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Application Of The Finite Difference Method For Solving Stationary One-Dimensional Flow

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Abstract. *The origin of viscosity should be sought in molecular nature of the structure of matter. Separate molecular of liquid at their proper motion take from one place of space to another ones certain quantity of matter, energy and motion. The equations of motion of viscous liquid are mathematical expressions of equilibrium of several forces. It is clear that when it is impossible to take into account all these forces i.e. when it is impossible to integrate the equations of motion of viscous liquid, one can attempt to disgraced one of these forces.*

If we consider exclusively the motion of ideal liquid, we see that surface forces applied to the elements of the surface dS of any volume of liquid are inward normal pressures. However all real liquids are viscous in this or other degree; in other words they possess internal friction property. In the case of motion of viscous liquid we should consider not only surface forces with only normal pressures and also tangential stresses. Liquid is said to be viscous if the surface forces applied to the elements of the surface of any volume of liquid, generally speaking, in addition to normal are also tangential components. The origin of liquid should be sought in molecular nature of the structure of matter. Separate molecules of liquid at their proper motion take from one place of space to another ones certain quantity of matter energy and motion. The quantities that we deal with in hydrodynamics, are mean quantities obtained at the result of total account, relating to very great quantity of molecules. The proper motion of molecules helps to equalize the values of these mean quantities in neighboring layers. So, for example, in the availability of two neighboring layers of one and the same liquid but of different density the transfer of molecules will promote equalization of these densities: a diffusion process happens. In the same way, if we have non-uniform distribution of temperature, then transfer of molecules will help to equalize temperature: So the heat conductivity process finds its clarification. Finally, in our case, in availability of viscosity, we deal with the process of transfer by molecules of own proper quantity of motion: this process reduces to equalization of velocities of neighboring layers of liquid. The viscous liquid equations have very complicated form; therefore, they may be integrated in comparatively few number of cases. Theory of motion of viscous liquid follows mainly the line of development of approximate methods of integration of viscous liquid motion equation. The viscous liquid motion equations are mathematical expression of equilibrium of some forces: 1) external forces applied to liquid; 2) inertia forces, 3) pressure forces, 4) internal friction forces. It is clear that when it is impossible to take into account all these forces, i.e. when it is impossible to integrate these viscous liquid equations completely, one can attempt to disregard one of these forces. But we can't disregard the external forces, because we put them into operation if and only if they are important. On the other hand, we can not disregard the pressure forces since these forces are internal forces by means of which equilibrium all remaining forces is realized. If we disregard the internal friction forces of hydrodynamics of ideal liquid. Vice-verse, having disregarded the inertia forces and having left

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the friction forces, we can get approximate solution of a number of viscous fluid motion problems. Let's consider a general case of stationary viscous flow. Assume that the motion is stationary and happens along the axis Oz so that .

$$v_x = v_y = 0, \ \ v_z = v(x, y, z).$$

Assuming no external forces, we conclude that

$$\frac{\partial p}{\partial z} = const,$$

the function ν depends only on x, y and satisfies the equation

$$\Delta\nu \equiv \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial z}, \tag{1}$$

where μ is an internal friction coefficient or viscosity coefficient. If the pressures at two points M_1 and M_2 on the axis Oz be l distant from each other, are denoted by p_1 and p_2, respectively, then obviously we have:

$$\frac{\partial p}{\partial z} = \frac{p_2 + p_1}{l} = -\frac{p_1 + p_2}{l}. \tag{2}$$

If the we divide the volume of liquid leaking in a per unit time through the cross section of the pipe determined from formula:

$$Q = \int_0^a 2\pi r\nu dr = \frac{\pi a^4}{8\mu}.\frac{p_1 - p_2}{l} \tag{3}$$

by πa^2, we find the mean velocity of the flow

$$\bar{\nu} = \frac{Q}{\pi a^2} = \frac{a^2}{8\mu}.\frac{p_1 - p_2}{l} = \frac{1}{2}\nu_0, \tag{4}$$

where $\nu_0 = \frac{(p_1 - p_2)a^2}{4\mu l}$. In experiments usually the quantity $p_1 - p_2 = \Delta p$ is defined. Therefore having solved equation (3) and (4) with respect to Δp, we have:

$$p_1 - p_2 = \Delta p = \frac{8\mu l Q}{\pi a^4}; \Delta p = \frac{8\mu l \bar{\nu}}{a^2}.$$

Thus , we get the Hagen-Poiseulle law: Under laminar flow the pressure drop is proportional to volume of fluid flowing for a second and the length of the pipe, and is inversely proportional to the fourth power of the pipe radius. Or other wise, the pressure drop is proportional to mean velocity of flow and the pipe length and is conversely proportional to square of the pipe radius. It is known that there exist two forms of liquid flows: laminar and turbulent. The laminary form of flow is characterized by tame motion of liquid's particles as it takes place in the Moselle flow . Vice-versa, in turbulent motion, the particles move in a very random manner both at turbulent motion in the pipe in the main motion in the direction of the pipe axis, and perpendicular to this direction. We can visually show the difference of two forms of flows if in some place of the pipe axis we inject some quantity of coloring substance; then at laminar flow we see one sharply colored stream of liquid, while at turbulent form of flow the whole liquid will be coloured that shows strong stirring of the liquid's particles. Now we distinguish two cases, it depends on whether $\partial p/\partial z$ vanishes or not.

1.$\partial p/\partial z = 0$ In this case the velocity ν satisfies the Laplace equation

$$\Delta\nu \equiv \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} = 0. \tag{5}$$

It is clear that in the present case the boundaries of liquid may be only the cylinders with generators parallel to the axis Oz that may remain fixed or move parallel to the axis Oz with constant velocity. Let for example consider the motion of liquid between two cylinders whose intersection with the plane Oxy are the curves C_1 and C_2 enclosing one another. Let the first cylinder displace parallel to the axis Oz

with velocity ν_1, the second one with velocity ν_2. In this case the boundary conditions satisfied by the harmonic function ν, will be:

$$\nu = \nu_1 \quad \text{on} \quad C_1$$

$$\nu = \nu_2 \quad \text{on} \quad C_2,$$

but then it is clear that the considered problem may be at once reduced to the equivalent problem on plane vortexless motion of incompressible liquid.

2.$\partial p/\partial z \neq 0$ In this case the velocity v satisfies the Poisson equation

$$\Delta\nu \equiv \frac{\partial^2 \nu}{\partial x^2} + \frac{\partial^2 \nu}{\partial y^2} = -k \tag{6}$$

if for brevity we introduce the denotation

$$k = \frac{1}{\mu} . \frac{\partial p}{\partial z} = \frac{(p_1 - p_2)a^2}{\mu l},$$

where $p_1 - p_2$ is the pressure drop on the segment l located parallel to the axis Oz

The issue on motion of viscous liquid in a fixed cylindrical pipe with generators parallel to the axis Oz is the most important case of this type. If the lateral cross section of this pipe is the curve C, then the boundary condition for the sought for function ν will be

$$\nu = 0 \quad on \quad C. \tag{7}$$

In the case we have the generalization of the Poiseulle flow for the case of arbitrary cross section pipe.

A problem of theory of elasticity on torsion of a prism, and also a problem on plane motion of ideal incompressible liquid in the domain S whose contour C rotates with constant angular velocity, and finally a problem on deflection of a membrane under uniform loading are reduced to the solution of equation (6) under boundary condition (7). In this connection, the equation integration problem (6) under boundary condition (7) is solved for a large number of contours.

Now, let's consider the case of motion of a more general form viscous liquid, more exactly, reject the stationary condition.

Assume that the motion of incompressible liquid at no external forces happens parallel to the axis Ox.

$$\nu_y = \nu_z = 0.$$

The continuity equations shows that ν_x is independent of x, i.e.

$$\nu_x = \nu(y, z, t).$$

Therefore the equations of hydromechanics are strongly simplified :

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 \nu}{\partial z^2}\right) - \frac{\partial \nu}{\partial t}; \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0.$$

From the last equations it is seen that p depends only on x and t. But then at the first equation the left side is independent of y and z, the right side is independent of x; therefore both the left and right sides are the functions of only one t:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = f(t).$$

If$f(t) = o$, the equation for ν takes the form :

$$\frac{\partial \nu}{\partial t} = \nu\left(\frac{\partial^2 \nu}{\partial y^2} + \frac{\partial^2 \nu}{\partial z^2}\right). \tag{8}$$

But if $f(t) \neq 0$, then instead of ν we introduce a new function $\bar{\nu}$ having put

$$\bar{\nu} = \nu + \int_0^t f(t)dt,$$

then

$$\frac{\partial \tilde{\nu}}{\partial t} = \frac{\partial \nu}{\partial t} + f(t)$$

and therefore $\tilde{\nu}$ will satisfy the same equation (8) : true , the boundary conditions therewith somethat change.

So, in all the cases of nonstatinary one- dimensional flow the matther is reduced to integration of equation (8). This equation is the main equation of heat theory; the solution of a great number of special problems connected with this equation is known and this enables to determine a large number of appropriate flows of viscous liquid. Of course, in the course of solution of equation (8) it is also necessary to take into account corresponding boundary and initial conditions; the last ones are reduced to representation of ν for the initial time $t = 0$.

It turns out well to obtain the exact solution of boundary value problems for elliptic and hyperbolic type equations only in special cases. Therefore, it is necessary to solve these problems approximately. The finite differences method or the netpoint method is a universal and effective method for solving boundary value problems of elliptic and hyperbolic type.

The finite differences method at present is one the most extended methods of approximate solution of boundary value problems for partial differential equations. Consequently, operation of the working finite - difference schemes for linear and nonlinear differential equations of elliptic and hyperbolic type is one of the most urgent problems of numerical analysis. Consider on equation of stationary motion of viscous liquid:

$$\frac{\partial \nu}{\partial t} = \frac{\partial^2 \nu}{\partial y} = -k$$

$$0 \le t < +\infty, 0 \le x \le a, 0 \le y \le b, \tag{9}$$

$$\nu = 0, \quad \text{on} \quad C$$

For approximation of equations (9) on the plane x, y we introduce a uniform rectangular net.

$$t^n = n\tau, n = 0, 1, 2, ...,$$

$$x_\nu = mh_1, h_1 = \frac{a}{M}, m = 1, 2, ..., M,$$

$$y_k = kh_2, h_2 = \frac{b}{K}, k = 1, 2, ..., K.$$

Denote

$$\nu^n_{m,k} = \nu(t^n, x_m, y_k).$$

We write the finite -difference approximation of the equation of stationary motion of viscous liquid in the form:

$$\frac{\nu^{n+1}_{m,k} - \nu^n_{m,k}}{\tau} = \frac{\nu^{n+1}_{m+1,k} - 2\nu^n_{m,k} + \nu^n_{m+1,k}}{h_1^2} + \frac{\nu^{n+1}_{m,k+1} - 2\nu^n_{m,k} + \nu^n_{m,k-1}}{h_2^2} - \frac{1}{\mu}\frac{p^{n+1} - p^n}{h_1}. \tag{10}$$

Here p^{n+1} and p^n is the value of the pressure on the $n+1$-th and n-th layer (transverse to layer p is constant,therefore in differences of the record the lower index is absent).

It is easy to see that the principal part of the approximation error for equation (10) is $O(\tau) + O(h_1^2) + O(h_2^2)$.

Recent years the investigation of finite -differences method for solving boundary value problems for mixed type equations and degenerate equations is of great interest. This is stipulated on one hand with a great applied value of these problems, on the other hand with increasing possibilities of ECM.

By writing differential schemes it is always necessary to keep in mind the calculational work that will be necessary for solving the obtained systems of difference equations. Therefore, we are restricted in studying the simplest schemes with minimal pattern on which the second (or higher) accuracy order is provided. Namely, such schemes are widely used in practice.

The quality of the difference scheme is determined first of all by its accuracy and efficiency.

For efficient realization of boundary value problems for mixed type equations, there exist various aspects of the finite-differences method: and different stable calculation algorithms are created on the basis of these aspects.

Solving equation (10) with respect $\nu_{m,k}^{n}$ and applying the Zeidel iterative method, we get the following system:

$$\nu_{m,j}^{n^{(i)}} = \sum_{j=1}^{n-1} a_{mj}^{n} \nu_{jm}^{n^{(i)}} + \sum_{j=n+1}^{N_p} a_{mj}^{n} \nu_{jm}^{n^{(i-1)}} - P^{n}$$

$$(n = \overline{1, N_p}, (m = \overline{1, m_p})$$

It is easy to prove that for any $\left\{\nu_{m,j}^{n^{(0)}}\right\}$ as $i \to \infty$

$$V_{m,j}^{n^{(i)}} = \nu_{m,j}^{n^{(i)}} - \nu_{m,j}^{n} \to 0.$$

By the induction method it is easy to be convinced that for any knot (m, j) the following estimation is valid:

$$\left|V_{m,j}^{n^{(i)}}\right| \le (1-\rho)^{i} M,$$

where ρ are some coefficients. Consequently,

$$\max\left|V_{m,j}^{n^{(i)}}\right| \le (1-\rho)^{i} M \to 0 \text{ as } i \to \infty.$$

Determine the number of iterations that provide the finding of the solution with the given accuracy $\varepsilon > 0$:

$$\left|V_{m,j}^{n^{(i)}}\right|, \le M(1-\rho)^{i} \le \varepsilon, \ln(1-\rho)^{i} \le \ln\frac{\varepsilon}{M}, i \ge \frac{\ln\frac{\varepsilon}{M}}{\ln(1-\rho)^{i}},$$

Consequently, for attaining the solution with the given accuracy, it is required $m_0(\varepsilon) = \left[\frac{\ln\frac{\varepsilon}{M}}{\ln(1-\rho)}\right]$ minimal quality of iterations. In this case the amount of arithmetical operators for finding the solution with the given accuracy is found as follows:

$$Q(\varepsilon) = \sum_{k=1}^{m_0(\varepsilon)} q_k,$$

where q_k amount of arithmetical operations for any k almost doesn't change, then

$$q_k = q^*, \quad \text{where} \quad q^* = o\left(\frac{1}{h^2}\right).$$

Then

$$Q(\varepsilon) = o\left(\frac{1}{h^2} \cdot \frac{\ln\frac{\varepsilon}{M}}{\ln(1-\rho)}\right)$$

Suppose that C is given by the equation

$$y = \sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^{2}},$$

$$\nu(x,y,t) = (x^2 + y^2)^{1/4} \sin\left(\frac{1}{2}\, arctg\frac{y}{x}\right) + x^2 + y^2.$$

By the model problem of the Zeidel algorithm, solving the corresponding difference problem on a computer, we get the following results:

Nodal points	Exact solution	Numerical solution	Absolute solution
v_{13}^{1}	0,0100	0,0089	0,0011
v_{15}^{1}	0,0400	0,0395	0,0005
v_{17}^{1}	0,0900	0,0896	0,0004
v_{19}^{1}	0,1600	0,1592	0,0008
v_{111}^{1}	0,2500	0,2489	0,0011
v_{113}^{1}	0,3600	0,3593	0,0007
v_{115}^{1}	0,4900	0,0008	0,0008
v_{117}^{1}	0,6400	0,6395	0,0005
v_{119}^{1}	0,8100	0,8115	0,0015
v_{121}^{1}	1,0000	0,9889	0,0111
v_{33}^{3}	0,1639	0,1618	0,0021
v_{35}^{3}	0,1586	0,1567	0,0019
v_{37}^{3}	0,1901	0,1917	0,0016

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