Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 3-12 (2015).
Series of Physical-Technical and Mathematical Sciences.

SOLVABILITY OF BOUNDARY VALUE PROBLEMS FOR SECOND ORDER ELLIPTIC DIFFERENTIAL-OPERATOR EQUATIONS WITH A SPECTRAL PARAMETER IN THE EQUATION AND BOUNDARY CONDITIONS

Bahram A. Aliev · Oktay R. Amiraslanly

Received: 16.01.2015 / Accepted: 28.10.2015

Abstract. *In the Hilbert space H we study the solvability of boundary value problem for a second order homogeneous elliptic differential-operator equation in the case when one and the same spectral parameter linearly enters into the equation and boundary conditions, moreover the spectral parameter in the boundary conditions participates simultaneously both in front of the sought for function itself and in front of the derivative taken respectively at the boundary points 0 and 1. Sufficient conditions for coercive solvability of the considered boundary value problem in the space $L_p((0,1);H)$ $(p>1)$ were found.*

Keywords. spectral parameter · elliptic differential-operator equations · strongly positive operator · interpolation spaces

Mathematics Subject Classification (2010): 35J25 · 47A10 · 58J40

1 Introduction

Second order boundary value problems for elliptic differential-operator equations when one and the same spectral parameter enters into the equation and the boundary conditions, were studied in different aspects in many papers (see e.i. [1]-[3], [5]-[7], [10]-[11]).

In all these papers [1]-[3], [5]-[7], [10]-[11], the spectral parameter in the equation and boundary conditions has the same order and the spectrum in the boundary conditions is either in front of the sought-for function or in the front of the first derivative of the sought-for function taken respectively at the boundary points 0 and 1.

In the Hilbert space H we study the solvability of boundary value problem for a second order homogeneous elliptic differential-operator equation in the case when one and the same spectral parameter linearly enters into the equation and boundary conditions, moreover the spectral parameter in the boundary conditions participates simultaneously both in front of the sought for function itself and in front of the derivative taken respectively at the boundary points 0 and 1.

Thus, in the present paper, in a separable Hilbert space H, we consider the following boundary value problem.

$$\lambda u(x) - u''(x) + Au(x) = 0, \quad x \in (0,1), \tag{1.1}$$

B.A. Aliev
E-mail: aliyevbakhram@yandex.ru
Institute of Mathematics and Mechanics, Baku, Azerbaijan

O.R. Amiraslanly
E-mail: amiraslanlioqtaysdu@gmail.com
Sumgayit State University 1, Baku str, 43 ward., AZ 5008, Sumgayit, Azerbaijan

$$(\alpha_1 + \lambda\beta_1)\, u'(0) + (\alpha_2 + \lambda\beta_2)\, u(0) = f_1,$$
$$(\sigma_1 + \lambda\gamma_1)\, u'(1) + (\sigma_2 + \lambda\gamma_2)\, u(1) = f_2, \tag{1.2}$$

where λ-is a spectral parameter, A is a linear close operator with everywhere dense in H domain of definition and a resolvent decreasing as $|\lambda|^{-1}$ for sufficiently large $|\lambda|$ in some angles containing a negative semi-axis; $\alpha_i, \beta_i, \sigma_i, \gamma_i$ $(i = 1, 2)$ are any fixed complex numbers, moreover $\alpha_1, \beta_1, \sigma_1, \gamma_1 \neq 0$.

In the given paper we find sufficient conditions for the solvability of boundary value problems (1), (2) in the space $L_p((0,1); H)$ $(p > 1)$, establish some estimates (with respect to u and λ) for the solution.

Note that boundary value problems for second order ordinary differential equations with a spectral parameter in the equation and boundary conditions were studied in a lot of papers. In particular, in the paper [8], spectral parameter exists in the equation as λ, while in the boundary condition as a linear function with respect to λ, where the asymptotic behavior of eigenvalue and eigenfunction of the considered boundary value problems are studied.

Now introduce some necessary denotation and notion that are used in the paper.

Let E_1 and E be two Banach spaces. Denote by $B(E_1, E)$ Banach space of all bounded operators acting from E_1 to E, with ordinary operator norm. In the special case, if $E_1 = E$, then $B(E, E) := B(E)$.

Definition 1.1. *A linear closed operator A is said to be strongly positive in the Hilbert space H, if the domain of definition $D(A)$ is dense in H, at some $\varphi \in [0, \pi)$, for all the points μ from the angle $|\arg \mu| \leq \varphi$ (including $\mu = 0$) there exists $(A + \mu I)^{-1}$ and for such μ it holds the estimation*

$$\left\| (A + \mu I)^{-1} \right\|_{B(H)} \leq C\, (1 + |\mu|)^{-1},$$

where I is a unit operator in H, $C = const > 0$. For $\varphi = 0$ the operator A is said to be positive.

Self-adjoint positive definite operators acting in the Hilbert space H are the simplest examples of strongly positive operators.

Note that the strong positivity of the operator A yields strong positivity of the operator $A^{\alpha}, \alpha \in (0, 1)$. Let A be a strongly-positive operator in H. As A^{-1} is bounded in H, then

$$H(A^n) := \left\{ u : u \in D(A^n), \|u\|_{H(A^n)} = \left\| A^n u \right\|_H \right\}, n \in \mathbb{N}$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A. If the operator A is strongly positive in H, then the operator $-A$ is a generating operator analytic for $t > 0$ of the semigroup e^{-tA} and this semigroup exponentially decreases, i.e. there exist two numbers $c > 0, \sigma_0 > 0$ such that $\left\| e^{-tA} \right\| \leq c e^{-\sigma_0 t}, 0 \leq t < +\infty$. By [[9], theorem 1.5.5] the operator $-A^{1/2}$ generates an analytic semigroup for $t > 0$, decreasing at infinity.

Definition 1.2. *[[12], theorem 1.14.5]. Interpolation spaces $(H(A^n), H)_{\theta,p}$ of Hilbert spaces $H(A^n)$ and H, where A is a strongly positive operator in H are determined by the equality*

$$\left(H(A^n), H \right)_{\theta,p} :$$
$$= \left\{ u : u \in H, \|u\|_{(H(A^n),H)_{\theta,p}} := \int_0^{+\infty} t^{-1+n\theta p} \left\| A^n e^{-tA} u \right\|_H^p dt < \infty \right\},$$
$$\theta \in (0, 1),\ p > 1,\ n \in \mathbb{N}.$$

Moreover, $(H(A^n), H)_{0,p} := H(A^n)$, $(H(A^n), H)_{1,p} := H$.

Denote by $L_p((0,1); H)$ $(1 < p < \infty)$, a Banach space (for $p = 2$ a Hilbert space) of functions $x \to u(x) : [0, 1] \to H$, strongly measurable and summable in the p-th degree, with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p\, dx \right)^{1/p} < \infty,$$

by

$$W_p^n\left((0,1);H(A^n),H\right) := \left\{u : A^n u, u^{(n)} \in L_p((0,1);H)\right\}$$

a space of vector-functions with the norm

$$\|u\|_{W_p^n((0,1);H(A^n),H)} := \left\|A^n u\right\|_{L_p((0,1);H)} + \left\|u^{(n)}\right\|_{L_p((0,1);H)}.$$

It is known that [[12], theorem 1.8.2], if $u \in W_p^n((0,1);H(A^n)H)$, then

$$u^{(j)}(\cdot) \in \left(H(A^n),H\right)_{\frac{j+\frac{1}{p}}{n},p}, j = \overline{0,(n-1)}.$$

2 Homogeneous equations

Consider now boundary value problem (1.1), (1.2) in separable Hilbert space H.

Theorem 2.1. *Let the following conditions be fulfilled:*

1) A is a strongly positive operator in H.

2) $\alpha_k, \beta_k, \sigma_k, \gamma_k$, $(k = 1,2)$ are any fixed complex values, moreover $\alpha_1, \beta_1, \delta_1, \gamma_1 \neq 0$.

Then for $f_k \in (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$, $k = 1,2$; $p \in (1,\infty)$ and for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi$, where $\varphi \in [0,\pi)$ is some number, problem (1.1), (1.2) has a unique solution $u \in W_p^2((0,1);H(A),H)$ and for the solution it holds the estimation

$$|\lambda|\,\|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)}$$
$$\leq C_\varphi\,|\lambda|^{-1}\left(\sum_{k=1}^{2}\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}}\,\|f_k\|_H\right). \tag{2.1}$$

Proof. From conditions 1) by [[13], theorem 5.4.2/6] for $|\arg\lambda| \leq \varphi < \pi$, there exists an analytic for $x > 0$ and strongly continuous for $x \geq 0$ semi group $e^{-x(A+\lambda I)^{1/2}}$. By [[13], theorem 5.3.2/1] the arbitrary solution of equation (1) belonging to $W_p^2\left((0,1);H(A),H\right)$, for $|\arg\lambda| \leq \varphi$ has the form

$$u(x) = e^{-x(A+\lambda I)^{1/2}}g_1 + e^{-(1-x)(A+\lambda I)^{1/2}}g_2, \tag{2.2}$$

where $g_1, g_2 \in (H(A),H)_{\frac{1}{2p},p}$, are arbitrary elements.

Require the function $u(x)$ of form (2.2) satisfy conditions (1.2).

$$\begin{aligned}
(\alpha_1+\lambda\beta_1)&\left[-(A+\lambda I)^{1/2}g_1 + (A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}g_2\right]\\
&+(\alpha_2+\lambda\beta_2)\left[g_1 I + e^{-(A+\lambda I)^{1/2}}g_2\right] = f_1,\\
(\sigma_1+\lambda\gamma_1)&\left[-(A+\lambda I)^{1/2}e^{-(A+\lambda I)^{1/2}}g_1 + (A+\lambda I)^{1/2}g_2\right]\\
&+(\sigma_2+\lambda\gamma_2)\left[e^{-(A+\lambda I)^{1/2}}g_1 + g_2 I\right] = f_2.
\end{aligned} \tag{2.3}$$

Rewrite system (2.3) in the form:

$$\begin{aligned}
&\left[-(\alpha_1+\lambda\beta_1)(A+\lambda I)^{1/2} + (\alpha_2+\lambda\beta_2)I\right]g_1\\
&+\left[(\alpha_1+\lambda\beta_1)(A+\lambda I)^{1/2} + (\alpha_2+\lambda\beta_2)I\right]e^{-(A+\lambda I)^{1/2}}g_2 = f_1,\\
&\left[-(\sigma_1+\lambda\gamma_1)(A+\lambda I)^{1/2} + (\sigma_2+\lambda\gamma_2)I\right]e^{-(A+\lambda I)^{1/2}}g_1\\
&+\left[(\sigma_1+\lambda\gamma_1)(A+\lambda I)^{1/2} + (\sigma_2+\lambda\gamma_2)I\right]g_2 = f_2.
\end{aligned}$$

We can write this system in the space $\mathbb{H} := (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ in the form of the operator equation

$$(A(\lambda)+R(\lambda))\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{2.4}$$

where $A(\lambda)$ and $R(\lambda)$ are operator-matrices of 2×2 dimension:

$$A(\lambda):$$

$$:= \begin{pmatrix} \left[-(\alpha_1+\lambda\beta_1)(A+\lambda I)^{1/2}+(\alpha_2+\lambda\beta_2 I)\right] & 0 \\ 0 & \left[(\sigma_1+\lambda\gamma_1)(A+\lambda I)^{1/2}+(\sigma_2+\lambda\gamma_2) I\right] \end{pmatrix}$$

$$D(A(\lambda)) := (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$$

$$R(\lambda) := e^{-(A+\lambda I)^{1/2}}$$

$$\times \begin{pmatrix} 0 & \left[(\alpha_1+\lambda\beta_1)(A+\lambda I)^{1/2}+(\alpha_2+\lambda\beta_2) I\right] \\ \left[-(\sigma_1+\lambda\gamma_1)(A+\lambda I)^{1/2}+(\sigma_2+\lambda\gamma_2) I\right] & 0 \end{pmatrix}$$

$$D(R(\lambda)) := \mathbb{H}.$$

Represent $A(\lambda)$ in the form

$$A(\lambda) = A_1(\lambda)+\lambda A_2(\lambda),$$

where

$$A_1(\lambda) := \begin{pmatrix} \left[-\alpha_1 (A+\lambda I)^{1/2}+\alpha_2 I\right] & 0 \\ 0 & \left[\sigma_1 (A+\lambda I)^{1/2}+\sigma_2 I\right] \end{pmatrix}$$

$$D(A_1(\lambda) = D(A(\lambda)),$$

$$A_2(\lambda) := \begin{pmatrix} \left[-\beta_1 (A+\lambda I)^{1/2}+\beta_2 I\right] & 0 \\ 0 & \left[\gamma_1 (A+\lambda I)^{1/2} I\right]+\gamma_2 I \end{pmatrix}$$

$$D(A_2(\lambda)) := D(A(\lambda)).$$

Show that the operator $A(\lambda)$ in the space $\mathbb{H}$ for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ has a bounded inverse, acting from $\mathbb{H}$ into $\mathbb{H}_1 := (H(A),H)_{\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2p},p}$ and it holds the estimation

$$\left\|A(\lambda)^{-1}\right\|_{B(\mathbb{H},\mathbb{H}_1)} \le C\,|\lambda|^{-1}, \tag{2.5}$$

where $C>0$ is a constant, independent of λ.

As formally

$$A(\lambda) = A_1(\lambda)+\lambda A_2(\lambda) = \lambda A_2(\lambda)\left[I+\lambda^{-1}A_1(\lambda)A_2(\lambda)^{-1}\right],$$

and

$$A(\lambda)^{-1} = \lambda^{-1}\left[I+\lambda^{-1}A_1(\lambda)A_2(\lambda)^{-1}\right]^{-1} A_2(\lambda)^{-1} \tag{2.6}$$

the from representation (2.6) it is seen that for that it suffices to show that

a) the operator $A_2^{-1}(\lambda)$, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ boundedly acts from $\mathbb{H}$ to $\mathbb{H}_1$ and it holds the estimation

$$\left\|A_2(\lambda)^{-1}\right\|_{B(\mathbb{H},\mathbb{H}_1)} \le C, \tag{2.7}$$

where $C>0$ is a constant, independent of λ;

b) the operator $\left(I+\lambda^{-1}A_1(\lambda)A_2(\lambda)^{-1}\right)^{-1}$ for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$ boundedly acts from $\mathbb{H}_1$ to $\mathbb{H}_1$ and it holds the estimation

$$\left\|\left(I+\lambda^{-1}A_1(\lambda)A_2^{-1}(\lambda)\right)^{-1}\right\|_{B(\mathbb{H}_1)}\le C, \tag{2.8}$$

where $C>0$ is a constant, independent of λ.

Prove a). As formally

$$A_2(\lambda)^{-1}:=\begin{pmatrix}\left[-\beta_1\left(A+\lambda I\right)^{1/2}+\beta_2 I\right]^{-1} & 0\\ 0 & \left[\gamma_1(A+\lambda I)^{1/2}+\gamma_2 I\right]^{-1}\end{pmatrix},$$

then in order to show a), it suffices to show that:

a_1) the operator $(A+\lambda I)^{-1/2}$ for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$ boundedly acts from $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimation

$$\left\|(A+\lambda I)^{-1/2}\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)}\le C, \tag{2.9}$$

where $C>0$ is a constant, independent of λ.

a_2) the operator $\left[I+s\left(A+\lambda I\right)^{-1/2}\right]^{-1}$ for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi$ boundedly act from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimation

$$\left\|\left[I+s\left(A+\lambda I\right)^{-1/2}\right]^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}\le q<1, \tag{2.10}$$

where s is any fixed complex number.

a_1) was proved in [4].

Prove a_2). By [[13], lemma 5.4.2/6] for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$ we have

$$\begin{gathered}\left\|s\left(A+\lambda I\right)^{-1/2}\right\|_{B(H)}\le\frac{C}{|\lambda|^{1/2}},\\ \left\|s\left(A+\lambda I\right)^{-1/2}\right\|_{B(H(A))}\le\frac{C}{|\lambda|^{1/2}}.\end{gathered} \tag{2.11}$$

Then by the interpolation theorem [[12], theorem 1.3.3] and [[13], lemma 5.4.2/6] the operator $s(A+\lambda I)^{-1/2}$ boundedly acts from $(H(A),H)_{\theta,p}$ into $(H(A),H)_{\theta,p}$ for any $\theta\in(0,1)$ and it holds the estimation

$$\begin{gathered}\left\|s\left(A+\lambda I\right)^{-1/2}\right\|_{B((H(A),H)_{\theta,p})}\\ \le C\left\|(A+\lambda I)^{-1/2}\right\|^{1-\theta}_{B(H(A))}\left\|(A+\lambda I)^{-1/2}\right\|^{\theta}_{B(H)}\le\frac{C}{|\lambda|^{1/2}}.\end{gathered} \tag{2.12}$$

We take $\theta=\dfrac{1}{2p}$. Then for sufficiently large $|\lambda|$ from the angle $|\arg\lambda|\le\varphi<\pi$, the operator $(A+\lambda I)^{-\frac{1}{2}}$ boundedly acts from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimation

$$\left\|s\left(A+\lambda I\right)^{-1/2}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}\le\frac{C}{|\lambda|^{1/2}}, \tag{2.13}$$

where $C > 0$ is a constant, independent of λ. So, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$, the operator $\left[I + s(A+\lambda I)^{1/2}\right]^{-1}$ boundedly acts from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds estimation (2.10), i.e. $a_2)$ is proved. From the representations we get.

$$\left[-\beta_1\,(A+\lambda I)^{1/2} + \beta_2 I\right]^{-1} = -\frac{1}{\beta_1}\left[I - \frac{\beta_2}{\beta_1}\,(A+\lambda I)^{-1/2}\right]^{-1}(A+\lambda I)^{-1/2}\,, \tag{2.14}$$

and

$$\left[\gamma_1\,(A+\lambda I)^{1/2} + \gamma_2 I\right]^{-1} = \frac{1}{\gamma_1}\left[I + \frac{\gamma_2}{\gamma_1}\,(A+\lambda I)^{-1/2}\right]^{-1}(A+\lambda I)^{-1/2}\,, \tag{2.15}$$

by estimation (2.9) and (2.10) it follows that the operators $\left[\beta_2 I - \beta_1\,(A+\lambda I)^{1/2}\right]^{-1}$ and $\left[\gamma_2 I + \gamma_1\,(A+\lambda I)^{1/2}\right]^{-1}$ for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ boundedly act from $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ and it holds the estimation:

$$\left\|\left[\beta_2 I - \beta_1\,(A+\lambda I)^{1/2}\right]^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)} \le C,$$

$$\left\|\left[\gamma_2 I + \gamma_1\,(A+\lambda I)^{1/2}\right]^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)} \le C,$$

where $C > 0$ is a constant, independent of λ.

Consequently for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ the operator $A_2(\lambda)^{-1}$ boundedly acts from $\mathbb{H}$ into $\mathbb{H}_1$ and it holds estimation (2.7).

Prove now b). From the representation of the operator $A_1(\lambda)$ and $A_2(\lambda)^{-1}$ and by equalities (2.14), (2.15) we have:

$$\lambda^{-1}A_1(\lambda)A_2^{-1}(\lambda)$$

$$= \lambda^{-1}\begin{pmatrix} \dfrac{\alpha_1}{\beta_1}\left[I - \dfrac{\beta_2}{\beta_1}(A+\lambda I)^{-1/2}\right]^{-1} & 0 \\ 0 & \dfrac{\sigma_1}{\gamma_1}\left[I + \dfrac{\gamma_2}{\gamma_1}(A+\lambda^2 I)^{-1/2}\right]^{-1} \end{pmatrix}$$

$$+\lambda^{-1}(A+\lambda I)^{-1/2}\begin{pmatrix} -\dfrac{\alpha_2}{\beta_1}\left[I - \dfrac{\beta_2}{\beta_1}(A+\lambda I)^{-1/2}\right]^{-1} & 0 \\ 0 & \dfrac{-\sigma_2}{\gamma_1}\left[I + \dfrac{\gamma_2}{\gamma_1}(A+\lambda I)^{-1/2}\right]^{-1} \end{pmatrix}.$$

Hence, by estimations (2.10), (2.13) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ we have

$$|\lambda|^{-1}\left\|A_1(\lambda)A_2(\lambda)^{-1}\right\|_{B(\mathbb{H}_1)} \le \frac{C}{|\lambda|} < q < 1. \tag{2.16}$$

From estimation (2.16) it follows that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ there exists

$$\left(I + \lambda^{-1}A_1(\lambda)A_2(\lambda)^{-1}\right)^{-1}$$

and boundedly acts from $\mathbb{H}_1$ into $\mathbb{H}_1$ and it holds estimation (2.8). This proves b). From representations (2.6) by estimations (2.7) and (2.8) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$ we get estimation (2.5). Then from equation (2.4) we have

$$\left(I + A(\lambda)^{-1}R(\lambda)\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A(\lambda)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{2.18}$$

Using the representations $A(\lambda)^{-1}$ and $R(\lambda)$ we can show that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$ the operator $A(\lambda)^{-1}R(\lambda)$ boundedly acts from $\mathbb{H}_1$ into $\mathbb{H}_1$ and it holds the estimation

$$\left\|A(\lambda)^{-1}R(\lambda)\right\|_{B(\mathbb{H}_1)} \leq \frac{C}{|\lambda|}\left(1+|\lambda|\right)e^{-\omega|\lambda|^{1/2}} \leq q < 1. \tag{2.19}$$

Hence, by the Neumann identity, for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda| \to \infty$,

$$\left(I + A(\lambda)^{-1}R(\lambda)\right) = I + \sum_{k=1}^{\infty}\left(-A(\lambda)^{-1}R(\lambda)\right)^{k}, \tag{2.20}$$

where the series in the right side of (2.20) converges in the norm of the space of bounded operators in $\mathbb{H}_1$.

By from (2.18), (2.19) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$ we have

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(I + A(\lambda)^{-1}R(\lambda)\right)^{-1} A(\lambda)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Consequently, for sufficiently $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi$ the elements g_1 and g_2 may be represented in the form:

$$g_k = \left(C_{k1}(\lambda) + R_{k1}(\lambda)\right)f_1 + \left(C_{k2}(\lambda) + R_{k2}(\lambda)\right)f_2, \quad k = 1, 2, \tag{2.21}$$

where

$$C_{11}(\lambda) = \left[-\left(\alpha_1 + \lambda\beta_1\right)\left(A + \lambda I\right)^{1/2} + \left(\alpha_2 + \lambda\beta_2\right)I\right]^{-1}, C_{12}(\lambda) = C_{21}(\lambda) = 0,$$

$$C_{22}(\lambda) = \left[(\sigma_1 + \lambda\gamma_1)(A + \lambda I)^{1/2} + (\sigma_2 + \lambda\gamma_2)I\right]^{-1}$$

R_{kj}-are some bounded operators acting from $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$ for $|\arg\lambda| \leq \varphi$ and $|\lambda| \to \infty$. Moreover, from estimations (2.5) and (2.19) it follows that for $|\arg\lambda| \leq \varphi < \pi$ and $|\lambda| \to \infty$

$$\left\|R_{kj}(\lambda)\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)} \leq \exists C^{-\omega|\lambda|^{1/2}}, \quad \exists C; \omega > 0. \tag{2.22}$$

From the representation $A(\lambda)^{-1}$ it also follows that for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$ for the operators $R_{kj}(\lambda)$ there hold the following estimations

$$\left\|R_{kj}(\lambda)\right\|_{B(H)} \leq Ce^{-\omega|\lambda|^{1/2}}, \quad \exists C; \omega > 0. \tag{2.23}$$

Substituting (2.21) into (2.2) we get

$$u(x)$$

$$= \sum_{k=1}^{2}\left\{e^{-x(A+\lambda I)^{1/2}}\left(C_{1k}(\lambda) + R_{1k}(\lambda)\right) + e^{-(1-x)(A+\lambda I)^{1/2}}\left(C_{2k}(\lambda) + R_{2k}(\lambda)\right)\right\}f_k.$$

Then, for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi$, we have

$$
\begin{aligned}
&|\lambda| \, \|u\|_{L_p((0,1);H)} + \left\|u''\right\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \\
&\le C \sum_{k=1}^{2} \Bigg\{ |\lambda| \left[\left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} C_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \right. \\
&\quad + \left(\int_0^1 \left\| e^{-(1-x)(A+\lambda I)^{1/2}} C_{2k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \\
&\left. + \int_0^1 \left(\left\| e^{-(1-x)(A+\lambda I)^{1/2}} R_{2k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \right] + \left(1 + \left\| A(A+\lambda I)^{-1} \right\| \right) \\
&\quad \times \left[\left(\int_0^1 \left\| (A+\lambda I) e^{-x(A+\lambda I)^{1/2}} C_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \right. \\
&\quad + \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} R_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \\
&\quad + \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} C_{2k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \\
&\quad \left. + \left(\int_0^1 \left\| (A+\lambda I) e^{-(1-x)(A+\lambda I)^{1/2}} R_{2k}(\lambda) f_k \right\|_H^p dx \right)^{1/p} \right] \Bigg\}.
\end{aligned}
\tag{2.24}
$$

Estimate the first term in the right hand side of inequality (2.24) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \le \varphi < \pi$.

As

$$
\begin{aligned}
&\left[-(\alpha_1 + \lambda\beta_1)(A+\lambda I)^{1/2} + (\alpha_2 + \lambda\beta_2) I \right] \\
&\left[\left(\alpha_2 I - \alpha_1 (A+\lambda I)^{1/2} \right) + \lambda \left(\beta_2 I - \beta_1 (A+\lambda I)^{1/2} \right) \right] \\
&= \lambda \left(\beta_2 I - \beta_1 (A+\lambda I)^{1/2} \right) \left[I + \frac{1}{\lambda} \left(\beta_2 I - \beta_1 (A+\lambda I)^{1/2} \right)^{-1} \left(\alpha_2 I - \alpha_1 (A+\lambda I)^{1/2} \right) \right] \\
&= -\lambda\beta_1 (A+\lambda I)^{1/2} \left(I - \frac{\beta_2}{\beta_1} (A+\lambda I)^{-1/2} \right) \\
&\times \left[I + \frac{1}{\lambda} \frac{\alpha_1}{\beta_1} \left(I - \frac{\beta_2}{\beta_1} (A+\lambda I)^{-1/2} \right)^{-1} \left(I - \frac{\alpha_2}{\alpha_1} (A+\lambda I)^{-1/2} \right) \right],
\end{aligned}
$$

$$
C_{11}(\lambda) = \left[-(\alpha_1 + \lambda\beta_1)(A+\lambda I)^{1/2} + (\alpha_2 + \lambda\beta_2) I \right]^{-1} = -\frac{1}{\lambda} \frac{1}{\beta_1} (A+\lambda I)^{-1/2} S(\lambda),
$$

where

$$
\begin{aligned}
S(\lambda) &= \left(I - \frac{\beta_2}{\beta_1} (A+\lambda I)^{-1/2} \right)^{-1} \\
&\times \left[I + \frac{1}{\lambda} \frac{\alpha_1}{\beta_1} \left(I - \frac{\beta_2}{\beta_1} (A+\lambda I)^{-1/2} \right)^{-1} \left(I - \frac{\alpha_2}{\alpha_1} (A+\lambda I)^{-1/2} \right) \right]^{-1}.
\end{aligned}
$$

From estimation (2.11) and (2.12) it follows that

$$\|S(\lambda)\|_{B(H)} \leq C, \ \|S(\lambda)\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}\right)} \leq C. \tag{2.25}$$

Therefore by [[13], theorem 5.4.2/1] and (2.25) for the first term of the right side of inequality (2.24) for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$ we have

$$|\lambda| \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} C_{11}(\lambda) f_1 \right\|_H^p dx \right)^{1/p}$$

$$= |\lambda| \frac{1}{|\lambda|} \frac{1}{|\beta_1|} \left(\int_0^1 \left\| (A+\lambda I)^{-1/2} e^{-x(A+\lambda I)^{1/2}} S(\lambda) f_1 \right\|_H^p dx \right)^{1/p}$$

$$\leq C \left\| (A+\lambda I)^{-1} \right\|_{B(H)} \left(\int_0^1 \left\| (A+\lambda I)^{1/2} e^{-x(A+\lambda I)^{1/2}} S(\lambda) f_1 \right\|_H^p dx \right)^{1/p}$$

$$\leq \frac{C}{1+|\lambda|} \left[\|S(\lambda) f_1\|_{\left(H(A),H_{\frac{1}{2}+\frac{1}{2p},p}\right)} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|S(\lambda) f_1\|_H \right]$$

$$\leq C\,|\lambda|^{-1} \left[\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_1\|_H \right].$$

By [[13], theorem 5.4.2/1] and estimation (2.22), (2.23) for the second term of the right hand side of inequality (2.24), for sufficiently large $|\lambda|$ from the angle $|\arg\lambda| \leq \varphi < \pi$ we have

$$|\lambda| \sum_{k=1}^{2} \left(\int_0^1 \left\| e^{-x(A+\lambda I)^{1/2}} R_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p}$$

$$\leq |\lambda| \left\| (A+\lambda I)^{-1} \right\|_{B(H)} \sum_{k=1}^{2} \left(\int_0^1 \left\| (A+\lambda I) e^{-x(A+\lambda I)^{1/2}} R_{1k}(\lambda) f_k \right\|_H^p dx \right)^{1/p}$$

$$\leq C \cdot \sum_{k=1}^{2} \left(\|R_{1k}(\lambda) f_k\|_{(H(A),H)_{\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|R_{1k}(\lambda) f_k\|_H \right)$$

$$\leq C e^{-\omega|\lambda|^{1/2}} \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{2p}} \|f_k\|_H \right)$$

$$\leq C|\lambda|^{-1} \cdot \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{\frac{1}{2}-\frac{1}{2p}} \|f_k\|_H \right).$$

The remaining terms of the right hand side of inequality (2.24) are estimated in the same way. The theorem is proved.

Acknowledgments. The authors would like to express their gratitude to the referee for his/her very valuable comments and suggestions.

References

1. Aliev, B.A.: *Asymptotic behavior of eigenvalues of a boundar value problem with spectral parameter in the boundary conditions for the second order elliptic diferential-operator equations*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., Mathematics **25** (7), 3-8 (2005).
2. Aliev, B.A.: *Asymptotic behavior of eigenvalues of a boundary value problem for a second order elliptic differential-operator equation*. Ukrainian Math. J., **58** (8), 1146-1152 (2006) (Russian).
3. Aliev, B.A., Yakubov, Ya.: *Elliptic diferential-operator problems with a spectral parameter in both the equation and boundary-operator conditions*. Adv. Differential Equations, **11** (10), 1081-1110 (2006). (Erratum in 12, 9 (2007), 1079).
4. Aliev, B.A., Yakubov, Ya.: *Second order elliptic differential-operator equations with unbounded operator boundary conditions in UMD Banach spaces*. Integral Equations Operator Theory, **69**, 269-300 (2011).
5. Aliev B.A.: *Solvability of a boundary value problem for a second order differential-operator equation with a spectral parameter both in the equation and in the boundary conditions*. Ukrainian Math. J., **62** (1), 3-14 (2010).
6. Bairamoglu, M., Aslanova, N.M.: *Distribution of eigenvalues and trace formula of Sturm-Liouville operator equation*. Ukrainian Math. J., **62** (7), 867-877 (2010).
7. Gorbachuk, V.I., Rybak, M.A.: On boundary value problems for Sturm-Liouville operator equation with a spectral parameter in the equation and in the boundary condition In: Direct and inverse problems of scattering theory. Kiev, 3-16 (1981) (Russian).
8. Kerimov, N.B., Allahverdiyev, T.I.: *On a boundary value problem. I.* Differential Equations, **29** (1), 54-60 (1993) (Russian).
9. Krein, S.G.: Linear differential equations in Banach space. *M., Nauka* (1967) (Russian).
10. Oleinik, L.A.: Nonhomogeneous boundary value problems for differential-operator equations with a spectral parameter in the boundary conditions. Spectral theory of differential-operator equations. *Kiev*, 25-28 (1986) (Russian).
11. Rybak, M.A.: *On asymptotic distribution of eigen values of boundary value problems for Sturm-Liouville operator equation*. Ukrainian Math. J., **32** (2), 248-252 (1980) (Russian).
12. Triebel, Kh.: Interpolation theory. Functional spaces. Differential operators. *M. Mir* (1980) (Russian).
13. Yakubov, S., Yakubov, Ya.: Differential-Operator Equations. Ordinary and Partial Differential Equations. Chapman and Hall. *CRC, Boca-Raton* (2000).