Transactions of National Academy of Sciences of Azerbaijan
Series of Physical-Technical and Mathematical Sciences, vol. XXXV, No 1, 2015.

On A Boundary Control Problem For A Thin Plate Oscillations Equation

Hamlet F. Kuliyev · Khayala I. Seyfullayeva

Received: 02.09.2014 / Revised: 10.12.2014

Abstract. *In the present paper, we suggest a boundary optimal control problem for a linear equation of thin plate oscillations. In the paper we prove a theorem on the existence and uniqueness of the optimal control, calculate the differential functional and derive necessary optimality condition in the form of an integral inequality.*

Introduction. It is known that some processes of mathematical physics are described by partial differential equations of fourth order. For instance, equations of oscillations of a bar, camerton, elastic plate, thin plate and so on are such equations [1]-[3]. Therefore, investigation of optimal control problems in the processes described by such equations are important [problems. When control functions are boundary functions, it becomes difficult to study the control problems. But note that the boundary control problem is very natural compared with distributed parameters problems from theoretical and practical point of view.

Note that in the papers [4]-[8], some close control problems were considered. More exactly, in the paper [5], some generalizations of the Pontryagin maximum principle for an inhomogeneous thin plate were set up. In [5] the feedback problem for a thin plate oscillations equation was considered, in [6] a problem on optimization with domain was studied, in [7], theorems on the existence of optimal control in some cases in the optimal control problem for thin plate oscillation equation were proved, and finally in [8], mainly the numerical realizations of the optimal control problem for a thin plate equation were considered.

Unlike the above mentioned papers, in the present paper we study the boundary optimal control.

1. Problem statement. Let the control process be described by a thin plate oscillations equation

$$\frac{\partial^2 u}{\partial t^2} + a^2 \Delta^2 u = 0 \ \text{ in } Q_T = \Omega \times (0,T), \ \ \Omega = (0,l_1) \times (0,l_2) \tag{1}$$

with initial

$$u(x_1,x_2,0) = \varphi_0(x_1,x_2), \frac{\partial u(x_1,x_2,0)}{\partial t} = \varphi_1(x_1,x_2), \ (x_1,x_2) \in \Omega \tag{2}$$

H.F. Kuliyev
Baku State University,
23, Z. Khalilov str., AZ 1148, Baku, Azerbaijan

Kh.I. Seyfullayeva
Sumgayit State University
1, Baku str, 43 ward.,
AZ 5008, Sumgayit, Azerbaijan
E-mail: seyfullayeva.xeyale@yandex.ru

and boundary conditions

$$u(0,x_2,t)=0,\ u(l_1,x_2,t)=0,\ \frac{\partial u(0,x_2,t)}{\partial x_1}=\upsilon(x_2,t),$$
$$\frac{\partial u(l_1,x_2,t)}{\partial x_1}=0,(x_2,t)\in(0,l_2)\times(0,T),$$
$$u(x_1,0,t)=0,\ u(x_1,l_2,t)=0,\ \frac{\partial u(x_1,0,t)}{\partial x_2}=0,$$
$$\frac{\partial u(x_1,l_2,t)}{\partial x_2}=0,(x_1,t)\in(0,l_1)\times(0,T), \tag{3}$$

where a^2,l_1,l_2,T are the given positive numbers, $\upsilon(x_2,t)$ is a boundary control function, $\varphi_0(x_1,x_2)\in W_2^2(\Omega)$, $\varphi_1(x_1,x_2)\in L_2(\Omega)$ are the given functions, Δ is the Laplace operator with respect to $x_1,\ x_2$.

Let's consider a space of controls $H=W_2^{4,2}\left((0,l_2)\times(0,T)\right)$.

For a class of admissible controls U_{ad} we take the set of functions $\upsilon(x_2,t)$ from H, for which $\upsilon(0,t)=\upsilon(l_2,t)=0,\ \dfrac{\partial\upsilon(0,t)}{\partial x_2}=\dfrac{\partial\upsilon(l_2,t)}{\partial x_2}=0,\upsilon(x_2,0)=\dfrac{\partial\upsilon(x_2,t)}{\partial t}=0,$ moreover

$$\left\|\frac{\partial^4\upsilon}{\partial x_2^4}\right\|_{L_2((0,l_2)\times(0,T))}\le M_1,\ \left\|\frac{\partial^2\upsilon}{\partial t^2}\right\|_{L_2((0,l_2)\times(0,T))}\le M_1,$$

where M_1 is a given number.

Here and in the sequel, by M_i, we'll denote different constants independent of admissible controls and estimated variables.

It is supposed that the functions $\varphi_0(x_1,x_2)$ and $\upsilon(x_2,t)$ satisfy the natural agreement conditions.

We state a problem: in the set U_{ad} find such a function that together with the solution of boundary value problem (1)-(3) it delivers minimum to the functional

$$J(\upsilon)=\frac{1}{2}\int\limits_{\Omega}\left[u(x_1,x_2,T)\right]^2dx_1dx_2+\frac{\alpha}{2}\int\limits_0^{l_2}\int\limits_0^{T}\upsilon^2(x_2,t)dx_2dt, \tag{4}$$

where $\alpha>0$ is a positive number.

Under the solution of problem (1)-(3) for each fixed admissible control $\upsilon(x_2,t)$ we understand a function $u(x_1,x_2,t)\in W_2^{2,1}(Q_T)$ such that for any function $\eta\in W_2^{2,1}(Q_T)$, $\eta(x_1,x_2,T)=0$,

$$\eta(0,x_2,t)=0,\ \eta(l_1,x_2,t)=0,\ \frac{\partial\eta(0,x_2,t)}{\partial x_1}=0,\ \frac{\partial\eta(l_1,x_2,t)}{\partial x_1}=0.$$
$$\eta(x_1,0,t)=0,\ \eta(x_1,l_2,t)=0,\ \frac{\partial\eta(x_1,0,t)}{\partial x_2}=0,\ \frac{\partial\eta(x_1,l_2,t)}{\partial x_2}=0$$

satisfies the integral identity

$$\int\limits_{Q_T}\left[-\frac{\partial u}{\partial t}\frac{\partial\eta}{\partial t}+a^2\Delta u\Delta\eta\right]dx_1dx_2dt-\int\limits_{\Omega}\varphi_1(x_1,x_2)\eta(x_1,x_2,0)dx_1dx_2=0 \tag{5}$$

and the conditions

$$u(x_1,x_2,0)=\varphi_0(x_1,x_2),\frac{\partial u(0,x_2,t)}{\partial x_1}=\upsilon(x_2,t) \tag{6}$$

in the ordinary sense.

Note that one can prove a theorem on the existence and uniqueness of the solution $u(x_1,x_2,t)$ of boundary value problem (1)-(3) for each fixed admissible control $\upsilon(x_2,t)$, and such a solution has the property $u\in C\left([0,T];W_2^2(\Omega)\right),\ u\in C\left([0,T];L_2(\Omega)\right)$ [9].

2. Existence of optimal control in problem (1)-(4).

Theorem 1. *In the optimal control problem (1)-(4) there exists a unique optimal control.*

Proof. Let $\{\upsilon_n\} \in U_{ad}$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} J(\upsilon_n) = \inf_{\upsilon\in U_{ad}} J(\upsilon). \tag{7}$$

Then from the definition of the class U_{ad} it follows that

$$\|\upsilon_n\|_{W_2^{4,2}((0,l_2)\times(0,T))} \le M_2.$$

Denote by $u_n(x_1, x_2, t)$ the solution of problem (1)-(3) that corresponds to $\upsilon_n(x_2, t)$.

As in the first part of the paper [10], for each function $u_n(x_1, x_2, t)$ one can construct Galerkin approximations $u_n^N(x_1, x_2, t)$ to the solution $u_n(x_1, x_2, t)$ and to obtain the estimation

$$\left\|u_n^N\right\|_{W_2^{2,1}(Q_T)} \le M_3.$$

Hence, by the weak lower semi-continuity of the norm in Hilbert spaces, we obtain the same estimation for $u_n(x_1, x_2, t)$, i.e.

$$\|u_n\|_{W_2^{2,1}(Q_T)} \le M_3.$$

Then, by the property of weak compactness in Hilbert spaces, we can consider that as $n \to \infty$

$$\upsilon_n \to \upsilon_0 \text{ weakly } W_2^{4,2}\left((0,l_2)\times(0,T)\right) \tag{8}$$

and

$$u_n \to u_0, \frac{\partial u_n}{\partial x_1} \to \frac{\partial u_0}{\partial x_1}, \frac{\partial u_n}{\partial x_2} \to \frac{\partial u_0}{\partial x_2}, \frac{\partial u_n}{\partial t} \to \frac{\partial u_0}{\partial t}$$

$$\frac{\partial^2 u_n}{\partial x_1^2} \to \frac{\partial^2 u_0}{\partial x_1^2}, \frac{\partial^2 u_n}{\partial x_1 \partial x_2} \to \frac{\partial^2 u_0}{\partial x_1 \partial x_2}, \frac{\partial^2 u_n}{\partial x_2^2} \to \frac{\partial^2 u_0}{\partial x_2^2} \text{ weakly } L_2(Q_T). \tag{9}$$

By the imbedding theorem [11], from relation (8) it follows that

$$\upsilon_n(x_2, t) \text{ uniformly converges to } \upsilon_0(x_2, t) \tag{10}$$

on $[0, l_2] \times [0, T]$ as $n \to \infty$.

From relations (9), by the imbedding theorem [12] it follows that as $n \to \infty$

$$u_n(x_1, x_2, t) \to u_0(x_1, x_2, t) \text{ strongly in } L_2(Q_T). \tag{11}$$

In the definition of the generalized solution of problem (1)-(3) we take $\upsilon = \upsilon_n,\ u = u_n$:

$$\int_{Q_T} \left[-\frac{\partial u_n}{\partial t}\frac{\partial \eta}{\partial t} + a^2 \Delta u_n \Delta \eta\right] dx_1 dx_2 dt - \int_{\Omega} \varphi_1(x_1, x_2)\eta(x_1, x_2, 0) dx_1 dx_2 = 0$$

and $u_n(x_1, x_2, 0) = \varphi_0(x_1, x_2)$ and $\dfrac{\partial u_n(0, x_2, t)}{\partial x_1} = \upsilon_n(x_2, t)$.

If in these equalities we pass to limit as $n \to \infty$, taking into account (9), (10) and (11) we get

$$\int_{Q_T} \left[-\frac{\partial u_0}{\partial t}\frac{\partial \eta}{\partial t} + a^2 \Delta u_0 \Delta \eta\right] dx_1 dx_2 dt - \int_{\Omega} \varphi_1(x_1, x_2)\eta(x_1, x_2, 0) dx_1 dx_2 = 0$$

$$u_0(x_1, x_2, 0) = \varphi_0(x_1, x_2), \frac{\partial u_n(0, x_2, t)}{\partial x_1} = \upsilon_0(x_2, t).$$

These relations show that the function $u_0(x_1, x_2, t)$ is the solution of problem (1)-(3) that corresponds to the admissible control $\upsilon_0(x_2, t)$.

Since the boundary value problem (1)-(3) is linear, the functional (4) is quadratic, then the functional (4) is weakly lower semicontinuous on U_{ad}, therefore

$$\underline{\lim}_{n\to\infty} J(\upsilon_n) \ge J(\upsilon_0). \tag{12}$$

Then from (7) and (12) it follows that

$$\inf_{v\in U_{ad}} J(v) \geq J(v_0).$$

Hence

$$\inf_{v\in U_{ad}} J(v) = J(v_0),$$

and this shows that $v_0(x_2,t)$ is an optimal control in the problem under consideration.

From the form of functional (4) it is clear that it is strongly convex in $L_2((0,l_2)\times(0,T))$, therefore the optimal control is unique. Theorem 1 is proved.

3. Calculation of differential functional (4) and necessary optimality condition

Introduce a conjugate problem for the given control $v(x_2,t)$:

$$\frac{\partial^2\psi}{\partial t^2} + a^2\Delta^2\psi = 0 \text{ in } Q_T, \tag{13}$$

$$\psi(x_1,x_2,T) = 0,\ \frac{\partial\psi(x_1,x_2,T)}{\partial t} = -u(x_1,x_2,T),\ (x_1,x_2)\in\Omega, \tag{14}$$

$$\psi(0,x_2,t) = \psi(l_1,x_2,t) = 0,\ \frac{\partial\psi(0,x_2,t)}{\partial x_1} = \frac{\partial\psi(l_1,x_2,t)}{\partial x_1} = 0,$$
$$(x_2,t)\in[0,l_2]\times[0,T],$$
$$\psi(x_1,0,t) = \psi(x_1,l_2,t) = 0,\ \frac{\partial\psi(x_1,0,t)}{\partial x_2} = \frac{\partial\psi(x_1,l_2,t)}{\partial x_2} = 0,$$
$$(x_1,t)\in[0,l_1]\times[0,T], \tag{15}$$

where $u(x_1,x_2,t)$ is the solution of problem (1)-(3) for the given control $v(x_2,t)$.

Since $u(x_1,x_2,T)\in W_2^2(\Omega)$, the conjugate problem (13)-(15) has a unique solution in the space $W_2^{4,2}(Q_T)$ [9].

Theorem 2. *Let the above assumed conditions be fulfilled on the data of problem (1)-(4). Then the functional (4) is Frechet continuously differentiable on H and its differential at the point $v(x_2,t)\in U_{ad}$ with the increment $\delta v(x_2,t)\in H,\ v(x_2,t)+\delta v(x_2,t)\in U_{ad}$ is determined by the expression*

$$\left\langle J'(v),\delta v\right\rangle_H$$
$$= \int_0^T\int_0^{l_2}\left[\alpha v(x_2,t) + a^2\left(\frac{\partial^2\psi(0,x_2,t)}{\partial x_1^2} + \frac{\partial^2\psi(0,x_2,t)}{\partial x_2^2}\right)\right]\delta v(x_2,t)dx_2dt. \tag{16}$$

Proof. Prove that the functional (11) is differentiable on H. For that we take arbitrary controls $v(x_2,t)\in U_{ad}$ and $v(x_2,t)+\delta v(x_2,t)\in U_{ad}$. Denote the appropriate solutions of problem (1)-(3) by $u(x_1,x_2,t)$ and $u(x_1,x_2,t)+\delta u(x_1,x_2,t)$. Then $\delta u(x_1,x_2,t)$ is the solution of the following boundary value problem

$$\frac{\partial^2(\delta u)}{\partial t^2} + a^2\Delta^2(\delta u) = 0 \text{ in } Q_T, \tag{17}$$

$$\delta u(x_1,x_2,0) = 0,\ \frac{\partial\delta u(x_1,x_2,0)}{\partial t} = 0,\ (x_1,x_2)\in\Omega, \tag{18}$$

$$\delta u(0,x_2,t) = 0,\ \delta u(l_1,x_2,t) = 0,\ \frac{\partial\delta u(0,x_2,t)}{\partial x_1} = \delta v(x_2,t),\ \frac{\partial\delta u(l_1,x_2,t)}{\partial x_1} = 0,$$
$$(x_2,t)\in[0,l_2]\times[0,T],$$
$$\delta u(x_1,0,t) = \delta u(x_1,l_2,t) = 0,\ \frac{\partial\delta u(x_1,0,t)}{\partial x_2} = \frac{\partial\delta u(x_1,l_2,t)}{\partial x_2} = 0, \tag{19}$$
$$(x_1,t)\in[0,l_1]\times[0,T].$$

The increment of the functional (4) is written in the form

$$\Delta J(\upsilon) = J(\upsilon+\delta\upsilon) - J(\upsilon) = \int_0^{l_1}\int_0^{l_2} u(x_1,x_2,T)\delta u(x_1,x_2,T)dx_1dx_2$$

$$+\alpha\int_0^{T}\int_0^{l_2} \upsilon(x_2,t)\delta\upsilon(x_2,t)dx_2dt + R, \tag{20}$$

where

$$R = \frac{1}{2}\int_0^{l_1}\int_0^{l_2} \left(\delta u(x_1,x_2,T)\right)^2 dx_1dx_2 + \frac{\alpha}{2}\int_0^{T}\int_0^{l_2} \left(\delta\upsilon(x_2,t)\right)^2 dx_2dt \tag{21}$$

is a remainder term.

Show that

$$R \le M_4 \left\|\delta\upsilon\right\|^2_{W_2^{4,2}((0,l_2)\times(0,T))} \tag{22}$$

For that it suffices to show that

$$\left\|\delta u(x_1,x_2,T)\right\|^2_{L_2((0,l_2)\times(0,T))} \le M_5 \left\|\delta\upsilon(x_1,x_2,T)\right\|^2_{W_2^{4,2}((0,l_2)\times(0,T))}. \tag{23}$$

In problem (17)-(19) we make the substitution

$$\delta u(x_1,x_2,t) = W(x_1,x_2,t) + z(x_1,x_2,t), \tag{24}$$

moreover $z(x_1,x_2,t)$ is such a sufficiently smooth function that

$$z(0,x_2,t)=0,\ z(l_1,x_2,t)=0,\ \frac{\partial z(0,x_2,t)}{\partial x_1} = \delta\upsilon(x_2,t),\ \frac{\partial z(l_1,x_2,t)}{\partial x_1}=0,$$

$$z(x_1,0,t) = z(x_1,l_2,t) = 0,\ \frac{\partial z(x_1,0,t)}{\partial x_2} = \frac{\partial z(x_1,l_2,t)}{\partial x_2} = 0.$$

As a function $z(x_1,x_2,t)$ we take $z(x_1,x_2,t) = f(x_1)\delta\upsilon(x_2,t)$, where $f(x_1)$ is a sufficiently smooth function on $[0,l_1]$ such that $f(0)=0,\ f(l_1)=0,\ f'(0)=1,\ f'(l_1)=0$.

Then for a new unknown function $W(x_1,x_2,t)$ we get the boundary value problem

$$\frac{\partial^2 W}{\partial t^2} + a^2\Delta^2 W = -\frac{\partial^2 z}{\partial t^2} - a^2\Delta^2 z \equiv F(x_1,x_2,t), \tag{25}$$

$$W(x_1,x_2,0)=0,\ \frac{\partial W(x_1,x_2,0)}{\partial t} = 0, \tag{26}$$

$$W(0,x_2,t) = W(l_1,x_2,t) = 0,\ \frac{\partial W(0,x_2,t)}{\partial x_1} = \frac{\partial W(l_1,x_2,t)}{\partial x_1} = 0,$$

$$W(x_1,0,t) = W(x_1,l_2,t) = 0,\ \frac{\partial W(x_1,0,t)}{\partial x_2} = \frac{\partial W(x_1,l_2,t)}{\partial x_2} = 0. \tag{27}$$

We get the estimation for solving the problem (25)-(27).

Let $\{\varphi_k(x_1)\}$ be a fundamental system in $\overset{\circ}{W}{}_2^2(\Omega)$, and

$$\int_\Omega \varphi_k(x_1,x_2)\varphi_l(x_1,x_2)dx_1dx_2 = \delta_k^l = \begin{cases} 1,\ k=l \\ 0,\ k\ne l. \end{cases}$$

We look for the approximate solution of problem (25-(27) in the form

$$W^N = \sum_{k=1}^{N} c_k^N(t)\varphi_k(x_1,x_2)$$

from the relations

$$\int_0^{l_1}\int_0^{l_2} \frac{\partial^2 W^N}{\partial t^2}\varphi_l(x_1,x_2)dx_1dx_2 + a^2\int_0^{l_1}\int_0^{l_2} \Delta W^N \Delta\varphi_l(x_1,x_2)dx_1dx_2$$

$$= \int_0^{l_1}\int_0^{l_2} F\varphi_l(x_1,x_2)dx_1dx_2,\ l=1,...,N, \tag{28}$$

$$c_k^N(0)=0,\ \frac{dc_k^N(0)}{dt}=0. \tag{29}$$

Equalities (28) are the systems of linear ordinary differential equations of second order for the unknowns $c_k^N(t)$, $k=1,...,N$, solved with respect to $\frac{d^2c_k^N}{dt^2}$. This system is uniquely solvable at initial data (29), moreover $\frac{d^2c_k^N}{dt^2}\in L_2(0,T)$.

Multiplying each of the equalities (28) by its $\frac{dc_l^N(t)}{dt}$ and summing with respect to l from 1 to N, we arrive at the equality

$$\int_0^{l_1}\int_0^{l_2} \frac{\partial^2 W^N}{\partial t^2}\frac{\partial W^N}{\partial t}dx_1dx_2 + a^2\int_0^{l_1}\int_0^{l_2}\Delta W^N\frac{\partial W^N}{\partial t}dx_1dx_2 = \int_0^{l_1}\int_0^{l_2}F\frac{\partial W^N}{\partial t}dx_1dx_2.$$

Hence we get

$$\frac{d}{dt}\int_0^{l_1}\int_0^{l_2}\left[\left|\frac{\partial W^N}{\partial t}\right|^2 + a^2\left|\Delta W^N\right|^2\right]dx_1dx_2 = 2\int_0^{l_1}\int_0^{l_2}F\frac{\partial W^N}{\partial t}dx_1dx_2.$$

Integrating this relation from 0 to t and taking into account condition (23), we have

$$\int_0^{l_1}\int_0^{l_2}\left[\left|\frac{\partial W^N}{\partial t}\right|^2 + a^2\left|\Delta W^N\right|^2\right]dx_1dx_2 = 2\int_0^{t}\int_0^{l_1}\int_0^{l_2}F\frac{\partial W^N}{\partial t}dx_1dx_2d\tau.$$

Hence, by the known inequality

$$\|u_{xx}\|_{2,\Omega}\le\|\Delta u\|_{2,\Omega}$$

from [13], p. 117, for the convex domains Ω, we get

$$\int_0^{l_1}\int_0^{l_2}\left[\left|\frac{\partial W^N}{\partial t}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1^2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1\partial x_2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_2^2}\right|^2\right]dx_1dx_2$$

$$\le M_6\int_0^{t}\int_0^{l_1}\int_0^{l_2}|F|^2dx_1dx_2d\tau$$

$$+M_7\int_0^{t}\int_0^{l_1}\int_0^{l_2}\left[\left|\frac{\partial W^N}{\partial t}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1^2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1\partial x_2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_2^2}\right|^2\right]dx_1dx_2d\tau.$$

Applying the Gronwall lemma to the last inequality, we have

$$\int_0^{l_1}\int_0^{l_2}\left[\left|\frac{\partial W^N}{\partial t}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1^2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_1\partial x_2}\right|^2+\left|\frac{\partial^2 W^N}{\partial x_2^2}\right|^2\right]dx_1dx_2$$

$$\leq M_7 \int_0^T \int_0^{l_1} \int_0^{l_2} |F|^2 \, dx_1 dx_2 dt, \ \forall t \in [0,T]. \tag{30}$$

Owing to this inequality, from the sequence $\left\{ W^N(x_1, x_2, t) \right\}$ we can choose a subsequence (for which we preserve the same denotation) weakly converging in $W_2^{2,1}(Q_T)$ to some element $W \in W_2^{2,1}(Q_T)$. it is easy to show that this element $W(x_1, x_2, t)$ is the generalized solution of problem (25)-(27).

By the weak lower semi-continuity of the norms in Banach spaces, the inequality (30) or just the same

$$\int_0^{l_1} \int_0^{l_2} \left[\left| \frac{\partial W}{\partial t} \right|^2 + \left| \frac{\partial^2 W^N}{\partial x_1^2} \right|^2 + \left| \frac{\partial^2 W^N}{\partial x_1 \partial x_2} \right|^2 + \left| \frac{\partial^2 W^N}{\partial x_2^2} \right|^2 \right] dx_1 dx_2$$

$$\leq M_7 \int_0^T \int_0^{l_1} \int_0^{l_2} |F|^2 \, dx_1 dx_2 dt, \ \forall t \in [0,T].$$

is valid.

Hence, in particular it follows that

$$\int_0^{l_1} \int_0^{l_2} \left| \frac{\partial W}{\partial t} \right|^2 dx_1 dx_2 \leq M_7 \int_0^T \int_0^{l_1} \int_0^{l_2} |F|^2 \, dx_1 dx_2 dt. \tag{31}$$

Taking into account

$$F(x_1, x_2, t) \equiv -\frac{\partial^2 z(x_1, x_2, t)}{\partial t^2} - a^2 \Delta^2 z(x_1, x_2, t) = -f(x_1) \frac{\partial^2 \delta v(x_2, t)}{\partial t^2}$$
$$-a^2 \left[f^{IV}(x_1) \delta v(x_2, t) + 2 f^{II}(x_1) \frac{\partial^2 \delta v(x_2, t)}{\partial x_2^2} + f(x_1) \frac{\partial^4 \delta v(x_2, t)}{\partial x_2^4} \right],$$

from inequality (31) we have

$$\int_0^{l_1} \int_0^{l_2} \left| \frac{\partial W(x_1, x_2, t)}{\partial t} \right|^2 dx_1 dx_2 \leq M_7 \int_0^T \int_0^{l_2} \left[\left| \frac{\partial^2 (\delta v)}{\partial t^2} \right|^2 + \left| \frac{\partial^2 (\delta v)}{\partial x_2^2} \right|^2 + \left| \frac{\partial^4 (\delta v)}{\partial x_2^4} \right|^2 \right] dx_2 dt. \tag{32}$$

Then taking into account substitution (24) and inequality (32), we get

$$\int_0^{l_1} \int_0^{l_2} \left| \frac{\partial (\delta u(x_1, x_2, t)}{\partial t} \right|^2 dx_1 dx_2 \leq M_8 \int_0^T \int_0^{l_2} \left[\left| \frac{\partial^2 (\delta v)}{\partial t^2} \right|^2 + \left| \frac{\partial^2 (\delta v)}{\partial x_2^2} \right|^2 + \left| \frac{\partial^4 (\delta v)}{\partial x_2^4} \right|^2 \right] dx_2 dt. \tag{33}$$

Since

$$\delta u(x_1, x_2, T) = \int_0^T \frac{\partial (\delta u)}{\partial t} d\tau,$$

hence and from (33) we have

$$\int_0^{l_1} \int_0^{l_2} |\delta u(x_1, x_2, T)|^2 \, dx_1 dx_2 \leq M_9 \int_0^T \int_0^{l_2} \left[\left| \frac{\partial^2 (\delta v)}{\partial t^2} \right|^2 + \left| \frac{\partial^2 (\delta v)}{\partial x_2^2} \right|^2 + \left| \frac{\partial^4 (\delta v)}{\partial x_2^4} \right|^2 \right] dx_2 dt. \tag{34}$$

Thus, taking into account estimation (34), we get

$$R = \frac{1}{2} \int_0^{l_1} \int_0^{l_2} |\delta u(x_1, x_2, T)|^2 \, dx_1 dx_2 + \frac{\alpha}{2} \int_0^T \int_0^{l_2} |\delta v(x_2, t)|^2 \, dx_2 dt \leq M_{10} \left\| \delta v \right\|^2_{W_2^{4,2}((0,l_2) \times (0,T))}, \tag{35}$$

i.e. estimation (22) is proved.

Now calculate the differential of the functional (4).

Since $\delta u(x_1,x_2,t)$ is a generalized solution of problem (17)-(19), for $\forall\eta\in W_2^{2,1}(Q_T)$, $\eta(x_1,x_2,T)=0$,

$$\eta(0,x_2,t)=0,\ \eta(l_1,x_2,t)=0,\ \frac{\partial\eta(0,x_2,t)}{\partial x_1}=0,\ \frac{\partial\eta(l_1,x_2,t)}{\partial x_1}=0,$$

$$\eta(x_1,0,t)=0,\ \eta(x_1,l_2,t)=0,\ \frac{\partial\eta(x_1,0,t)}{\partial x_2}=0,\ \frac{\partial\eta(x_1,l_2,t)}{\partial x_2}=0,$$

we take

$$\int\limits_{Q_T}\left[-\frac{\partial\delta u}{\partial t}\frac{\partial\eta}{\partial t}+a^2\Delta(\delta u)\Delta\eta\right]dx_1dx_2dt=0 \tag{36}$$

and

$$\delta u(0,x_2,t)=\delta\upsilon(x_2,t). \tag{37}$$

Since $\psi(x_1,x_2,t)$ is the solution of the conjugate problem (13)-(15) from $W_2^{4,2}(Q_T)$ for any function $g\in W_2^{2,1}(Q_T)$ it holds the equality

$$\int\limits_{Q_T}\left[\frac{\partial^2\psi}{\partial t^2}+a^2\Delta^2\psi\right]g\,dx_1dx_2dt=0$$

or in particular, the equality

$$\int\limits_{Q_T}\left[-\frac{\partial\psi}{\partial t}\frac{\partial g}{\partial t}+a^2\Delta\psi\Delta g\right]dx_1dx_2dt+\int\limits_{\Omega}\frac{\partial\psi(x_1,x_2,T)}{\partial t}g(x_1,x_2,T)dx_1dx_2$$

$$+a^2\int\limits_0^T\int\limits_0^{l_2}\frac{\partial^2\psi(0,x_2,t)}{\partial x_1^2}\frac{\partial g(0,x_2,t)}{\partial x_1}dx_2dt+a^2\int\limits_0^T\int\limits_0^{l_2}\frac{\partial^2\psi(0,x_2,t)}{\partial x_2^2}\frac{\partial g(0,x_2,t)}{\partial x_1}dx_2dt=0, \tag{38}$$

for the arbitrary function $g(x_1,x_2,t)$ from $W_2^{2,1}(Q_T)$,

$$g(x_1,x_2,0)=0,\ g(0,x_2,t)=0,\ g(l_1,x_2,t)=0,\ \frac{\partial g(l_1,x_2,t)}{\partial x_1}=0,$$

$$g(x_1,0,t)=0,\ g(x_1,l_2,t)=0,\ \frac{\partial g(x_1,0,t)}{\partial x_2}=0,\ \frac{\partial g(x_1,l_2,t)}{\partial x_2}=0.$$

If in (36) instead of η we take ψ, and in (38) instead of g we take δu and from (36) subtract (38), we have

$$-\int\limits_{\Omega}\frac{\partial\psi(x_1,x_2,T)}{\partial t}\delta u(x_1,x_2,T)dx_1dx_2$$

$$-a^2\int\limits_0^T\int\limits_0^{l_2}\left[\frac{\partial^2\psi(0,x_2,t)}{\partial x_1^2}+\frac{\partial^2\psi(0,x_2,t)}{\partial x_2^2}\right]\frac{\partial\delta u(0,x_2,t)}{\partial x_1}dx_2dt=0.$$

Hence, taking into account the second condition from (14) and (37), we have

$$\int\limits_{\Omega}u(x_1,x_2,T)\delta u(x_1,x_2,T)dx_1dx_2$$

$$=a^2\int\limits_0^T\int\limits_0^{l_2}\left[\frac{\partial^2\psi(0,x_2,t)}{\partial x_1^2}+\frac{\partial^2\psi(0,x_2,t)}{\partial x_2^2}\right]\partial\upsilon(x_2,t)dx_2dt. \tag{39}$$

Thus, from (20) and (39), for the increment of the functional we get the following expression

$$\Delta J(v)=\int_0^T\int_0^{l_2}\left[\alpha v(x_2,t)+a^2\left(\frac{\partial\psi^2(0,x_2,t)}{\partial x_1^2}+\frac{\partial\psi^2(0,x_2,t)}{\partial x_2^2}\right)\right]\delta v(x_2,t)dx_2dt+R \qquad (40)$$

moreover for R it holds the estimation (22).

So, the functional is Frechet differentiable on H and its differential is determined by the expression (16).

By the continuity of embeddings $W_2^{4,2}((0,l_2)\times(0,T))\subset L_2((0,l_2)\times(0,T))$ and $W_2^{4,2}(Q_T)\subset W_2^{3,0}((0,l_2)\times(0,T))$, the mapping $v\to J'(v)$ determined by equality (16), continuously acts from U_{ad} to H^*, where H^* is a conjugated space to H. Theorem 2 is proved.

Theorem 3. *Let all the above conditions imposed on the data of problem (1)-(4) be fulfilled.*

Then for the optimality of the control $v_*(x_2,t)\in U_{ad}$ in problem (1)-(4) it is necessary and sufficient that the inequality

$$\int_0^T\int_0^{l_2}\left[\alpha v(x_2,t)+a^2\left(\frac{\partial^2\psi_*(0,x_2,t)}{\partial x_1^2}+\frac{\partial^2\psi_*(0,x_2,t)}{\partial x_2^2}\right)\right]$$
$$\times(v(x_2,t)-v_*(x_2,t))dx_2dt\geq 0,\ \forall v\in U_{ad}, \qquad (41)$$

be fulfilled, where $\psi_*(x_1,x_2,t)$ is the solution of problem (13)-(15) for $u=u_*(x_1,x_2,t)$, while $u_*(x_1,x_2,t)$ is the solution of problem (1)-(3) for $v=v_*(x_2,t)$.

Proof. The set U_{ad} is convex. Furthermore, we showed that the functional is continuously differentiable on U_{ad}. Then by virtue of the theorem from [14] p. 28, on the element it is necessary that the inequality

$$\left\langle J'(v_*),v-v_*\right\rangle\geq 0$$

for all $v\in U_{ad}$ be fulfilled.

Hence and (16) it follows the validity of inequality (41). Since the optimal control problem is linear-quadratic, and in formula (40) the remainder term $R\geq 0$, the obtained condition (40) is also sufficient for optimality of the control $v_*(x_2,t)$. Theorem 3 is proved.

References

1. Tikhonov A.N., Samarskiy A.A.: Mathematical physics equations. M.; "Nauka", 736p. (1972), Russian.
2. Komkov V.: Theory of optimal control of vibrations damping of prime elastic systems. M. "Mir", 160p. (1975), Russian.
3. Arman J.-L.P.: Applications of theory of optimal control of the systems with distributed parameters to the problems of optimization of constructions. M. "Mir", 144p. (1977), Russian.
4. Komkov V.: *Optimal control of vibrating thin plates.* SIAM Journal on Control, **8**, No 2, 273-304 (2006).
5. Sadek S., Adali S., Sloss J.M., Bruchjr J.C.: *Vibration damping of a thin plate by optimal open-and closed-loop control forces.* Journal of the Franklin Institute Pergamon Presa ple, **329**, No 2, 207-214 (1992). Printed in Great Britain.
6. Daqun Tong, Williams R.L. II, Agrawal S.K.: *Optimal shape control of composite thin plates with piezoelectric actuators.* Journal of Intelligent Material Systems and Structures, **9**, 458-467, (1998).
7. Deineka V.S.: *Optimal control of the dynamic state of a thin compound plate.* Cybernetics and Systems Analysis, **42**, No 4, 151-175 (2006).
8. Blanton M.L., Sadek I.S.: *Optimal active pointwise of thin plates via state-control parametrization.* International Journal of Systems Science, **25**, Issue 11 (1994).
9. Lomovtsev F.E., Yurchuk N.I.: *Cauchy problem for hyperbolic operator-differential equations of second order.* Differentsialniye Uravnenia, **12**, No 12, 2242-2250 (1976), Russian.
10. Kuliyev H.F., Seyfullayeva Kh.: *Optimal control problem for the equation of thin plate vibrations.* News of Baku University Series of physical-mathematical science. No 3, 64-73 (2013), Azerbaijan.
11. Beosv O.V., Il'in V.P, Nikolskii S.M.: Integral representation of functions and imbedding theorems. M., "Nauka", 482p. (1975), Russian.
12. Lions J.L.: Control of singular distributed systems. M., "Nauka", 367p. (1987), Russian.
13. Ladyjenskaya O.A.: Boundary value problems of mathematical physics. M., "Nauka", 408p. (1973), Russian.
14. Vasil'ev F.P.: Methods for solving extremal problems. M., "Nauka", 300p. (1981), Russian.