Transactions of National Academy of Sciences of Azerbaijan
Series of Physical-Technical and Mathematical Sciences, vol. XXXV, No 1, 2015.

Connectivity and semigroups of homeomorphic, local homeomorphic and open continuous mappings of topological spaces

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Received: 11.06.2014 / Revised: 24.09.2014

Abstract. *In this paper connections between connectivity of topological spaces and semigroups of homeomorphic, local homeomorphic and open continuous mappings of these spaces into itself are studied.*

Let X be a Tychonov space containing such an open local compact Hausdorff subspace Ω_X that for any two points $\xi, \eta \in \Omega_X$ and every neighbourhood V_ξ of the point ξ there exists a homeomorphism a of X into itself such that $aX \subseteq V_\xi$, $a\eta = \xi$ and $a\Omega_X$ is an open subspace of X. Let us denote the class of all such spaces M. Let $X \in M$ and for every compact $K \subset \Omega_X$ and an arbitrary point $\xi \in \Omega_X \setminus K$ there exist homeomorphisms a, b of X into X such that $a\Omega_X \subseteq \Omega_X, b\Omega_X \subseteq \Omega_X$ and $a\Omega_X, b\Omega_X$ are open, $a\eta = b\eta$ for $\eta \in K$ but $a\xi \neq b\xi$. We denote the class of all such spaces M'. Finite dimensional Euclidian spaces and the cube $D^\tau, \tau \geq \aleph_0$ [1] belong to the class M'.

Proposition 1. *Let $X \in M'$ and Y is such a subspace of X that $Int_X(Y\cap$ $\cap\Omega_X) \neq \oslash$, then $Y \in M'$.*

Let $X \in M$ and a is such a homeomorphism X into Ω_X that $a\Omega_X$ is open. Let us denote $H(X, \Omega_X)$ the semigroup of all such homeomorphisms. $H_K(X, \Omega_X)$ is a subsemigroup of $H(X, \Omega_X)$ consisting of all $a \in H(X, \Omega_X)$ for which there exists a compact K_a such that $K_a \subseteq \Omega_X, aX \subseteq K_a$. $\{K_i\}$ is a system of compacts $K_i \subset \Omega_X,\ i \in I$ provided that $\bigcup\limits_{i\in I} Int\ K_i = \Omega_X$. $H_{\{K_i\}}(X, \Omega_X)$is a subsemigroup of $H_K(X, \Omega_X)$ consisting of all $a \in H_K(X, \Omega_X)$ for which there exists $i_a \in I$ such that $aX \subseteq K_{i_a}$. Let a be such a homeomorphism X into X that $a\Omega_X \subseteq \Omega_X$ and $a\Omega_X$ is open. $H_{\Omega_X}(X)$ is a semigroup of all such homeomorphisms. It is clear that $H(X, \Omega_X) \subseteq H_{\Omega_X}(X)$. D_{Ω_X} is a subsemigroup of $H_{\Omega_X}(X)$ provided that $H_{\{K_i\}}(X, \Omega_X) \subseteq D_{\Omega_X} \subseteq H_{\Omega_X}(X)$. $D^\circ_{\Omega_X}$ is a set of all such elements $a \in D_{\Omega_X}$ that $\overline{aX} \subseteq \Omega_X$ and $\overline{aX}$ is a compact. Obviously $D^\circ_{\Omega_X}$ is an ideal [2] of D_{Ω_X}. We denote φ an isomorphism between semigroups D_{Ω_X} and D_{Ω_Y} and f homeomorphism between Ω_X and Ω_Y induced by φ.

Theorem 1. *Let $X, Y \in M'$. If semigroups D_{Ω_X} and D_{Ω_Y} are isomorphic and X is represented as the union of finite numbers of its disjoint open-closed subsets then Y can be represented as the union of the same numbers of its disjoint open-closed subsets.*

Proof. Suppose that X is represented as the union of finite numbers of its non-empty disjoint open-closed subsets X_m, i.e. $X = \bigcup\limits_{m=1}^{n} X_m$. In the interior of some compact $K_{i_0},\ i_0 \in I$ of our system of compacts we choose n different points $\xi_m,\ m = 1, 2, ..., n$ and separate them from each other with mutually disjoint neighbourhoods $V_{\xi_m} \subset Int\ K_{i_0}$, $m = 1, 2, ..., m$. For each m there exists a compact

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neighbourhood $V'_{\xi_m} \subset V_{\xi_m}$ and a homeomorhism $g_m \in D_X$ provided that $g_m X \subset V'_{\xi_m}$. We denote g_{X_m} the restriction g_m to X_m. Let g be the mapping X into K_{i_0} such that $gX = \{g_m x, x \in X_m\}$. It's clear that g is a homeomorphism and $\overline{gX} \subset K_{i_0}$. As $\bigcup\limits_{m=1}^{n} (X_m \cap \Omega_X) = \Omega_X$ and $X_m \cap \Omega_X$, $m = 1, 2, ..., n$ are open in X we obtain that $g_{X_m}(X_m \cap \Omega_X)$ are open in X. Hence $g\Omega_X = \bigcup\limits_{m=1}^{n} g_{X_m}(X_m \cap \Omega_X)$is open. Besides it $g\Omega_X \subset Int\ K_{i_0}$. It follows that $g \in D^{\circ}_{\Omega_X}$. Let $d \in \varphi^{-1}\left(D^{\circ}_{\Omega_Y}\right)$. As $\varphi^{-1}\left(D^0_{\Omega_Y}\right) D^0_{\Omega_X} \subseteq \varphi^{-1}\left(D^0_{\Omega_Y}\right)$, $\varphi^{-1}\left(D^0_{\Omega_Y}\right) D^0_{\Omega_X} \subseteq \left(D^0_{\Omega_X}\right)$, so $dg \in D^0_{\Omega_X}$, $\varphi(dg) \in D^0_{\Omega_X}$. Besides it $\overline{\varphi(dg)Y} = \overline{f(dg)X}$. It is clear that fdV_{ξ_m}, $m = 1, 2, ..., n$ are open and mutually disjoint. Besides it $fdgX_m = fdg_m X_m = fdg_m X_m$, $m = 1, 2, ..., n$ are contained in compacts $fdV'_{\xi_m} \subset fdV_{\xi_m}$, $m = 1, 2, ..., n$. That's why $\overline{fdgX_m}$, $m = 1, 2, ..., n$ are open-closed in topology of the space $\overline{\varphi(dg)Y} = \overline{f(dg)X} = \overline{\bigcup\limits_{m=1}^{n} f(dg)X_m} = \bigcup\limits_{m=1}^{n} \overline{f(dg)X_m}$. Hence $\overline{\varphi(dg)Y}$ can be represented as the union of n number of its non-empty disjoint open-closed subsets.

Corollary. *Let $X, Y \in M'$. If semigroups D_{Ω_X} and D_{Ω_Y} are isomorphic then X and Y are both connected or unconnected.*

Let S be an abstract semigroup. All right ideals of S and an empty set we call open sets. It is clear that S will be a topological space. We denote it S_R.

Let X be a topological space having an open base provided that every element of this base is homeomorphic to X. We denote the class of all such spaces L. We denote $\{\Omega_i\}_{i \in I}$ the family of all component of connectivity of X. $OH(X)$ (respectively $LH(X)$, respectively $OC(X)$) is a semigroup of all homemorphic (respectively local homeomorphic, respectively open continuous) mappings of $X \in L$ into itself. $OH^c(X)$ (respectively $LH^c(X)$, respectively $OC^c(X)$) is a subsemigroup of the semigroup $OH(X)$ (respectively of the semigroup $LH(X)$, respectively of the semigroup $OC(X)$) consisting of all $a \in OH(X)$ (respectively of all $a \in LH(X)$, respectively of all $a \in OC(X)$) such that $aX \subseteq \Omega_i$, $i \in I$. For each fixed $i' \in I$ the set consisting of all $a \in OH(X)$ (respectively of all $a \in LH(X)$, respectively of all $a \in OC(X)$) such that $aX \subseteq \Omega_{i'}$ is a right ideal of the semigroup $OH^c(X)$ (respectively of the semigroup $LH^c(X)$, respectively of the semigroup $OC^c(X)$).

We denote $\theta(X)$ the next cardinal invariant:

$$\theta(X) \stackrel{def}{=} \sup \{|\gamma| : \gamma \subseteq \tau \backslash \{\phi\}, \cup\gamma = X \text{ and } \gamma \text{ is disjoint}\},$$

where τ is the topology of the space X.

Obviously, for connected spaces the cardinal invariant $\theta(X)$ is always equal to 1.

Let $X \in L$ and D_X be a subsemigroup of $OC(X)$ containing such a family $\{f_\alpha\}_{\alpha \in A} \subset OH(X)$ that $\{f_\alpha X\}_{a \in A}$ is a base of X and if $g \in OC(X)$, $gX \subseteq f_\alpha X$, $\alpha \in A$ then $g \in D_X$.

Proposition 2. $\theta(X) \geq \theta((D_X)_R)$.

Proof. Let $\{\Omega_\beta\}_{\beta \in B}$ be a family non-empty disjoint open sets of $(D_X)_R$ such that $\bigcup\limits_{\beta \in B} \Omega_\beta = (D_X)_R$. As the family $\{f_\alpha X\}_{\alpha \in A}$ is a base of X then $\bigcup\limits_{\alpha \in A} f_\alpha X = X$, $\bigcup\limits_{\beta \in B} \Omega_\beta X = X$ where $\Omega_\beta X = \bigcup\limits_{g \in \Omega_\beta} gX$. Suppose that some sets $\Omega_{\beta'} X$ and $\Omega_{\beta''} X$, $\beta', \beta'' \in B$ intersect. It's clear that there exist $g_1 \in \Omega_{\beta'}$, $g_2 \in \Omega_{\beta''}$ such that $g_1 X \cap g_2 X \neq \varnothing$. Let $\xi_1 \in g_1 X \cap g_2 X$. There exists $f_{\alpha_1} \in \{f_\alpha\}_{\alpha \in A}$ such that $\xi_1 \in f_1 X \subseteq g_1 X \cap g_2 X$. Let $f_{\alpha_2} \in \{f_\alpha X\}_{\alpha \in A}$ and $f_{\alpha_1}^{-1} \xi_1 \in f_{\alpha_2} X$. Obviously, $\xi_1 \in f_{\alpha_1} f_{\alpha_2} X \subseteq g_1 X \cap g_2 X$. Let $f_{\alpha_3} \in \{f_\alpha X\}_{\alpha \in A}$, $f_{\alpha_4} \in \{f_\alpha X\}_{\alpha \in A}$ provided that $g_1^{-1} \xi_1 \in f_{\alpha_3} X \subseteq g_1^{-1} f_{\alpha_1} f_{\alpha_2} X$ and $g_2^{-1} \xi_1 \in f_4 X \subseteq g_2^{-1} f_{\alpha_1} f_{\alpha_2} X$. It follows that $\xi_1 \in g_1 f_{\alpha_3} X \subseteq f_{\alpha_1} f_{\alpha_2} X$, $\xi_1 \in g_2 f_{\alpha_4} X \subseteq f_{\alpha_1 \alpha_2} X$. As $f_{\alpha_1}^{-1} \xi_1 \in f_{\alpha_1}^{-1} g_1 f_{\alpha_3} X \subseteq f_{\alpha_2} X$, $f_{\alpha_1}^{-1} \xi_1 \in f_{\alpha_1}^{-1} g_2\, f_{\alpha_4} X \subseteq f_{\alpha_2} X$ then $f_{\alpha_1}^{-1} g_1 f_{\alpha_3} \in D_X$, $f_{\alpha_1}^{-1} g_2 f_{\alpha_4} \in D_X$. Hence $g_1 f_{\alpha_3} \in f_{\alpha_1} D^1_X$, $g_2 f_{\alpha_4} \in f_{\alpha_1} D^1_X$. Besides it $g_1 f_{\alpha_3} \in g_1 D^1_X$, $g_2 f_{\alpha_4} \in g_2 D^1_X$. But $f_{\alpha_1} D^1_X$ belongs to some $\Omega_{\beta'''}$. The element $g_1 f_{\alpha_3}$ belongs to $\Omega_{\beta'}$. Besides it $g_1 f_{\alpha_3}$ belongs to $f_{\alpha_1} D^1_X$. Hence $g_1 f_{\alpha_3}$ belongs to $\Omega_{\beta'''}$. So $\Omega_{\beta'} \cap \Omega_{\beta'''} \neq \varnothing$. As the decomposition is disjoint then $\Omega_{\beta'} = \Omega_{\beta'''}$. Analogously, $\Omega_{\beta''} = \Omega_{\beta'''}$. Hence $\Omega_{\beta'} = \Omega_{\beta''}$. But $\Omega_{\beta'} \cap \Omega_{\beta''} = \varnothing$. So the family open sets $\{\Omega_\beta X\}_{\beta \in B}$ of the space X is disjoint.

Theorem 2. *Let X be a local connected space of the class L. Then*

$$\theta(X) = \theta\left(OH^c_R(X)\right) = \theta\left(LH^c_R(X)\right) = \theta\left(OC^c_R(X)\right).$$

Proof. As X is a local connected space then its every component of connectivity is open. We prove the equality $\theta(X) = \theta\left(OH^c_R(X)\right)$. Let us denote $R_{i'}$ the right ideal consisting of all $a \in OH^c(X)$ such that $aX \subseteq \Omega_{i'}$, where i' is fixed. Obviously, $OH^c_R(X) = \bigcup\limits_{i\in I} R_i$. So $\theta\left(OH^c_R(X)\right) \geq \theta(X)$. From proposition 2 it follows that $\theta(X) \geq \theta\left(OH^c_R(X)\right)$. Hence $\theta(X) = \theta\left(OH^c_R(X)\right)$. Analogously, $\theta(X) = \theta\left(LH^c_R(X)\right)$, $\theta(X) = \theta\left(OC^c_R(X)\right)$.

Corollary. *Local connected space of the class L is connected if and only if the semigroup $OH^c(X)$ (respectively the semigroup $LH^c(X)$, respectively the semigroup $OC^c(X)$ can not be represented as the union of its mutually disjoint right ideals.*

References

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