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An Optimal Conrtol Problem For The Equations Of Flexural-Torsional Oscillations Of A Bar

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Abstract. *It is known that a number of problems of mathematical physics, engineering, mechanics, etc are described by fourth order partial equations as equations of oscillations of a tuning fork, a bar, the equation of oscillations of rotating shafts, of the rolling of the vessel, plate oscillations equation and so on [1,2,3]. Therefore, investigation of optimal control problems in processes described by such equations is actual. In the paper [3], for linear equations of flexural- torsional oscillations of a bar with quadratic functional, when two control functions are only time -dependent, the Pondryagin maximum principle is proved. In the present paper, the Pontryagin maximum principle is proved in an optimal control problem for weakly nonlinear equations of flexural- torsional oscillations of a bar, when two control functions depend both on time and spatial coordinate . Note that for simplicity we take two control functionals, but their number may be taken arbitrarily.*

Keywords. optimal control · flexural- torsional oscillations · maximum principle · rod

1. Statement of optimal control problem and well-posedness of the solution of boundary value problem

Let's consider the oscillations of a bar described by a system of two differential equations in the domain $Q = \{0 < x < l,\ 0 < t < T\}$

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 y}{\partial x^2}\right) + \rho(x)A(x)\frac{\partial^2 y}{\partial t^2}$$

$$-\rho(x)A(x)e(x)\frac{\partial^2 \theta}{\partial t^2} = f_1(x,t,y,\theta,v_1,v_2), \tag{1}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)C_\omega(x)\frac{\partial^2 \theta}{\partial x^2}\right) - \frac{\partial^2}{\partial x^2}(G(x)C(x)\theta) - \rho(x)A(x)e(x)\frac{\partial^2 y}{\partial t^2}$$

$$+\rho(x)(I(x) + A(x)e^2(x))\frac{\partial^2 \theta}{\partial t^2} = f_2(x,t,y,\theta,v_1,v_2), \tag{2}$$

where $l > 0, T > 0$ are the given numbers, $y(x,t)$ is the lateral displacement of the bar, $\theta(x,t)$ is the turning angle of the cross-section of the bar, $E(x)$ is the Young modulus, $I(x)$ the is polar inertia moment of the cross section with respect to its gravity center, $\rho(x)$ is the density of the bar material, $A(x)$ is the area of cross section, $e(x)$ is the distance from the gravity center to the torsion center, $C_\omega(x)$ is the sectorial inertia moment of the cross section, $G(x)$ is the shear modulus, $C(x)$ is geometrical rigidity of free torsion, $E(x)C_\omega(x)$ is the rigidity of flexural torsion, $G(x)C(x)$ is the rigidity of free torsion,

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$v_1(x,t), v_2(x,t)$ are control functions, $f_i(x,t,y,\theta,v_1,v_2), i = 1,2$ are the functions given in domain $Q \times R^2 \times R^2$.

Let the bar be simply built-in . Then for $x = 0$ and $x = l$ we have the boundary conditions

$$y\,|_{x=0} = y\,|_{x=l} = 0,\ \frac{\partial y}{\partial x}\,|_{x=0} = \frac{\partial y}{\partial x}\,|_{x=l} = 0, \tag{3}$$

$$\theta\,|_{x=0} = \theta\,|_{x=l} = 0,\ \frac{\partial \theta}{\partial x}\,|_{x=0} = \frac{\partial \theta}{\partial x}\,|_{x=l} = 0. \tag{4}$$

Give the initial conditions

$$y\,|_{t=0} = \varphi_0(x)\ ,\ \frac{\partial y}{\partial t}\,|_{t=0} = \varphi_1(x), \tag{5}$$

$$\theta\,|_{t=0} = g_0(x)\ ,\ \frac{\partial \theta}{\partial t}\,|_{t=0} = g_1(x). \tag{6}$$

In place of a class of admissible controls U_d we take a set of vector functions $v(x,t) = (v_1(x,t), v_2(x,t))$ measurable in Q and for almost all $(x,t) \in Q$ taking the value from V where $V \subset R^2$ is an arbitrary set.

It is required to minimize the functional

$$J(v) = \iint\limits_Q f_0(x,t,y,\theta,v_1,v_2)dxdt \tag{7}$$

in the class U_d under restraints (1)-(6), where $(y(x,t;v), \theta(x,t;v))$ is the solution of problem (1)-(6), corresponding to the control $v(x,t) = (v_1(x,t), v_2(x,t))$, $f_0(x,t,y,\theta,v_1,v_2)$ is a function given in $Q \times R^2 \times R^2$. We call the optimal control problem (1)-(7) the problem (1)-(7).

Suppose that data of problem (1)-(7) satisfy the following conditions:

1) $E(x)$,$I(x)$,$\rho(x)$,$A(x)$,$e(x)$,$C_\omega(x)$,$G(x)$,$C(x)$ are measurable, bounded and positive functions on the interval $[0,l]$.

2) $\varphi_0, \varphi_1, g_0, g_1$ are the given functions, moreover $\varphi_0 \in \overset{\circ}{W}{}_2^2(0,l), g_0 \in \overset{\circ}{W}{}_2^2(0,l), \varphi_1, g_1 \in L_2(0,l)$.

3) the functions $f_i(x,t,y,\theta,v_1,v_2), i = 0,1,2,$ are continuous in $Q \times R^2 \times R^2$ and have continuous derivatives $\frac{\partial f_i}{\partial y}, \frac{\partial f_i}{\partial \theta}$, moreover $\frac{\partial f_i}{\partial y}, \frac{\partial f_i}{\partial \theta}, i = 1,2$ are bounded, and $\frac{\partial f_i}{\partial y}, \frac{\partial f_i}{\partial \theta}$, $i = 0,1,2$ satisfy the Lipschits condition with respect to (y, θ) .

For each control $v(x,t) \in U_d$ under the problem solution we will understand the vector- function $(y(x,t;v)\,, \theta(x,t;v)) \in W_2^{2,1}(Q)\times\, W_2^{2,1}(Q)$, whose components have the traces

$$y(\cdot,t), \frac{\partial y(\cdot,t)}{\partial t}, \theta(\cdot,t), \frac{\partial \theta(\cdot,t)}{\partial t} \in L_2[0,l] \text{ for all}$$

$$t \in [0,T], y(x,\cdot), \frac{\partial y(x,\cdot)}{\partial t}, \theta(x,\cdot), \frac{\partial \theta(x,\cdot)}{\partial x} \in L_2[0,T] \text{ for all}$$

$x \in [0,l]$ and satisfy conditions (3), (4),(5),(6) in the sense of equality of appropriate traces and integral identities

$$\iint\limits_Q \left(E(x)I(x)\frac{\partial^2 y}{\partial x^2}\frac{\partial^2 \eta_1}{\partial x^2} - \rho(x)A(x)\frac{\partial y}{\partial t}\frac{\partial \eta_1}{\partial t} + \rho(x)A(x)e(x)\frac{\partial \theta}{\partial t}\frac{\partial \eta_1}{\partial t} \right) dxdt$$

$$+\int_0^l \rho(x)A(x)\frac{\partial y(x,T)}{\partial t}\eta_1(x,T)dx - \int_0^l \rho(x)A(x)\varphi_1(x)\eta_1(x,0)dx$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial \theta(x,T)}{\partial t}\eta_1(x,T)dx + \int_0^l \rho(x)A(x)e(x)g_1(x)\eta_1(x,0)dx$$

$$= \iint\limits_Q f_1(x,t,y,\theta,v_1,v_2)\eta_1 dxdt, \tag{8}$$

$$\iint\limits_{Q} \left(E(x)C_w(x)\frac{\partial^2\theta}{\partial x^2}\frac{\partial^2\eta_2}{\partial x^2} - G(x)C(x)\theta\frac{\partial^2\eta_2}{\partial x^2} + \rho(x)A(x)e(x)\frac{\partial y}{\partial t}\frac{\partial\eta_2}{\partial t} \right.$$

$$\left. -\rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial\theta}{\partial t}\frac{\partial\eta_2}{\partial t}\right)dxdt$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial y(x,T)}{\partial t}\eta_2(x,T)dx + \int_0^l \rho(x)A(x)\varphi_1(x)\eta_2(x,0)dx$$

$$+\int_0^l \rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial\theta(x,T)}{\partial t}\eta_2(x,T)dx$$

$$-\int_0^l \rho(x)\left(I(x)+A(x)e^2(x)\right)\varphi_1(x)\eta_2(x,0)dx = \iint\limits_{Q} f_2(x,t,y,\theta,v_1,v_2)\eta_2 dxdt \tag{9}$$

for all

$$\eta_1 = \eta_1(x,t) \in W_2^{2,1}(Q),\ \eta_2 = \eta_2(x,t) \in W_2^{2,1}(Q),$$

moreover

$$\eta_1\,|_{x=0} = \eta_1\,|_{x=l} = 0,\ \frac{\partial\eta_1}{\partial x}\,|_{x=0} = \frac{\partial\eta_1}{\partial x}\,|_{x=l} = 0, \tag{10}$$

$$\eta_2\,|_{x=0} = \eta_2\,|_{x=l} = 0,\ \frac{\partial\eta_2}{\partial x}\,|_{x=0} = \frac{\partial\eta_2}{\partial x}\,|_{x=l} = 0. \tag{11}$$

Such a solution of boundary value problem (1)-(6) is called a generalized solution. We can show that [4,5,6,7], for each admissible control $v = v(x,t)$ problem (1)-(6) has a unique generalized solution $(y(x,t;v)\,, \theta(x,t;v))\ W_2^{2,1}(Q)\times\ W_2^{2,1}(Q)$, and such a solution has the property:

$$y, y_t, y_x, y_{xx} \in C\left([0,T], L_2(0,l)\right)\,,\ \theta, \theta_t, \theta_x, \theta_{xx} \in C\left([0,T], L_2(0,l)\right).$$

For $v_1^0(x,t)\,, v_2^0(x,t)\,, y_0(x,t), \theta_0(x,t)$ we introduce the conjugate problem:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2\psi_1}{\partial x^2}\right) + \rho(x)A(x)\frac{\partial^2\psi_1}{\partial t^2} - \rho(x)A(x)e(x)\frac{\partial^2\psi_2}{\partial t^2}$$

$$= \frac{\partial H(x,t,y_0(x,t),\theta_0(x,t),v_1^0(x,t),v_2^0(x,t),\psi_1(x,t),\psi_2(x,t))}{\partial y}, \tag{12}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)C_w(x)\frac{\partial^2\psi_1}{\partial x^2}\right) - G(x)C(x)\frac{\partial^2\psi_2}{\partial x^2} - \rho(x)A(x)e(x)\frac{\partial^2\psi_2}{\partial t^2}$$

$$+\rho(x)\left(I(x)+A(x)e^2(x)\right)\frac{\partial^2\psi_1}{\partial t^2}$$

$$= \frac{\partial H(x,t,y_0(x,t),\theta_0(x,t),v_1^0(x,t),v_2^0(x,t),\psi_1(x,t),\psi_2(x,t))}{\partial\theta}, \tag{13}$$

$$\psi_1|_{x=0} = \psi_1|_{x=l} = 0,\ \left.\frac{\partial\psi_1}{\partial x}\right|_{x=0} = \left.\frac{\partial\psi_1}{\partial x}\right|_{x=l} = 0, \tag{14}$$

$$\psi_2|_{x=0} = \psi_2|_{x=l} = 0,\ \left.\frac{\partial\psi_2}{\partial x}\right|_{x=0} = \left.\frac{\partial\psi_2}{\partial x}\right|_{x=l} = 0, \tag{15}$$

$$\psi_1|_{t=T} = 0,\ \left.\frac{\partial\psi_1}{\partial t}\right|_{t=T} = 0,\ \psi_2|_{t=T} = 0,\ \left.\frac{\partial\psi_2}{\partial t}\right|_{t=T} = 0. \tag{16}$$

Here

$$H(x,t,y,\theta,v_1,v_2,\psi_1,\psi_2) = \psi_1 f_1(x,t,y,\theta,v_1,v_2)$$

$$+\psi_2 f_2(x,t,y,\theta,v_1,v_2) - f_0(x,t,y,\theta,v_1,v_2) \tag{17}$$

is the Hamilton function for the problem (1)-(7).

2. Impulse variation and estimation of the increment of the solution

Here and in the sequel, we will use the scheme of the proof suggested in the paper [8].

Introduce the pulse variation of the control $v^0(x,t) = \left(v_1^0(x,t), v_2^0(x,t)\right)$:

$$v^\varepsilon(x,t) = \begin{cases} v_i, \ (x,t) \in \Pi^\varepsilon, \\ v_i^0(x,t), \ (x,t) \in Q \backslash \Pi^\varepsilon \end{cases} \qquad i = 1,2, \tag{18}$$

where $\Pi^\varepsilon = \{(x,t) | \sigma < x < \sigma + \varepsilon, \tau < t < \tau + \varepsilon\}$, moreover $\varepsilon > 0$ is so small that $\Pi^\varepsilon \subset Q$ and $(\sigma, \tau) \in Q$ is the Lebesgue point of all functions participating in the problem, $v = (v_1, v_2) \in V$ is an arbitrary fixed vector. Denote by $(y_\varepsilon(x,t), \theta_\varepsilon(x,t))$ the solution of problem (1)-(6) corresponding to the control $v^\varepsilon(x,t) = (v_1^\varepsilon(x,t), v_2^\varepsilon(x,t))$. Then $\delta y_\varepsilon = y_\varepsilon - y_0, \ \delta\theta_\varepsilon = \theta_\varepsilon - \theta_0$ will be a generalized solution of the following problem:

$$\frac{\partial^2}{\partial x^2}\left(E(x)I(x)\frac{\partial^2 \delta y_\varepsilon}{\partial x^2}\right) + \rho(x)A(x)\frac{\partial^2 \delta y_\varepsilon}{\partial t^2} - \rho(x)A(x)e(x)\frac{\partial^2 \delta\theta_\varepsilon}{\partial t^2}$$

$$= f_1(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon) - f_1(x,t,y_0,\theta_0,v_1^0,v_2^0), \tag{19}$$

$$\frac{\partial^2}{\partial x^2}\left(E(x)C_w(x)\frac{\partial^2 \delta\theta_\varepsilon}{\partial x^2}\right) - \frac{\partial^2}{\partial x^2}(G(x)C(x)\delta\theta_\varepsilon) - \rho(x)A(x)e(x)\frac{\partial^2 \delta y_\varepsilon}{\partial t^2}$$

$$+\rho(x)\left(I(x) + A(x)e^2(x)\right)\frac{\partial^2 \delta\theta_\varepsilon}{\partial t^2}$$

$$= f_2(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon) - f_2(x,t,y_0,\theta_0,v_1^0 v_2^0), \tag{20}$$

$$\delta y_\varepsilon|_{x=0} = \delta y_\varepsilon|_{x=l} = 0, \ \left.\frac{\partial \delta y_\varepsilon}{\partial x}\right|_{x=0} = \left.\frac{\partial \delta y_\varepsilon}{\partial x}\right|_{x=l} = 0, \tag{21}$$

$$\delta\theta_\varepsilon|_{x=0} = \delta\theta_\varepsilon|_{x=l} = 0, \ \left.\frac{\partial \delta\theta_\varepsilon}{\partial x}\right|_{x=0} = \left.\frac{\partial \delta\theta_\varepsilon}{\partial x}\right|_{x=l} = 0, \tag{22}$$

$$\delta y_\varepsilon|_{t=0} = 0, \ \left.\frac{\partial \delta y_\varepsilon}{\partial t}\right|_{t=0} = 0, \ \delta\theta_\varepsilon|_{t=0} = 0, \ \left.\frac{\partial \delta\theta_\varepsilon}{\partial t}\right|_{t=0} = 0. \tag{23}$$

Lemma. *Subject to conditions 1)-3), for the solution of problem (19)-(23) the following estimation is valid:*

$$\|\delta y_\varepsilon\|^2_{W_2^{2,1}(Q)} + \|\delta\theta_\varepsilon\|^2_{W_2^{2,1}(Q)} \le c\varepsilon^3, \tag{24}$$

Here and in the sequel, by c *we denote different constants that are independent of estimated values and admissible controls.*

Proof. As the functions $f_i(x,t,y,\theta,v_1,v_2), i = 1,2,$ with respect to arguments y, θ satisfy the Lipschits condition, it holds a uniqueness theorem for the solution of problem (19)-(23). But $\delta y_\varepsilon(x,t) = 0$, $\delta\theta_\varepsilon(x,t) = 0$ satisfies this system in domain $(0,l) \times (0,\tau)$, therefore boundary value problem (19)-(23) in domain $(0,l) \times (0,\tau)$ has only a trivial solution i.e.

$$\delta y_\varepsilon(x,t) = 0, \delta\theta_\varepsilon(x,t) = 0, \ (x,t) \in [0,l] \times [0,\tau]. \tag{25}$$

Now let $(x,t) \in [0,l] \times [\tau, \tau+\varepsilon]$. Let $\{\varphi_i(x)\}_{i=1}^\infty$ be a basis in the space $\overset{\circ}{W}{}_2^2(0,l)$ and orthonormalized in $L_2(0,l)$. We look for the approximate solutions of problem (19)-(23) in the form $\delta y_\varepsilon^N(x,t) = \sum\limits_{k=1}^N c_{1k}^N(t)\varphi_k(x)$, $\delta\theta_\varepsilon^N(x,t) = \sum\limits_{k=1}^N c_{2k}^N(t)\varphi_k(x)$ from the following relations :

$$\int_0^l E(x)I(x)\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\frac{d^2\varphi_p}{dx^2}dx + \int_0^l \rho(x)A(x)\frac{\partial^2 \delta y_\varepsilon^N}{\partial t^2}\varphi_p(x)dx$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial^2\delta\theta_\varepsilon^N}{\partial t^2}\varphi_p(x)dx$$

$$=\int_0^l \Big[f_1(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_1(x,t,y_0,\theta_0,v_1^0,v_2^0)\Big]$$

$$\times\varphi_p(x)dx,\ \ p=\overline{1,N}, \tag{26}$$

$$\int_0^l E(x)C_w(x)\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\frac{d^2\varphi_p}{dx^2}dx-\int_0^l G(x)C(x)\delta\theta_\varepsilon^N\frac{d^2\varphi_p}{dx^2}dx$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial^2\delta y_\varepsilon^N}{\partial t^2}\varphi_p(x)dx+\int_0^l \rho(x)(I(x)+A(x)e^2(x))\frac{\partial^2\delta\theta_\varepsilon^N}{\partial t^2}\varphi_p(x)dx$$

$$=\int_0^l \Big[f_2(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,t,y_0,\theta_0,v_1^0,v_2^0)\Big]$$

$$\varphi_p(x)dx,\ \ p=\overline{1,N}, \tag{27}$$

$$c_{1k}^N\,|_{t=0}\,=0,\ \frac{dc_{1k}^N(t)}{dt}\,|_{t=0}\,=0, \tag{28}$$

$$c_{2k}^N\,|_{t=0}\,=0,\ \frac{dc_{2k}^N(t)}{dt}\,|_{t=0}\,=0. \tag{29}$$

Equalities (26) and (27) are ordinary differential equations of second order for the functions $c_{1k}^N(t)$ and $c_{2k}^N(t)$ $k=\overline{1,N}$. These equations are solvable with respect to $\frac{d^2c_{1k}^N(t)}{dt^2}$ and $\frac{d^2c_{2k}^N(t)}{dt^2}$. The system (26),(27) under conditions (28), (29) is uniquely solvable, and $\frac{d^2c_{1k}^N(t)}{dt^2}$, $\frac{d^2c_{2k}^N(t)}{dt^2}\in L_2(0,T)$.

Multiply the both hand sides of (26) (27) by $\frac{dc_{1k}^N(t)}{dt}$ and $\frac{dc_{2k}^N(t)}{dt}$ respectively, and sum with respect to p from 1 to N .

Then

$$\int_0^l E(x)I(x)\frac{\partial^2\delta y_\varepsilon^N}{\partial x^2}\frac{\partial^3\delta y_\varepsilon^N}{\partial t\partial x^2}dx+\int_0^l \rho(x)A(x)\frac{\partial^2\delta y_\varepsilon^N}{\partial t^2}\frac{\partial\delta y_\varepsilon^N}{\partial t}dx$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial^2\delta\theta_\varepsilon^N}{\partial t^2}\frac{\partial\delta y_\varepsilon^N}{\partial t}dx$$

$$=\int_0^l \Big[f_1(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_1(x,t,y_0,\theta_0,v_1^0,v_2^0)\Big]\frac{\partial\delta y_\varepsilon^N}{\partial t}dx, \tag{30}$$

$$\int_0^l E(x)C_w(x)\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\frac{\partial^3\delta\theta_\varepsilon^N}{\partial t\partial x^2}dx-\int_0^l G(x)C(x)\delta\theta_\varepsilon^N\frac{\partial^2\delta\theta_\varepsilon^N}{\partial t\partial x^2}dx$$

$$-\int_0^l \rho(x)A(x)e(x)\frac{\partial^2\delta y_\varepsilon^N}{\partial t^2}\frac{\partial\delta\theta_\varepsilon^N}{\partial t}dx+\int_0^l \rho(x)(I(x)+A(x)e^2(x))\frac{\partial^2\delta\theta_\varepsilon^N}{\partial t^2}\frac{\partial\delta\theta_\varepsilon^N}{\partial t}dx$$

$$=\int_0^l \Big[f_2(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,t,y_0,\theta_0,v_1^0,v_2^0)\Big]\frac{\partial\delta\theta_\varepsilon^N}{\partial t}dx. \tag{31}$$

Suppose that $G(x), C(x)$ are independent of x.
Then from the previous equalities we have:

$$\frac{1}{2}\frac{d}{dt}\int_0^l \left[E(x)I(x)\left(\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right)^2 + \rho(x)A(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2 - \rho(x)A(x)e(x)\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial t}\right)^2\right]dx$$

$$= \int_0^l \left[f_1(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon) - f_1(x,t,y_0,\theta_0,v_1^0,v_2^0)\right]\frac{\partial \delta y_\varepsilon^N}{\partial t}dx,$$

$$\frac{1}{2}\frac{d}{dt}\int_0^l \left[E(x)C_w(x)\left(\frac{\partial^2 \delta \theta_\varepsilon^N}{\partial x^2}\right)^2 + GC\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial x}\right)^2\right.$$

$$\left. - \rho(x)A(x)e(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2 + \rho(x)\Big(I(x)+A(x)e^2(x)\Big)\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial t}\right)^2\right]dx$$

$$= \int_0^l \left[f_2(x,t,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon) - f_2(x,t,y_0,\theta_0,v_1^0,v_2^0)\right]\frac{\partial \delta \theta_\varepsilon^N}{\partial t}dx.$$

Hence, by integrating with respect to t from τ to t, we have:

$$\int_0^l \left[E(x)I(x)\left(\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right)^2 + \rho(x)A(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2 - \rho(x)A(x)e(x)\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial t}\right)^2\right]dx$$

$$= 2\int_\tau^l \left[\int_0^l f_1(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon) - f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\right]\frac{\partial \delta y_\varepsilon^N}{\partial t}dxds, \qquad (32)$$

$$\int_0^l \left[E(x)C_w(x)\left(\frac{\partial^2 \delta \theta_\varepsilon^N}{\partial x^2}\right)^2 + GC\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial x}\right)^2 - \rho(x)A(x)e(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2\right.$$

$$\left. + \rho(x)\Big(I(x)+A(x)e^2(x)\Big)\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial t}\right)^2\right]dx$$

$$= 2\int_\tau^l\int_0^l [f_2(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon) - f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\Big]\frac{\partial \delta \theta_\varepsilon^N}{\partial t}dxds. \qquad (33)$$

In the right hand sides of (32) and (33) make some transformations :

$$\int_0^l \left[E(x)I(x)\left(\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right)^2 + \rho(x)A(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2 - \rho(x)A(x)e(x)\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial t}\right)^2\right]dx$$

$$= 2\int_\tau^l\int_0^l \Big\{\Big[f_1(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon) - f_1(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)\Big]$$

$$+ \Big[f_1(x,s,y_0,v_1^\varepsilon,v_2^\varepsilon) - f_1(x,s,y_0,\theta_0,v_1^0,v_2^0)\Big]\Big\}\frac{\partial \delta y_\varepsilon^N}{\partial t}dxds, \qquad (34)$$

$$\int_0^l \left[E(x)C_w(x)\left(\frac{\partial^2 \delta \theta_\varepsilon^N}{\partial x^2}\right)^2 + GC\left(\frac{\partial \delta \theta_\varepsilon^N}{\partial x}\right)^2 - \rho(x)A(x)e(x)\left(\frac{\partial \delta y_\varepsilon^N}{\partial t}\right)^2\right.$$

$$+\rho(x)\left(I(x)+A(x)e^2(x)\right)\left(\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right)^2\Bigg]dx$$

$$=2\int_\tau^l\int_0^l\{\Big[f_2(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\Big]$$

$$+\Big[f_2(x,s,y_0,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\Big]\}\frac{\partial\delta\theta_\varepsilon^N}{\partial t}dxds. \tag{35}$$

Summing up the relations (34) and (35) , by some elementary transformations we have:

$$\int_0^l\Bigg[E(x)I(x)\left(\frac{\partial^2\delta y_\varepsilon^N}{\partial x^2}\right)^2+\rho(x)A(x)\left(\frac{\partial\delta y_\varepsilon^N}{\partial t}\right)^2+E(x)C_w(x)\left(\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\right)^2$$

$$+GC\left(\frac{\partial\delta\theta_\varepsilon^N}{\partial x}\right)^2+\rho(x)(I(x)+A(x)e^2(x))\left(\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right)^2$$

$$-\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y_\varepsilon^N}{\partial t}\right)^2+\left(\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right)^2\right)\Bigg]dx$$

$$=2\int_\tau^t\int_0^l\Big\{\Big[(f_1(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_1(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)$$

$$+((f_1(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)-f_1(x,s,y_0,\theta_0,v_1^0,v_2^0))\Big]\frac{\partial\delta y_\varepsilon^N}{\partial t}$$

$$+\Big[(f_2(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0))$$

$$+(f_2(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0))\Big]\frac{\partial\delta\theta_\varepsilon^N}{\partial t}dxds. \tag{36}$$

As the functions $f_i(x,t,xy,\theta,v_1,v_2)\ \ i=1,2$ with respect to arguments y,θ satisfy the Lipschits condition, hence by some elementary transformations we have:

$$\int_0^l\Bigg[E(x)I(x)\left|\frac{\partial^2\delta y_\varepsilon^N}{\partial x^2}\right|^2dx+\int_0^l(\rho(x)A(x)-\rho(x)A(x)e(x))\left|\frac{\partial\delta y_\varepsilon^N}{\partial t}\right|^2dx$$

$$+\int_0^l E(x)C_w(x)\left|\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\right|^2dx+\int_0^l(\rho(x)(I(x)+A(x)e^2(x)-A(x)e(x)))\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right|^2dx$$

$$+\int_0^l GC\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial x}\right|^2dx\le c\int_\tau^t\int_0^l\Big[\left|\delta y_\varepsilon^N\right|+\left|\delta\theta_\varepsilon^N\right|\Big]\left|\frac{\partial\delta y_\varepsilon^N}{\partial t}\right|+\Big[\left|\delta y_\varepsilon^N\right|+\left|\delta\theta_\varepsilon^N\right|\Big]\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right|dxds$$

$$+\int_\tau^t\int_0^l\Big[\left|f_1(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)-f_1(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|\left|\frac{\partial\delta y_\varepsilon^N}{\partial t}\right|$$

$$+\left|f_2(x,s,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right|\Big]dxds. \tag{37}$$

Suppose that $0 < \alpha_0 \leq e(x) < 1$, $I(x) + A(x)e(x)(e(x) - 1) \geq \alpha_1 > 0$, $\forall x \in [0, l]$ where α_0, α_1 are some numbers. As $E(x), I(x), A(x), C_w(x), \rho(x)$ are positive functions on the interval $[0, l]$, by the equivalence of the norm in $\overset{\circ}{W}{}_2^2(0, l)$ and strengthening the right side of the last inequality, we have:

$$\int_0^l \left[\left|\delta y_\varepsilon^N\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial x^2}\right|^2 + \left|\delta\theta_\varepsilon^N\right|^2 \right.$$

$$\left. + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta\theta_\varepsilon^N}{\partial x^2}\right|^2 \right] dx \leq c \int_\tau^t \int_0^l \left[\left|\delta y_\varepsilon^N\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial t}\right|^2 \right.$$

$$\left. + \left|\frac{\partial \delta y_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right|^2 + \left|\delta\theta_\varepsilon^N\right|^2 + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta\theta_\varepsilon^N}{\partial x^2}\right|^2 \right] dxds$$

$$+ c \int_\tau^t \left(\int_0^l \left[\left|\delta y_\varepsilon^N\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right|^2 + \left|\delta\theta_\varepsilon^N\right|^2 \right.\right.$$

$$\left.\left. + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta\theta_\varepsilon^N}{\partial x^2}\right|^2 \right) dx \right)^{1/2}$$

$$\times \left[\left(\int_0^l \left| f_1(x, s, y_0, \theta_0, v_1^\varepsilon, v_2^\varepsilon) - f_1(x, s, y_0, \theta_0, v_1^0, v_2^0) \right|^2 dx \right)^{1/2} \right.$$

$$\left. + \int_0^l \left(\int_0^l \left| f_2(x, s, y_0, \theta_0, v_1^\varepsilon, v_2^\varepsilon) - f_2(x, s, y_0, \theta_0, v_1^0, v_2^0) \right|^2 dx \right)^{1/2} \right] ds. \tag{38}$$

Accept the following denotation:

$$A^N(t) = \int_0^l \left[\left|\delta y_\varepsilon^N\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta y_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta y_\varepsilon^N}{\partial x^2}\right|^2 + \left|\delta\theta_\varepsilon^N\right|^2 \right.$$

$$\left. + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial t}\right|^2 + \left|\frac{\partial \delta\theta_\varepsilon^N}{\partial x}\right|^2 + \left|\frac{\partial^2 \delta\theta_\varepsilon^N}{\partial x^2}\right|^2 \right] dx, \tag{39}$$

$$g(t) = c \left[\left(\int_0^l \left| f_1(x, t, y_0, \theta_0, v_1^\varepsilon, v_2^\varepsilon) - f_1(x, t, y_0, \theta_0, v_1^0, v_2^0) \right|^2 dx \right)^{1/2} \right.$$

$$\left. + \left(\int_0^l \left| f_2(x, t, y_0, \theta_0, v_1^\varepsilon, v_2^\varepsilon) - f_2(x, t, y_0, \theta_0, v_1^0, v_2^0) \right|^2 dx \right)^{1/2} \right]$$

Then from (38) we get:

$$A^N(t) \leq c \int_\tau^t A^N(s) ds + c \int_\tau^t \left(A^N(s) \right)^{1/2} g(s) ds. \tag{40}$$

Let

$$a(t) = c\int_{\tau}^{t} A^N(s)ds + c\int_{\tau}^{t} g(s)\sqrt{A^N(s)}ds. \tag{41}$$

Then $a(\tau) = 0$ and for almost for all $t \in [\tau, \tau + \varepsilon]$

$$\dot{a}(t) = cA^N(t) + g(t)\sqrt{A^N(t)} \le ca(t) + g(t)\sqrt{a(t)}.$$

Multiply the both hand sides of this inequality by $a(t)^{-1/2}\exp\left(\frac{-c(t-\tau)}{2}\right)$. Then we get:

$$\frac{d}{dt}\left(2a^{1/2}(t)\exp\left(\frac{-c(t-\tau)}{2}\right)\right) \le g(t)\exp\left(\frac{-c(t-\tau)}{2}\right). \tag{42}$$

Integrate this inequality from τ to t and transform the obtained result:

$$2a^{1/2}(t)\exp\left(\frac{-c(t-\tau)}{2}\right) \le \int_{\tau}^{t} g(s)\exp\left(\frac{-c(s-\tau)}{2}\right)ds$$

or

$$a(t) \le \left(\frac{1}{2}\int_{\tau}^{t} g(s)\exp\left(\frac{-c(t-\tau)}{2} - \frac{c(s-\tau)}{2}\right)ds\right)^2 \le \left(\frac{1}{2}\int_{\tau}^{t} g(s)\exp\frac{c(t-s)}{2}ds\right)^2$$

Hence we have:

$$a(t) \le \left(\frac{1}{2}\int_{\tau}^{t} g(s)\exp\frac{c(t-s)}{2}ds\right)^2$$

$$\le c\left(\int_{\tau}^{t}\left[\left(\int_0^l \left|f_1(x,s,y_0,\theta_0,v_1^{\varepsilon},v_2^{\varepsilon}) - f_1(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|^2 dx\right)^{1/2}\right.\right.$$

$$\left.\left.+\left(\int_0^l \left|f_2(x,s,y_0,\theta_0,v_1^{\varepsilon},v_2^{\varepsilon}) - f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|^2 dx\right)^{1/2}\right]ds\right)^2$$

$$\le c\left(\int_{\tau}^{\tau+\varepsilon}\left[\left(\int_0^l \left|f_1(x,s,y_0,\theta_0,v_1^{\varepsilon},v_2^{\varepsilon}) - f_1(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|^2 dx\right)^{1/2}\right.\right.$$

$$\left.\left.+\left(\int_0^l \left|f_2(x,s,y_0,\theta_0,v_1^{\varepsilon},v_2^{\varepsilon}) - f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|^2 dx\right)^{1/2}\right]dt\right)^2. \tag{43}$$

By the definition of impulse variation of controls $v_1^{\varepsilon}, v_2^{\varepsilon}$, we get:

$$a(t) \le c\left(\int_{t}^{\tau+\varepsilon}\left[\left(\int_{\sigma}^{\sigma+\varepsilon} \left|f_1(x,s,y_0,\theta_0,v_1,v_2) - f_1(x,s,y_0,\theta_0,v_1^0(x,t),v_2^0(x,t)\right|^2 dx\right)^{1/2}\right.\right.$$

$$\left.\left.+\int_{\sigma}^{\sigma+\varepsilon} \left|f_2(x,s,y_0,\theta_0,v_1,v_2) - f_2(x,s,y_0,\theta_0,v_1^0(x,t),v_2^0(x,t)\right|^2 dx\right]dt\right)^2. \tag{44}$$

As (σ, τ) is the Lebesgue point of all functions participating in the problem, and $A^N(t) \leq a(t)$, hence we have:

$$A^N(t) \leq c\varepsilon^2, \ \ \forall t \in [\tau, \tau+\varepsilon], \ \varepsilon > 0. \tag{45}$$

Now we get estimation (24) in the rectangle $(x,t) \in [0,l] \times [\tau+\varepsilon, T]$.

From (19)-(23) as in the previous case we have:

$$\int_0^l \left[E(x)I(x) \left(\frac{\partial \delta y_\varepsilon^N(x,t)}{\partial t^2} \right)^2 + \rho(x)A(x) \left(\frac{\partial \delta y_\varepsilon^N(x,t)}{\partial t} \right)^2 - \rho(x)A(x)e(x) \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial t} \right)^2 \right] dx$$

$$= \int_0^l \left[E(x)I(x) \left(\frac{\partial^2 \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial x^2} \right)^2 + \rho(x)A(x) \left(\frac{\partial \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial t} \right)^2 - \rho(x)A(x)e(x) \left(\frac{\partial \delta \theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial t} \right)^2 \right] dx$$

$$+ 2\int_{\tau+\varepsilon}^t \int_0^l \left[f_1(x,s,y_0+\delta y_\varepsilon^N, \theta_0+\delta\theta_\varepsilon^N, v_1^0, v_2^0) - f_1(x,s,y_0,\theta_0,v_1^0,v_2^0) \right] \frac{\partial \delta y_\varepsilon^N}{\partial t} dxds, \tag{46}$$

$$\int_0^l \left[E(x)C_w(x) \left(\frac{\partial^2 \delta \theta_\varepsilon^N(x,t)}{\partial x^2} \right)^2 + GC \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial x} \right)^2 -\rho(x)A(x)e(x) \left(\frac{\partial \delta y_\varepsilon^N(x,t)}{\partial t} \right)^2 + \rho(x)(I(x)+A\ (x)e^2(x)) \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial t} \right)^2 \right] dx$$

$$= \int_0^l \left[E(x)C_w(x) \left(\frac{\partial^2 \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial x^2} \right)^2 + GC \left(\frac{\partial \delta \theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial x} \right)^2 -\rho(x)A(x)e(x) \left(\frac{\partial \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial t} \right)^2 + \rho(x)(I(x)+A\ (x)e^2(x)) \left(\frac{\partial \delta \theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial t} \right)^2 \right] dx$$

$$+2\int_\tau^t \int_0^l \left[f_2(x,s,y_0+\delta y_\varepsilon^N, \theta_0+\delta\theta_\varepsilon^N, v_1^0, v_2^0) - f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)) \right] \frac{\partial \delta \theta_\varepsilon^N}{\partial t} dxds. \tag{47}$$

Summing up relations (46) and (47), we get:

$$\int_0^l \left[E(x)I(x) \left(\frac{\partial^2 \delta y_\varepsilon^N(x,t)}{\partial x^2} \right)^2 + \rho(x)A(x) \left(\frac{\partial \delta y_\varepsilon^N(x,t)}{\partial t} \right)^2 +\rho(x)(I(x)+A(x)e^2(x)) \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial t} \right)^2 + GC \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial x} \right)^2 +E(x)C_w(x) \left(\frac{\partial^2 \delta \theta_\varepsilon^N(x,t)}{\partial x^2} \right)^2 - \rho(x)A(x)e(x) \left(\left(\frac{\partial \delta y_\varepsilon^N(x,t)}{\partial t} \right)^2 + \left(\frac{\partial \delta \theta_\varepsilon^N(x,t)}{\partial t} \right)^2 \right) \right] dx$$

$$= \int_0^l \left[E(x)I(x) \left(\frac{\partial^2 \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial x^2} \right)^2 + \rho(x)A(x) \left(\frac{\partial \delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial t} \right)^2 \right.$$

$$+\rho(x)(I(x)+A(x)e^2(x))\left(\frac{\partial\delta\theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial t}\right)^2+GC\left(\frac{\partial\delta\theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial x}\right)^2$$

$$+E(x)C_w(x)\left(\frac{\partial^2\delta\theta_\varepsilon^N(x,\tau+\varepsilon)}{\partial x^2}\right)^2-\rho(x)A(x)e(x)\left(\left(\frac{\partial\delta y_\varepsilon^N(x,\tau+\varepsilon)}{\partial t}\right)^2\right.$$

$$\left.\left.+\left(\frac{\partial\delta\theta_\varepsilon^N(x,t)}{\partial t}\right)^2\right)\right]dx+2\int\limits_{\tau+\varepsilon}^{t}\int\limits_0^l\left[(f_1(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^0,v_2^0)\right.$$

$$\left.\times\frac{\partial\delta y_\varepsilon^N}{\partial t}\right]+\left[f_2(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^0,v_2^0)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0))\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right]dxds. \tag{48}$$

As the functions $f_1(x,t,y,\theta,v_1,v_2)$, $f_2(x,t,y,\theta,v_1,v_2)$ with respect to arguments y,θ satisfy the Lipschits condition, we have

$$\left|f_1(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^0,v_2^0)-f_1(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|\le L\left\lfloor\left|\delta y_\varepsilon^N\right|+\left|\delta\theta_\varepsilon^N\right|\right\rfloor,$$

$$\left|f_2(x,s,y_0+\delta y_\varepsilon^N,\theta_0+\delta\theta_\varepsilon^N,v_1^0,v_2^0)-f_2(x,s,y_0,\theta_0,v_1^0,v_2^0)\right|$$

$$\le L\left\lfloor\left|\delta y_\varepsilon^N\right|+\left|\delta\theta_\varepsilon^N\right|\right\rfloor, \tag{49}$$

If we have in view inequality (49) in (48), taking into account estimation (45) at the point $t=\tau+\varepsilon$, by means of elementary transformations we get:

$$\int\limits_0^l\left[\left|\delta y_\varepsilon^N\right|^2+\left|\frac{\partial\delta y_\varepsilon^N}{\partial t}\right|^2+\left|\frac{\partial\delta y_\varepsilon^N}{\partial x}\right|^2+\left|\frac{\partial^2\delta y_\varepsilon^N}{\partial x^2}\right|^2+\left|\delta\theta_\varepsilon^N\right|^2\right.$$

$$\left.+\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right|^2+\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial x}\right|^2+\left|\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\right|^2\right]dx\le c\varepsilon^3+\int\limits_{\tau+\varepsilon}^{t}\left(\int\limits_0^l\left[\left|\delta y_\varepsilon^N\right|^2+\left|\frac{\partial\delta y_\varepsilon^N}{\partial t}\right|^2\right.\right.$$

$$\left.\left.+\left|\frac{\partial\delta y_\varepsilon^N}{\partial x}\right|^2+\left|\frac{\partial^2\delta y_\varepsilon^N}{\partial x^2}\right|^2+\left|\delta\theta_\varepsilon^N\right|^2+\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial t}\right|^2+\left|\frac{\partial\delta\theta_\varepsilon^N}{\partial x}\right|^2+\left|\frac{\partial^2\delta\theta_\varepsilon^N}{\partial x^2}\right|^2\right]dx\right)ds.$$

Here taking into account the denotation of the function $A^N(t)$,we have:

$$A^N(t)\le c\varepsilon^3+c\int\limits_{\tau+\varepsilon}^{t}A^N(s)ds,\forall t\in[\tau,\tau+\varepsilon],\ \varepsilon>0.$$

Hence, applying the Gronwall lemma, we get

$$A^N(t)\le c\cdot\varepsilon^3,0<c<\infty,t\in[\tau,+\varepsilon T]. \tag{50}$$

Thus, taking into account that the norm in the Banach space is weakly lower semicontinuous, from (25), (45) and (50) we get that estimation (24) is valid.

3. Proof of the maximum principle

Under the generalized solution of the conjugate problem (12) -(16) we understand the pair of functions $(\psi_1(x,t),\psi_2(x,t))\in W_2^{2,1}(Q)\times W_2^{2,1}(Q)$, that satisfies the integral identities:

$$\iint\limits_Q\left[E(x)I(x)\frac{\partial^2\psi_1}{\partial x^2}\frac{\partial^2 g_1}{\partial x^2}-\rho(x)A(x)\frac{\partial\psi_1}{\partial t}\frac{\partial g_1}{\partial t}+\rho(x)A(x)e(x)\frac{\partial\psi_2}{\partial t}\frac{\partial g_1}{\partial t}\right.$$

$$\left.-\frac{\partial H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial y}g_1\right]dxdt=0, \tag{51}$$

$$\iint\limits_Q \left[E(x)C_w(x)\frac{\partial^2\psi_2}{\partial x^2}\frac{\partial^2 g_2}{\partial x^2} - GC\frac{\partial^2\psi_1}{\partial x^2}g_2 \right.$$

$$+\rho(x)A(x)e(x)\frac{\partial\psi_2}{\partial t}\frac{\partial g_2}{\partial t} - \rho(x)(I(x)+A(x)e^2(x))\frac{\partial\psi_1}{\partial x^2}\frac{\partial g_2}{\partial x^2}$$

$$\left. - \frac{\partial H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial y}g_2 \right] dxdt = 0, \tag{52}$$

for any functions $g_1 \in W_2^{2,1}(Q)$, $g_1 \in W_2^{2,1}(Q)$,

$$g_1|_{t=0} = 0, \qquad g_2|_{t=0} = 0,$$

$$g_1|_{x=0} = g_1|_{x=l} = 0, \qquad g_2|_{x=0} = g_2|_{x=l} = 0,$$

$$\left.\frac{\partial g_1}{\partial x}\right|_{x=0} = \left.\frac{\partial g_1}{\partial x}\right|_{x=l} = 0, \quad \left.\frac{\partial g_2}{\partial x}\right|_{x=0} = \left.\frac{\partial g_2}{\partial x}\right|_{x=l} = 0.$$

Let the vector- functions $\delta y_\varepsilon, \delta\theta_\varepsilon$ be generalized solutions of problems (19)-(23) i.e. the following identities be satisfied

$$\iint\limits_Q \left[\left(E(x)I(x)\frac{\partial^2\delta y_\varepsilon}{\partial x^2}\frac{\partial^2\eta_1}{\partial x^2} - \rho(x)A(x)\frac{\partial\delta y_\varepsilon}{\partial t}\frac{\partial\eta_1}{\partial t} \right.\right.$$

$$\left. + \rho(x)A(x)e(x)\frac{\partial\delta\theta_\varepsilon}{\partial t}\frac{\partial\eta_1}{\partial t} \right) - (f_1(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)$$

$$\left. - f_1(x,t,y_0,\theta_0,v_1^0,v_2^0))\eta_1 \right] dxdt = 0 \tag{53}$$

$$\iint\limits_Q \left[E(x)C_w(x)\frac{\partial^2\delta\theta_\varepsilon}{\partial x^2}\frac{\partial^2\eta_2}{\partial x^2} - GC\frac{\partial^2\delta\theta_\varepsilon}{\partial x^2}\eta_2 + \rho(x)A(x)e(x)\ \frac{\partial\delta y_\varepsilon}{\partial t}\frac{\partial\eta_2}{\partial t} \right.$$

$$-\rho(x)(I(x)+A(x)e^2(x))\frac{\partial\delta\theta_\varepsilon}{\partial t}\frac{\partial\eta_2}{\partial t}$$

$$\left. -(f_2(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon) - f_2(x,t,y_0,\theta_0,v_1^0,v_2^0))\eta_2 \right] dxdt = 0 \tag{54}$$

for any functions $\eta_1 \in W_2^{2,1}(Q)$, $\eta_2 \in W_2^{2,1}(Q)$,

$$\eta_1|_{t=T} = 0, \qquad \eta_2|_{t=T} = 0,$$

$$\eta_1|_{x=0} = \eta_1|_{x=l} = 0, \qquad \eta_2|_{x=0} = \eta_2|_{x=l} = 0,$$

$$\left.\frac{\partial\eta_1}{\partial x}\right|_{x=0} = \left.\frac{\partial\eta_1}{\partial x}\right|_{x=l} = 0, \quad \left.\frac{\partial\eta_2}{\partial x}\right|_{x=0} = \left.\frac{\partial\eta_2}{\partial x}\right|_{x=l} = 0. \tag{55}$$

Calculate the increment of the functional (7):

$$\Delta J(v_0) = J(v_\varepsilon) - J(v_0) = \iint\limits_Q \left[f_0(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon) - f_0(x,t,y_0,\theta_0,v_1^0,v_2^0) \right] dxdt.$$

In identities (51) and (52) instead of g_1 and g_2 we take δy_ε and $\delta\theta_\varepsilon$, in identities (53) and (54) instead of η_1 and η_2 we take $\psi_1$and ψ_2, subtract the obtained relations and add the results to the increment of the functional. Then after some transformations the increment of the functional has the form:

$$\Delta J(v_0) = \iint\limits_Q (f_0(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon) - f_0(x,t,y_0,\theta_0,v_1^0,v_2^0))dxdt$$

$$+\iint\limits_Q \left(\frac{\partial H}{\partial y}(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)\delta y_\varepsilon + \frac{\partial H}{\partial\theta}(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)\delta\theta_\varepsilon \right) dxdt$$

$$-\iint\limits_Q (f_1(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon,\psi_1,\psi_2)-f_1(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2))\psi_1 dxdt$$
$$-\iint\limits_Q (f_2(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon,\psi_1,\psi_2)-f_2(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2))\psi_2 dxdt. \quad (56)$$

Taking into account the expansion

$$f_i(x,t,y_\varepsilon,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)-f_i(x,t,y_0,\theta_0,v_1^0,v_2^0)$$
$$=\frac{\partial f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}\delta y_\varepsilon+\frac{\partial f_i(x,t,y_0,\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}\delta\theta_\varepsilon+\omega^i(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon),\ i=0,1,2,$$

we write the formulas of increments of functional (56) in the from

$$\Delta J(v_0)=\iint\limits_Q \Big[H(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon,\psi_1,\psi_2)-H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)\Big]dxdt$$
$$+\eta(\varepsilon)=-\iint\limits_Q \Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)dxdt+\eta(\varepsilon), \quad (57)$$

where

$$\Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)=H(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon,\psi_1,\psi_2)$$
$$-H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2),$$
$$\eta(\varepsilon)=\iint\limits_Q \left[\sum_{i=1}^{2}\psi_i(x,t)\omega^i(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon)+\omega^0(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon)\right.$$
$$+\frac{\partial\Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial y}\delta y_\varepsilon$$
$$\left.=\frac{\partial\Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial \theta}\delta\theta_\varepsilon\right]dxdt. \quad (58)$$

Here, taking into account the form $\omega^i(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon)$ the conditions imposed on $f_i(x,t,y_\varepsilon,\theta,v_1,v_2)$, boundedness of the functions $\psi_1(x,t),\psi_2(x,t)$ in Q, the mean value theorem and estimation (24), we have:

$$\left|\iint\limits_Q\sum_{i=1}^{2}\psi_i(x,t)\omega^i(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon)dxdt\right|\le\iint\limits_Q\sum_{i=1}^{2}|\psi_i(x,t)|$$
$$\times\Big|f_i(x,t,y_0+\delta y_\varepsilon,\theta_0+\delta\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)-f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)$$
$$-\frac{\partial f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}\delta y_\varepsilon-\frac{\partial f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}\delta\theta_\varepsilon\Big|dxdt$$
$$\le c\sum_{i=1}^{2}\iint\limits_Q\sup_{0\le\lambda\le1}\left[\left|\frac{\partial f_i(x,t,y_0+\lambda\delta y_\varepsilon,\theta_0+\lambda\delta\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}\right.\right.$$
$$\left.-\frac{\partial f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}\right||\delta y_\varepsilon|$$
$$\left.+\left|\frac{\partial f_i(x,t,y_0+\lambda\delta y_\varepsilon,\theta_0+\lambda\delta\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}-\frac{\partial f_i(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}\right||\delta\theta_\varepsilon|\right]dxdt$$
$$\le c\iint\limits_Q\Big[|\delta y_\varepsilon|^2+|\delta\theta_\varepsilon|^2\Big]dxdt=c\varepsilon^3,\ \ 0\le\lambda\le1. \quad (59)$$

Similarly

$$\left|\iint_Q w^0(y_0,\theta_0,\delta y_\varepsilon,\delta\theta_\varepsilon)dxdt\right| \le c\iint_Q \sup_{0\le\lambda\le 1}$$

$$\times\left[\left|\frac{\partial f_0(x,t,y_0+\lambda\delta y_\varepsilon,\theta_0+\lambda\delta\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}-\frac{\partial f_0(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial y}\right||\delta y_\varepsilon|\right.$$

$$\left.+\left|\frac{\partial f_0(x,t,y_0+\lambda\delta y_\varepsilon,\theta_0+\lambda\delta\theta_\varepsilon,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}-\frac{\partial f_0(x,t,y_0,\theta_0,v_1^\varepsilon,v_2^\varepsilon)}{\partial \theta}\right||\delta \theta_\varepsilon|\right]dxdt$$

$$\le c\iint_Q\left(|\delta y_\varepsilon|^2+|\delta\theta_\varepsilon|^2\right)dxdt = c\varepsilon^2. \tag{60}$$

Taking into account estimation (24) and definition of impulse variation of functions $v_1^0(x,t)$,$v_2^0(x,t)$, we have:

$$\iint_Q\left[\frac{\partial\Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial y}\ \delta y_\varepsilon\right.$$

$$\left.\frac{\partial\Delta_{v_\varepsilon}H(x,t,y_0,\theta_0,v_1^0,v_2^0,\psi_1,\psi_2)}{\partial \theta}\delta\theta_\varepsilon\right]dxdt = o(\varepsilon^2). \tag{61}$$

Therefore, from equality (57), taking into account relations (58),(59),(60),(61), for the first variation of the functional $J(v)$ we get the final expression:

$$\lim_{\varepsilon\to 0}\frac{\Delta J(v)}{\varepsilon^2} = -[H(\sigma,\tau,y_0(\sigma,\tau),\theta_0(\sigma,\tau),v_1,v_2,\psi_1(\sigma,\tau),\psi_2(\sigma,\tau))$$

$$\left.-H(\sigma,\tau,y_0(\sigma,\tau),\theta_0(\sigma,\tau),v_1^0(\sigma,\tau),\ v_2^0(\sigma,\tau),\psi_1(\sigma,\tau),\psi_2(\sigma,\tau))\right] = \delta J(v_1^0,v_2^0).$$

If $v_1^0(x,t)$,$v_2^0(x,t)$ is the optimal control in problem (1)-(7), then $\delta J(v_1^0,v_2^0)\ge 0$. Hence we get:

$$H(\sigma,\tau,y_0(\sigma,\tau),\theta_0(\sigma,\tau),v_1,v_2,\psi_1(\sigma,\tau),\psi_2(\sigma,\tau))$$

$$\le H(\sigma,\tau,y_0(\sigma,\tau),\theta_0(\sigma,\tau),v_1^0(\sigma,\tau),v_2^0(\sigma,\tau),\psi_1(\sigma,\tau),\psi_2(\sigma,\tau))$$

$$(\sigma,\tau)\in Q, \forall v=(v_1,v_2)\in V$$

or

$$\max_{v=(v_1,v_2)\in V} H(x,t,y_0(x,t),\theta_0(x,t),v_1,v_2,\psi_1(x,t),\psi_2(x,t))$$

$$= H(x,t,y_0(x,t),\theta_0(x,t),v_1^0(x,t),v_2^0(x,t),\psi_1(x,t),\psi_2(x,t)) \tag{62}$$

almost for all

$$(x,t)\in Q.$$

Thus, the following theorem is proved.

Theorem (maximum principle). *Let the data of problem (1)-(7) satisfy the above conditions 1) - 3). Then for optimality of the control* $(v_1^0(x,t),v_2^0(x,t))$ *in the problem (1)-(7) condition (62) should be fulfilled, where* $y_0(x,t),\theta_0(x,t)$ *is the solution of problem (1)-(6) for* $v(x,t)=v_0(x,t)$ *and* $\psi_1(x,t),\psi_2(x,t)$ *of the conjugate problem (12)-(16).*

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