the spectrum of unitary operators [3, p. 365]; the Weyl theorem on essential spectra [2, p. 390], and also extended problem N8 [5a, p. 129] and the paper [8] is the motivation for the present paper.

In [7], a class of operators $T \in B(H)$ whose spectrum is von Neumann's spectral set for T, is considered. Later they were called spectroidal operators (briefly, spectroids). Spectral set (s.s) was introduced for extending spectral theory to non normal operators in H [2,4,6]. Its definition is suitable for $T \in B(X)$, but by passing in to X, the situation changes. We note only the following one. One of the basic theorems on s.s. says: von Neumann's theorem [2, p. 465]; a closed unit circle D in $\mathbf{C}$ will be a s.s. for $T \in B(H)$ if and only if T is a contraction, i.e. $\|T\| \le 1$. In Banach space X this fact is valid only to one side [2, p. 460]. Furthermore, if for any contraction $T \in B(H)$ s.s. will be D, then X is a Hilbert space [2. p. 463].

Denote by $R(E)$ all rational functions without poles in $E \subset \mathbf{C}$.

Definition 1. *A closed set $E \subset C$ called a spectral set (s.s) for $T \in B(X)$ if the spectrum $\sigma(T) \subset E$ and for all $f \in R(E)$ the inequality $\|f(T)\| \le \|f\|_E$ is valid, where $\|f\|_E = \sup\{|f(z)| : z \in E\}$. The operator $T \in B(X)$ is called a spectroid if $\sigma(T)$ is s.s. for T.*

It is known that unitarity of $T \in B(H)$ is equivalent to spectroidness of T with a unimodular spectrum, i.e. $\sigma(T)$ lies in a unit circle in $\mathbf{C}$ [2.p. 468]. Therefore the following definition is relevant.

Definition 2. *The operator $T \in B(X)$ is called σ-unitary if T is a spectroid with a unimodular spectrum.*

§ **1** of the present paper contains the proof of the auxiliary criterion of spectroidness of $T \in B(X)$. Its necessary part for $T \in B(H)$ was shown in [7. p. 421], and sufficient part in [9 b),p. 202, theorem 2]. Prime conclusion on relation of spectroidness with (almost) normal operators are derived. An example is given for example, when a normal operator in a finite-dimensional Banach space X unlike the finite-dimensional H will not be a spectroid.

The way how to construct spectroids by means of the direct sum is also shown.

§ **2** is devoted to the place of spectroids, σ-unitary and Hermitian operators in the hierarchy of the classes of operators from $B(X)$. In the paper [7.p. 421] it is shown that the spectroids $T \in B(H)$ will be transloidal and therefore convexoidal. For that normal dilatation of operators and spectral expansion of normal operators in H are used. But they are not suitable for Banach space X. For the spectoids $T \in B(X)$ in theorems 1 (a), (b) we prove these results by another way. In the same place we show that the spectrods lie in the class $G_1(X)$.

It is known that for spectroidness of $T \in B(H)$ with a real spectrum, the Hermitian property of T is necessary and sufficient [2, p. 468]. For unitarity of $T \in B(H)$ the norm-unitarity of T is necessary and sufficient, i.e. T is inversible, and $\|T\| = \left\|T^{-1}\right\| = 1$. In theorem 1 (c), (d) we will show for $T \in B(X)$ the validity of necessary parts of these facts and give counter examples to their sufficient parts. In both examples, Banach spaces X are finite-dimensional.

In § 3 the structure of the spectrum of σ-unitary and spectroidal operator is given. When studying the properties of the spectrum of the operator, along with other questions there arise two problems; description of thin structure of the spectrum and essential spectra. It is known that for a unitary $T \in B(H)$ the residual spectrum $\sigma_r(T)$ is empty and $\sigma(T) = \sigma_\pi(T)$ [3, p. 365]. In theorem 2 (a) we prove this for σ-unitary operators. The description of the essential spectrum get started from the paper of H. Weyl for Hermite operators $T \in B(H)$ [2, p. 390]. Only after nearly 60 years in [11] the first extension of the Weyl theorem to nonnormal operators $T \in B(H)$ was given. Then a number of works followed this theme.

In theorem 2 (b) (c) we prove the Weyl theorem for essential spectra of Browder and Weyl for the spectroids and σ-unitary $T \in B(X)$.

§ **4** is connected with the problem No 8 [5a, p. 129] : for which operators $T \in B(X)$ and spaces X is the numerical range $V(T)$ closed. The first result for the Hermitian $T \in B(H)$ was obtained in 1936 by Lengyel and Stone [1(b), p. 84] and after 40 years by Meng [8] for a normal $T \in B(H)$. In the both cases, the spectral expansion of the operator working only in H was used. The first passage to the Banach space for normal $T \in B(X)$ was represented by the author [17 c.] and then for different classes of operators $T \in B(X)$ [17 b) e) f)]. In [8] the criteria of closure of numerical range $W(T)$ for $T \in B(H)$ were obtained for unitary operators, while in [7] for spectroids. They use the tool suitable only in H. In theorem 3, we use the general scheme of the proof of such criteria suggested by the author in [17, d], get a result for σ-unitary and spectroidal operators in uniformly roundish X.

In [9 a) p. 480], cramped operators $T \in B(H)$ were introduced. There it was proved that [9. a) p 482], if the normal inversible operator has a cramped operator U in polar expansion $T = UR$, then $0 \notin \overline{W}(U)$. For any inversible $T \in B(H)$ this is not always the case. Some authors gave positive answer for the classes of operators from hypernormal to convexoid ones. In all these papers the following proposition is used: if the operator U is a cramped operator, then $0 \notin \overline{W}(U)$. The original of the proof [9 a) p. 480] is only in H. In theorem 3(c) in the Banach space we give a more brief proof of this geometrically visible fact.

§ 1. Criterion of spectroidness and (almost) normality.

We will prove the criterion of spectroidness, discuss relation of spectroidness with (almost) normality of the operator $T \in B(X)$ and give the construction of spectroidal operators.

Lemma 1. *For the operator $T \in B(X)$ the following conditions are equivalent: a) T is a specroid; b) $r(f(T)) = \|f(T)\|$ for all $f \in R(\sigma(T))$.*

Proof. At first we note that for any $T \in B(X)$ and all $f \in R(\sigma(T))$ by the spectral mapping theorem $\sigma(f(T)) = f(\sigma(T))$ it follows the equality

$$r(f(T)) = \|f\|_{\sigma(T)} \tag{1.1}$$

a) implies b) By definition of spectroidness of T for all $f \in R(\sigma(T))$ we have

$$\|f(T)\| \leq \|f\|_{\sigma(T)}.$$

According to intermediate inequality (see preliminaries) $r(f(T)) \leq \|f(T)\|$. Radiality of $f(T)$ for all $f \in R(\sigma(T))$ follows from the last three relations.

b) implies a) If for T for all $f \in R(\sigma(T))$ we have $r(f(T)) = \|f(T)\|$, then taking into account (1.1), we get $\|f(T)\| = \|f\|_{\sigma(T)}$. Consequently, T is a spectroid. The lemma is proved.

Note some conclusions from the lemma. Remind that $T \in B(X)$ is called paranormal if $\|Tx\|^2 \leq \left\|T^2x\right\|$ for all $x \in S(X)$. The proper part of this class are the transparanormal (t.p.n) operators introduced in [17 b]. $T \in B(X)$ is called a t.p.n. operator if $T - \lambda$ is paranormal for all $\lambda \in \mathbf{C}$. It is easy to show that hypornormal operators $T \in B(H)$ [4, p. 109] are always t.p.n.

We don't know if hyponormal ones constitute the proper part of t.p.n. operators.

The result [16, p. 619] or [9 b), p. 202] on hyponormal $T \in B(H)$ immediately follows for t.p.n. operators $T \in B(X)$.

Corollary 1. *If $f(T)$ is t.p.n. for all $f \in R(\sigma(T))$, then $T \in B(X)$ is a spectroid.*

The proof follows from the lemma, since t.p.n. $f(T) \in B(X)$ is radialoidal. In the general case, the classes of spectroidal and t.p.n operators in X are independent.

Example 1. Nontrivial examples of hyponormal $T \in B(H)$ that are not spectroidal [16, p. 619] are known. The contrary one is simpler. Let $T = A \oplus B$, where A is an operator with the elements of the matrix $a_{11} = a_{21} = a_{22} = 0$, $a_{12} = 1$ in two-dimensional Hilbert space $\mathbf{C}^2$, B is the right shift in l_2. Then $\sigma(A) = \{0\}$, $\|A\| = 1$, $\sigma(B) = D$ and $\|B\| = 1$ [4, p. 49]. Therefore $\|T\| = 1$ and by the von Neumann theorem from §0, the spectrum $\sigma(T) = D$ will be s.s. for T. But the spectroid T is not paranormal, since in the contrary care A would be paranormal. But it is not the case: for the vector $x = (0,1)$ we have $\left\|A^2x\right\| = 0$ and $\|Ax\|^2 = 1$ and the inequality $\|Ax\|^2 \leq \left\|A^2x\right\|$ is violated.

A theorem proved by von Neumann by means of spectral expansion of the normal $T \in B(H)$ is derived from the lemma in [9 b)] by elementary way.

Corollary 2. The normal operator $T \in B(H)$ *is spectroid.*

Proof. For the normal $T \in B(H)$ the operator $f(T)$ is normal for all $f \in R(\sigma(T))$, and the normal operator in H is radioloid $r(f(T)) = \|f(T)\|$. By the lemma, T is a spectroid.

In this connection, the reversibility of corollary 2 is affirmed in the monograph [1 b), p. 90]. But in infinite-dimensional H it is not so. The right shift in l_2 (see example 1) that is a spectroid is not normal is a counterexample.

If H is a finite dimensional, then the classes of spectroidal and normal operators coincide [2, p. 469]. But in the Banach space X the normal operator may be not spectroidal even at finite-dimensionality of X.

Example 2. We use example [15, p. 734, example 2] and the previous lemma. In this the example, a four-dimensional Banach algebra with the normal element $u+iv$ of which $r(u+iv) = V\overline{2}$ and $\|u+iv\| = 2$, was represented. Let us express this normal element in the matrix form. Then in four-dimensional Banach space we have the normal operator $A = \left(a_{kj}\right)$, $k,j = 1,2,3,4$, where $a_{kj} = 1$ if $|k-j| = 1$, $a_{kj} = i$ is an imaginary unit, if $|k-j| = 2$, the remaining elements are zeros. The normal operator A is not a spectroid, otherwise by the lemma, for $f(z) = z$ the equality $r(A) = \|A\|$ would be fulfilled.

In theorem 1 we will show that in finite-dimensional Banach space there exists even Hermition operator T that is not a spectroid.

Give the following fact remarked in [6, p. 83] without proof.

Corollary 3. *The subnormal operator $T \in B(H)$ is spectroidal.*

If T is subnormal and N is its minimal normal extension, then $\sigma(N) \subset \sigma(T)$ [14, p. 108]. Then by the spectral mapping theorem and corollary 2 we have $\|f(T)\| \leq \|f(N)\| = \|f\|_{\sigma(N)} \leq \|f\|_{\sigma(T)}$, and T is a spectroid.

Note that a wider class of hyponormal operators $T \in B(H)$ becomes independent of the class of spectroids (see above example 1).

Now be means of the direct sum of operators we give a way for constructing spectroids (see example 1).

Proposition: Let $X = X_1 \oplus X_2$ be a direct sum of Banach spaces X_1 and X_2 and $T = T_1 \oplus T_2$, where $T_j \in B\left(X_j\right)$, $j = 1,2$. If one of the following conditions fulfilled, then T is spectroid: a) T_1 and T_2 are both spectroids; b) T_1 is a spectroid, and $\sigma(T_1)$ contains some s.s. of the operator T_2.

The scheme of the proof. For $f \in R(\sigma(T))$ we show the equality

$$f(T) = f(T_1) \oplus f(T_2) \tag{1.2}$$

a) for $f \in R(\sigma(T))$ and from the spectroidness of T_j, $j = 1,2$ it follows that $\left\|f\left(T_j\right)\right\| \leq \|f\|_{\sigma(T)}$, and application of (1.2) gives spectroidness. b) from the conditions of the point b) for $f \in R(\sigma(T))$ we have $\left\|f\left(T_j\right)\right\| \leq \|f\|_{\sigma(T_j)}$, $j = 1,2$. Hence it follows $\|f(T)\| \leq \|f\|_{\sigma(T)}$ i.e. the spectroidness of .

§**2. Spectroids and hierarchy of the class of operators.**

Let us establish relation of spectroids with other classes of operators in Banach space, that is not always identical to he Hilbert case. Such a distinction is seen for example in theorem 1 c) d).

Theorem 1. a) The spectroids $T \in B(X)$ form the proper part of both of the class of transloids and the class G_1 of the operators. b) Spectroids are always convexoids. c) a spectroid with a real spectrum is Hermitian, but there exists an Hermitian operator that is not a spectroid. d) σ-unitary operators compose a proper part of the class of norm-unitatiary ones.

Proof. a) Transloidness of the spectroid T immediately follows from the lemma of § 1. If for the function $R(\sigma(T))$ we take $f(z) = z - \lambda$ for any $\lambda \in \mathbf{C}$, then $r(T-\lambda) = \|T-\lambda\|$. Having taken $f(z) = (z-\lambda)^{-1}$ for $\lambda \notin \sigma(T)$ we can show the belonging of T to the class $G_1(X)$. Since $f \in R(\sigma(T))$, and $\sigma(T)$ s.s. for T, then $\left\|(T-\lambda)^{-1}\right\| \leq \sup\left\{|z-\lambda|^{-1} : z \in \sigma(T)\right\}$ (see, definition 1). Hence allowing for $(\inf M)^{-1} = \sup\left(M^{-1}\right)$ for any set M of positive numbers, it follows the inequality $\left\|(T-\lambda)^{-1}\right\| \leq d^{-1}(\lambda, \sigma(T))$ for all $\lambda \notin \sigma(T)$. Here $d(\lambda, \sigma(T)) = \inf\{|z-\lambda| : z \in \sigma(T)\}$. On the other hand, for any $T \in B(X)$ it is valid the lower estimation of the norm of the resolvent [1 a), p. 606, corollary 3] $d^{-1}(\lambda, \sigma(T)) \leq \left\|(T-\lambda)^{-1}\right\|$ for all $\lambda \notin \sigma(T)$. Consequently, $T \in G_1(X)$:

$$\|(T-\lambda)\| = d^{-1}(\lambda, \sigma(T)) \quad (\lambda \notin \sigma(T)). \tag{2.1}$$

Let us be convinced that there exist transloids and G_1 operators that are not spectroids.

Example 3. Since any Hermitian $T \in B(X)$ is transloidal [5 b) p. 73], then example 4 from the proof of theorem 1 c) gives a transloid which is not a spectroid. The example of non-spectroidal $T \in G_1$ will be $T = A \oplus B$, where $A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ in $\mathbf{C}^2$ and B is the right shift in l_2 (see example 1 above). Since $\sigma(T) = \overline{W}(T)$, then from the bilateral estimation of the norm of the resolvent of any operator $T \in B(H)$ (Winter A) (1929)

$$d^{-1}(\lambda, \sigma(T)) \leq \left\|(T-\lambda)^{-1}\right\| \leq d^{1}\left(\lambda \overline{W}(T)\right), \quad \lambda \notin \overline{W}(T) \tag{2.2}$$

it follows $T \in G_1$. But $r(T) = 1$, $\|T\| = 2$ and by theorem 1 a) T is not a spectroid. These examples show that the classes of transloids and G_1 operators are, general speaking, independent between themselves.

b) The convexoidness of the spectrum may be proved in two ways.

Firsty, to use that any convex compact in the plane will be an intersection of all possible closed cirles containing it. Then for a convex shell of the spectrum of any operator $T \in B(X)$ it is valid the equality

$$co\sigma(T) = \cap\{\lambda \in \mathbf{C} : |\lambda - \mu| \leq r(T-\mu), \mu \in \mathbf{C}\}.$$

According to [5 b) p. 42, lemma 1] the algebraic numerical range $\mathcal{V}(T)$ of any operator $T \in B(X)$ has the following description

$$\mathcal{V}(T) = \cap\{\lambda \in \mathbf{C} : |\lambda - \mu| \leq \|T-\mu\|, \mu \in \mathbf{C}\}.$$

According to the first part of theorem 1 a), for all $\mu \in \mathbf{C}$ we have an equality for the spectraoid $T \in B(X)$

$$r(T-\mu) = \|T-\mu\|$$

and from the three previous relations we get the coincidence of $\mathcal{V}(T)$ and $co\sigma(T)$.

The second way of the proof is based on the estimation of the norm of the resolvent of the spectroid. According to the first part of theorem 1, for the spectroid T we have equality (2.1), that fall all $\lambda \notin co\sigma(T)$ yields the inequality

$$\left\|(T-\mu)^{-1}\right\| \leq d^{-1}(\lambda, co\sigma(T)). \tag{2.3}$$

The Orland criterion [10, p. 27, theorems 2,3] (inequality (2.3) is equivalent to convexoidness of any operator $T \in B(H)$) is valid in Banach space.

Note that theorem 3 from [10] immediately follows from the right inequality in (2.2) for $T \in B(H)$ We can show that this inequality is preserved in Banach space with participation of $\overline{V}(T)$.

c) Hermitian property of the spectroid $T \in B(X)$ with a real spectrum immediately follows from theorem 1 b) .

Show that unlike the Hilbert case, in the Banach space the Hermitian operator may not be a spectroid.

Example 4. We use the Crabb example that in the Banach space the product of commuting Hermitian operators, generally speaking, is not Hermitian [15, p.743].

Let us consider a space $X = \mathbf{C}^3$ with the norm $\nu(x,y,z) = \sup\left\{\left|\lambda^{-1}x + y + \lambda z\right| : |\lambda| = 1\right\}$, $(x,y,z) \in \mathbf{C}^3$ and an operator $A \in B(X)$ with the matrix a_{ij}, $1 \leqslant i, j \leqslant 3$, where $a_{11} = -1$, $a_{33} = 1$ and $a_{ij} = 0$ for remaining i, j. As Crabb has shown, the operator A is Hermitian. We will show that A is not a spectroid. Assume the contrary one. Then by theorem 1 a) for all $\lambda \notin \sigma(A)$ it is valid the equality $\left\|(A-\lambda)^{-1}\right\| = d^{-1}(\lambda, (\sigma(A)))$. Let's be convinced that it is not so.

Having taking the number $1/2 \notin \sigma(A) = \{-1, 0, 1\}$ we consider the operator $(A - 1/2)^{-1}$. Calculations show that for $\lambda_0 = 1/2$ we have $\left\|(A-\lambda_0)^{-1}\right\| > 2$. On the other hand, $d^{-1}(\lambda_0, \sigma(A)) = 2$. Consequently we found $\lambda_0 \notin \sigma(A)$ such that $\left\|(A-\lambda_0)^{-1}\right\| > d^{-1}(\lambda_0, (\sigma(T)))$, and A is not a G_1-operator. Thus, A is not a spectroid.

d) Let's prove that any σ unitary operator $T \in B(X)$ is norm-unitary, but not vice-versa. By theorem 1, a) $\|T\| = r(T) = 1$. Having taken the function $f \in R(\sigma(T))$, where $f(z) = z^{-1}$, $\left\|T^{-1}\right\| \leq \sup\left\{\left|z^{-1}\right| : z \in \sigma(T)\right\} = 1$. Since $\left\|T^{-1}\right\| \leq 1$ and $\|T\| = 1$, from the relations $1 = \left\|T \cdot T^{-1}\right\| \leq \left\|T^{-1}\right\| \leq 1$ it follows $\left\|T^{-1}\right\| = 1$. So, T is norm-unitary.

Show that unlike the Hilbert case, in Banach space the norm-unitarity does not always yield σ-unitarity.

Example 5. Let's consider two-dimensional Banach space $X = \left\{z = (z_1, z_2) : z_j \in \mathbf{C}, j = 1, 2\right\}$ with the norm $\|z\| = \max\{|z_1|, |z_2|\}$. Let's take the operator $A \in B(X)$ with the matrix $\left(a_{jk}\right)$, j, $k = 1, 2$, where $a_{11} = a_{22} = 0$, $a_{12} = a_{21} = 1$. Then A is inversible $A = A^{-1}$ and $\|A\| = \left\|A^{-1}\right\| = 1$, i.e. A is norm-unitary. Let's bee convinced that A will not be σ-unitary

For any $u = (u_1, u_2) \in X$ and $z = (z_1, z_2) \in X$ determine the function $[u, z] = u_1 z_1^*$ for $\|z\| = |z_1| \geq |z_2|$ and $[u, z] = u_2 z_2^*$ for $\|z\| = |z_2| > |z_1|$. The routine verification shows that $[u, z]$ will be s.i.p. generating the norm X. One can count that the appropriate Lumerian numerical range of $W_{[]}(A)$ will be the closed unit circle $D = \{z \in \mathbf{C} : |z| \leq 1\}$, and therefore $D \subset V(A)$. If we don't carry out calculations for $W_{[]}(A)$, then we use the inclusion $G(A) \subset V(A)$ for any A in n-dimensional $\mathbf{C}^n$ with the l_∞-norm [17 f), p.76, theorem 4.6]. Here $G(A)$ is the union of Gershgorin's circles

$$G(A) = \bigcup_{j=1}^{n} \left\{z : \left|z - a_{jj}\right| \leq \sum_{k=1}^{n} \left|a_{jk}\right| - \left|a_{jj}\right|\right\}.$$

In or example for $n = 2$ it is easy to see that $G(A) = D$ again shows the inclusion $D \subset V(A)$. On the other hand, $co\sigma(A)$ will be a segment [-1,1]. Consequently, A is not a convexoid with $co\sigma(A) \neq V(A)$ and according to theorem 1 b), the operator A is not σ-unitary.

Theorem 1 is completely proved.

§ **3. Structure of the spectrum of spectroids.**

For proving theorems on the structure of the spectrum of spectroids, recall some notion. The operator $T \in B(X)$ is said to be invariant-spectroidal if the restriction T on its any invariant subspace will be a spectroid.

Following [11], we determine the Weyl spectrum $\sigma_W(T)$ for $T \in B(X)$ as follows: $\sigma_W(T) = \cap \{\sigma(T+K) : K \in K(X)\}$, where $K(X)$ is the ideal of compact operators in algebra $B(X)$. It is said that the Weyl type theorem is valid for T if $\sigma_W(T) = \sigma(T) - \Pi_{00}(T)$. Here $\Pi_{00}(T)$ is the set of isolated numbers $\lambda \in \sigma_p(T)$ of finite geometrical multiplicity, i.e. the kernel $Ker(T-\lambda)$ is finite-dimensional. In the paper [12], another interpretation of the Weyl type theorem is given. It is said that for T the Weyl type theorem is valid if $\sigma_W(T) = \sigma(T) - \widehat{\Pi}_{00}(T)$. Here $\widehat{\Pi}_{00}(T)$ is the set of isolated numbers $\lambda \in \sigma_p(T)$ of finite algebraic multiplicity, i.e. the image $RanP(\lambda, T)$ of the appropriate spectral projector of Riesz-Danford $P(\lambda, T)$ is finite-dimensional.

The Browder spectrum $\sigma_b(T)$ of the operator $T \in B(X)$ may be determined as $\sigma_b(T) = \sigma_W(T) \cup \sigma(T)^a$, where $\sigma(T)^a$ is the set of limit points of the spectrum $\sigma(T)$. In the general case $\sigma_W(T) \subset \sigma_b(T)$, and the inclusion may be strong. For example $T = S_R \oplus S_L$, where S_R is the right, S_L is the left shift in l_2. For T the point $0 \in \sigma_b(T)$, but $0 \notin \sigma_W(T)$.

Theorem 2. *a) For σ-unitary operator $U \in B(X)$ its spectrum $\sigma(U)$ coincides with approximate-pointwise spectrum $\sigma_\pi(U)$, and in uniformly roundish X the residual spectrum $\sigma_r(U)$ is empty. b) Browder's spectrum of invariant-spectroid $T \in B(X)$ satisfies the Weyl theorem c) The Weyl spectrum of σ-unitary operator $U \in B(X)$ satisfies the Weyl theorem.*

Proof. a) At σ-unitary operator $\sigma(U)$ is unimodular and therefore $\sigma(U) = \partial\sigma(U)$ is the boundary of the spectrum. Since for any $T \in B(X)$ we have $\partial\sigma(T) \subset \sigma_\pi(T)$, then $\sigma(U) = \sigma_\pi(U)$. If we assume non-emptiness of $\sigma_r(U)$, then for $\lambda \in \sigma_r(U)$ by theorem 1d) we have $|\lambda| = \|U\| = 1$. Prove that $\lambda \in V(U)$. From $\lambda \in \sigma_r(U)$ it follows that $R = Ran(U-\lambda)$ is a nontrivial closed space in X and there exists a unit vector $x \in X \backslash R$. By the lemma on nontrivial property of an annihilator [3, p. 72], there will be found a functional $f \in X^*$ such that $f(R) = 0$ and $f(x) = 1$. If $\|f\| \le 1$, from $f(x) = 1$ we have $\|f\| = 1$. Then (x, f) form a dual pair $x \in S(X)$, $f \in D(x, X)$ moreover $f[(U-\lambda)x]$, i.e. $\lambda = f(Ux)$ and $\lambda \in V(U)$. If $\|f\| > 1$ we retouch f, having taken $g(x) = \|f\|^{-1} f(x)$ for all $x \in X$. Then $\|g\| = 1$ and from reflexivity of X, by the Mazur lemma [3, p. 368], there will be found a vector $y \in S(X)$ such that $g(y) = \|g\| = 1$. Since as before g annihilates R, as above we have the dual pair (y, g) and $\lambda = g(Uy) \in V(U)$. Thus, $\lambda \in perV(U)$ is a peripheral part of $V(U)$, where for any $T \in B(X)$

$$perV(T) = V(U) \cap \{z \in \mathbf{C} : |z| = \|T\|\}. \tag{3.1}$$

At roundness of X, by the Winter-Lumer theorem [5 a), p. 93, theorem 8] for any $T \in B(X)$ the inclusion $perV(T) \subset \sigma_p(T)$ is valid. Therefore $\lambda \in \sigma_p(U)$ that contradicts the disjointness of $\sigma_r(U)$ and $\sigma_p(U)$ [1 a), p. 620]. Thus, $\sigma_r(U)$ is empty.

b) Show that for invariant-spectroids $T \in B(X)$ the following equality is valid

$$\sigma_b(T) = \sigma(T) - \Pi_{00}(T). \tag{3.2}$$

According to [14, p.775], for all $T \in B(X)$ the Weyl-type theorem is valid:

$$\sigma_b(T) = \sigma(T) - \widehat{\Pi}_{00}(T). \tag{3.3}$$

Therefore, for validity of (3.2) for invariant-spectroids T the following equality should be proved

$$\widehat{\Pi}_{00}(T) = \Pi_{00}(T). \tag{3.4}$$

Since for any $T \in B(X)$ always $\widehat{\Pi}_{00}(T) \subset \Pi_{00}(T)$, then for equality (3.4) it remains to show the inclusion $\Pi_{00}(T) \subset \widehat{\Pi}_{00}(T)$ at invariant spectroidness of T.

Prove the last inclusion $\Pi_{00}(T) \subset \widehat{\Pi}_{00}(T)$. Let $\lambda \in \Pi_{00}(T)$ and $E(\lambda, T)$ be appropriate spectral projector of Riesz-Danford [1. a), p. 612], where $E(\lambda, T) = \frac{1}{2\pi i} \int\limits_{\gamma} (T-\lambda)^{-1} d\lambda$, $\gamma = \{z \in \mathbf{C} : |z-\lambda| = \varepsilon\}$

and $\varepsilon > 0$ be such that $\sigma(T) \setminus \{\lambda\}$ is outside of the circle with the boundary γ. Since the image $R = RanE(\lambda, T)$, of the projector is invariant with respect to T, one can consider the restriction T_R of the operator T on R [1 a), p. 614 theorem 20]. Then $\sigma(T_R) = \{\lambda\}$ and the operator $T_R - \lambda$ is quasinilpotent $r(T_R - \lambda) = 0$. By the theorem condition, T_R is a spectroid and by theorem 1 a) T_R is a transloid $r(T_R - \lambda) = \|T_R - \lambda\|$. Therefore $T_R - \lambda$ is a zero operator and consequently $(T_R - \lambda) E(\lambda, T) = 0$. Hence it follows that $RanE(\lambda, T) \subset Ker(T - \lambda)$. Since $\lambda \in \Pi_{00}(T)$, then $Ker[T - \lambda)$ is a finite-dimensional space and the finite-dimensionality of $RanE(\lambda, T)$ follows from the previous inclusion. Thus, $\lambda \in \widehat{\Pi}_{00}(T)$ and equality (3.4) is shown. Weyl's theorem for $\sigma_b(T)$ invariant-spectroid $T \in B(X)$ is proved.

c) Let's be convinced that $\sigma_W(U) = \sigma_b(U)$ for σ-unitary $U \in B(X)$ and for $\sigma_W(U)$ the Weyl theorem is true. Since $\sigma(U)$ is unimodular, then $\sigma(U)$ in nowhere dense in the complex plane. Therefore, according to [12, p. 470, corollary 1], the operator U satisfies the equality

$$\sigma_W(U) = \sigma(U) - \widehat{\Pi}_{00}(U). \tag{3.5}$$

Comparing (3.3) with (3.5), we get the coincidence $\sigma_W(U) = \sigma_b(U)$ for σ-unitary $U \in B(X)$. In order to get the Weyl theorem for $\sigma_W(U)$ as in the point b), must show $\Pi_{00}(U) \subset \widehat{\Pi}_{00}(U)$. For that as in the point b), we consider the operator U_R with the spectrum $\sigma(U_R) = \{\lambda\}$. Since $\lambda \neq 0$, the operator $S = \lambda^{-1} U_R$ has the spectrum, $\sigma(S) = \{1\}$, and $\sigma(S - I) = \{0\}$. Since the operator $Q = S - I$ is quasinilpotent, the operator $S = I + Q$ will we a Danford spectral operator [1 c)]. By theorem 1 d) it is easy to see that $\|S^n\| \leq 1$ for all integers n. For Danford spectral operator S with uniformly bounded entire powers, according to [13, point 5, theorem 3] we have that S will be a unit operator I. Therefore, $QE(\lambda, T) = 0$. As in the point b) we get the inclusion $RanE(\lambda, U) \subset Ker(U - \lambda)$ and arrive at $\lambda \in \widehat{\Pi}_{00}(U)$. Thus, $\widehat{\Pi}_{00}(U) = \Pi_{00}(U)$ and the Weyl theorem is true for $\sigma_W(U)$. Theorem 2 is completely proved.

§**4. Numerical ranges of spectroids.**

Motivations and references to the paper connected with the following theorem on numerical ranges of spectroids were stated in §0.

Following the Hilbert case, we call the operator $U \in B(X)$ a cramped operator if U is σ-unitary and $\sigma(U)$ lies in the open arch of a unique circle with a central angle less than π

$$\sigma(U) \subset \{\exp it : t_0 < t < t_0 + \pi\}.$$

Theorem 3. *Let X be a uniform round Banach space.*

a) For the closure of the numerical range $V(T)$ of the spectroid $T \in B(X)$ it is necessary and sufficient that $\exp \overline{V}(T) \subset \sigma_p(T)$.

b) The closeness $V(U)$ of σ-unitary operator $U \in B(X)$ is equivalent to $\sigma(U) = \sigma_p(U)$.

c) Let in any X the operator $C \in B(X)$ be σ-unitary. Then C is a cramped operator if and only if $0 \notin \overline{V}(C)$.

Proof a) necessity: $V(T) = \overline{V}(T) \to \exp \overline{V}(T) \subset \sigma_p(T)$. For any $\lambda \in \exp \overline{V}(T)$ we have $\lambda \in \exp V(T)$. By definition of the exposed point there will be found a circle centered at some point $\mu \in \mathbf{C}$ such that outside of it there are no points from $V(T)$ and λ lies on the boundary of this circle. Therefore, the equality $|\lambda - \mu| = \nu(T - \mu)$ is fulfilled. From the spectroidness of T by theorem 1 a) it follows $r(T - \mu) = \|T - \mu\|$, and according to intermediate inequality $r(T - \mu) \leq \nu(T - \mu) \leq \|T - \mu\|$ we get $|\lambda - \mu| = \|T - \mu\|$. By translational property of $V(T - \mu) = V(T) - \mu$ from the inclusion $\lambda \in V(T)$ it follows $\lambda - \mu \in V(T - \mu)$. Consequently, $\lambda - \mu \in perV(T - \mu)$ see (3.1) in the proof of theorem 2 a). In the rounded X, according to [5, a) p. 93, theorem 8], we have $perV(T - \mu) \subset \sigma_p(T - \mu)$ that implies $\lambda - \mu \in \sigma_p(T - \mu)$ or $\lambda \in \sigma_p(T)$. The inclusion $\exp \overline{V}(T) \subset \sigma_p(T)$ is proved.

Sufficiency: $\exp \overline{V}(T) \subset \sigma_p(T) \to V(T) = \overline{V}(T)$. From the inclusion $\exp \overline{V}(T) \subset \sigma_p(T)$ it follows

$$co \exp \overline{V}(T) \subset co\sigma_p(T). \tag{4.1}$$

By the Zenger theorem [5 b), p. 21, theorem 3] for any $T \in B(X)$ the inclusion

$$co\sigma_p(T) \subset V(T) \tag{4.2}$$

is true.

Since by theorem 1 b) the operator T is a convexoid $\overline{V}(T) = co\sigma(T)$, then by lemma 3 [10, p. 78], for the convex compact $\overline{V}(T)$ we have

$$\overline{V}(T) = co\exp\overline{V}(T). \tag{4.3}$$

From (4.3), (4.1) and (4.2) it follows the inclusion $\overline{V}(T) \subset V(T)$ i.e. $V(T)$ is closed.

b) Implication: $V(U) = \overline{V}(U) \to \sigma(U) = \sigma_p(U)$. By the Williams localization theorem [5, a), p. 88, theorem 1], for any $T \in B(X)$ it is valid $\sigma(T) \subset \overline{V}(T)$ and by the condition we have $\sigma(U) \subset V(U)$. Furthermore, for any $\lambda \in \sigma(U)$ from the unimodularity $\sigma(U)$ and theorem 1 d) we have $|\lambda| = \|U\| = 1$. Consequently, $\sigma(U) \subset perV(U)$ and as in the above point a) we get $\sigma(U) \subset \sigma_p(U)$. Coincidence $\sigma(U) = \sigma_p(U)$ is proved.

Implication: $\sigma(U) = \sigma_p(U) \to V(U) = \overline{V}(U)$ is valid in any X. We conduct the proof by means of extremal points. By the condition we have

$$\sigma(U) = \sigma_p(U). \tag{4.4}$$

By theorem 1 b) the operator U is convexoidal: $\overline{V}(U) = co\sigma(U)$ and therefore $ext\overline{V}(U) = extco\sigma(U)$. Since for any compact $K \subset \mathbf{C}$ it is valid $extcoK \subset K$ [3, p. 86, theorem 3.22], then $extco\sigma(U) \subset \sigma(U)$. Consequently, we have the inclusion

$$ext\overline{V}(U) \subset \sigma(U). \tag{4.5}$$

From (4.4) and (4.5) it follows the inclusion

$$ext\overline{V}(U) \subset \sigma_p(U). \tag{4.6}$$

At the same time, by the Krein-Millmann theorem [3, p. 85, theorem 3.21] by the convexoidness of U for the convex compact we have

$$\overline{V}(U) = coext\overline{V}(U). \tag{4.7}$$

(4.6) and (4.7) yield $\overline{V}(U) \subset co\sigma_p(U)$, and by the Zinger theorem mentioned in point a) $co\sigma_p(U) \subset V(U)$. From the last two inclusions it follows $\overline{V}(U) \subset V(U)$.

c) Implication: $0 \notin \overline{V}(C) \to C$ is cramped. If we assume noncrampedeness of C, then there exist $\mu_j \in \sigma(C)$, $j = 1,2,3$ such that $0 \in co\{\mu_1, \mu_2, \mu_3\}$. By the Crabb theorem [5,b) p.22 theorem 4] $co\sigma(C) \subset \overline{V}(C)$ we have $0 \in \overline{V}(C)$. This assumption gives contradication.

Implication: cramped $C \to 0 \notin \overline{V}(C)$. Mutiplying C by the suitable $z \in \mathbf{C}$, $|z| = 1$ we can assume that

$$\sigma(C) \subset \{\exp it : 0 < t < \pi\}. \tag{4.8}$$

Let $t_1 = \inf\{t : \exp it \in \sigma(C)\}$ and $t_2 = \sup\{t : \exp it \in \sigma(C)\}$. Then $\exp it_j \in \sigma(C)$, $j = 1,2$. If we assume $0 \in \overline{V}(C)$, by theorem 1 b) we have $co\sigma(C) = \overline{V}(C)$ and therefore $0 \in co\sigma(C)$. Then $0 \in [\exp it_1, \exp it_2]$ is a segment with the ends $\exp it_j$, $j = 1,2$. Consequently $t_1 = 0$, $t_2 = \pi$, that contradicts to $0 < t < \pi$ in (4.8). Therefore $0 \notin \overline{V}(C)$. The theorem is proved.

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