Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 13-18 (2015).
Series of Physical-Technical and Mathematical Sciences.

Bifurcation from infinity for some nonlinear eigenvalue problems which are not linearizable

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Received: 25.10.2015 / Accepted: 30.10.2015

Abstract. *In this paper we consider bifurcation from infinity for a some class of nonlinear eigenvalue problems in Hilbert space with nonlinearizable nonlinearities. We show the existence of unbounded continua of nontrivial solutions bifurcating from the interval at infinity. These global continua have properties similar to those of the continua found in Rabonowitz' well-known global bifurcation theorem from infinity.*

Keywords. asymptotically linear mapping, asymptotic bifurcation point, global bifurcation, eigenvalue odd multiplicity, continua of solutions.

Mathematics Subject Classification (2010): 34B15, 47H11, 47J10, 47J15

1 Introduction

Let H be a real separable Hilbert space with norm denoted by $||\cdot||$, and $L : D(L) \subset H \to H$ be a linear semi-bounded below self-adjoint operator with compact resolvent, i.e. $(L - \lambda I)^{-1}$ is compact for some (and hence for all) λ not belonging to the spectrum $\sigma(L)$ of L, and $D(L)$ is dense in H.

We consider the nonlinear eigenvalue problem

$$Lu = \lambda u + F(\lambda, u) + G(\lambda, u), \tag{1.1}$$

where $F : \mathbb{R} \times H \to H$ and $G : \mathbb{R} \times H \to H$ are continuous operators satisfying the following conditions:

$$||F(\lambda, u)|| \leq M||u||, \ \forall \lambda \in \mathbb{R}, \ \forall u \in H, \ ||u|| > 1, \tag{1.2}$$

where M is a positive constant; for any bounded interval $\Lambda \subset \mathbb{R}$,

$$G(\lambda, u) = o(||u||) \ \text{ at } \ u = \infty, \tag{1.3}$$

uniformly with respect to $\lambda \in \Lambda$.

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As norm in $\mathbb{R} \times H$, we take $||(\lambda, u)|| = \left\{|\lambda|^2 + ||u||^2\right\}^{1/2}$.

Note that, due to the assumption on L, every eigenvalue of L is real and necessarily isolated and of finite multiplicity, and the whole spectrum $\sigma(L)$ consists of infinite nondecreasing sequence of only such points.

We say (λ, ∞) is a bifurcation point for (1.1) if every neighborhood of (λ, ∞) contains solutions of (1.1), i.e., there exists a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ of solutions of (1.1) such that $\lambda_n \to \lambda$ and $||u_n|| \to \infty$ as $n \to \infty$ [3, 7].

In the case when $F \equiv 0$ it follows from [3, Ch. 4, § 3, Theorem 3.1] that if λ be an eigenvalue of odd multiplicity of the operator L, then (λ, ∞) is an bifurcation point of problem (1.1) and this bifurcation point corresponds to a continuous branch of solutions a leaving to infinity. In [7] shows that if $F \equiv 0$ and $\lambda \in \sigma(L)$ is of odd multiplicity, then the set of nontrivial solutions of problem (1.1) possesses an unbounded component $\mathcal{D}_\lambda$ which meets (λ, ∞). Moreover if $\Lambda \subset \mathbb{R}$ is an interval such that $\Lambda \cap \sigma(L) = \{\lambda\}$ and $\mathcal{M}$ is a neighborhood of (λ, ∞) whose projection on $\mathbb{R}$ lies in Λ and whose projection on H is bounded away from 0, then either (i) $\mathfrak{D}_\lambda \backslash \mathcal{M}$ is bounded in $\mathbb{R} \times H$, in which case $\mathcal{D}_\lambda \backslash M$ meets $\mathcal{R} = \{(\lambda, 0) : \lambda \in \mathbb{R}\}$, or (ii) $\mathcal{D}_\lambda \backslash \mathcal{M}$ is unbounded in $\mathbb{R} \times H$; if additionally $\mathcal{D}_\lambda \backslash \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathcal{D}_\lambda \backslash \mathcal{M}$ meets $(\hat{\lambda}, 0)$ where $\lambda \neq \hat{\lambda} \in \sigma(L)$.

The bifurcation from infinity for noninearizable Sturm-Liouville problems have been considered in [6, 9-11]. These papers prove the existence of global continua of nontrivial solutions in $\mathbb{R} \times C^1$ corresponding to the usual nodal properties and emanating from "bifurcation intervals" at infinity surrounding the eigenvalues of the linear problem.

In the present paper, we give generalization of result from [7] in the case $F \equiv 0$ and not differentiable at infinity, and we shows the existence of global continua bifurcating from intervals containing eigenvalues of the linear problem.

2 Preliminary

Alongside with the problem (1.1) we shall consider the following nonlinear problem

$$Lu = \lambda u + \tilde{F}(\lambda, u) + \tilde{G}(\lambda, u), \tag{2.1}$$

where L be as above and $\tilde{F} : \mathbb{R} \times H \to H$ and $\tilde{G} : \mathbb{R} \times H \to H$ are continuous operators satisfying the following conditions:

$$||\tilde{F}(\lambda, u)|| \leq M||u||, \ \forall \lambda \in \mathbb{R}, \ \forall u \in H, \ ||u|| < 1, \tag{2.2}$$

where M is a positive constant; for any bounded interval $\Lambda \subset \mathbb{R}$,

$$\tilde{G}(\lambda, u) = o(||u||) \ \text{at} \ u = 0, \tag{2.3}$$

uniformly with respect to $\lambda \in \Lambda$.

Let $\tilde{\mathcal{T}}$ denote the closure of the set of nontrivial solutions of (2.1) in $\mathbb{R} \times H$ and let

$$I_\mu = [\mu - M, \mu + M].$$

Let $\tilde{\mathbf{B}}$ denote the set of bifurcation points of problem (1.1).

Theorem 2.1 *[4] (see also [5]). Let $\mu \in \mathbb{R}$ be an eigenvalue of the operator L of odd multiplicity and*

$$\text{dist}\,(\mu\,;\,\sigma(L) \backslash \{\mu\}) > 2M. \tag{2.4}$$

Then the set $\tilde{\mathbf{B}}$ is nonempty, and $\tilde{\mathbf{B}} \cap (I_\mu \times \{0\}) \neq \emptyset$.

We define the set $\tilde{\mathcal{C}}_\mu^* \in \tilde{\mathcal{T}}$ to be the union of all the components of $\tilde{\mathcal{T}}$ which meet $I_\mu \times \{0\}$ (by theorem 2.1 this set is nonempty). Note that the set $\tilde{\mathcal{C}}_\mu^*$ may not be connected in $\mathbb{R} \times H$, but the set $\tilde{\mathcal{C}}_\mu = \tilde{\mathcal{C}}_\mu^* \cup (I_\mu \times \{0\})$ is connected in $\mathbb{R} \times H$.

Theorem 2.2 *[4]. If $\mu \in \sigma(L)$ is odd multiplicity and condition* (2.4) *holds, then either (i) $\tilde{\mathcal{C}}_\mu$ is unbounded in $\mathbb{R} \times H$ or (ii) $\tilde{\mathcal{C}}_\mu$ contains the set $I_{\hat{\mu}} \times \{0\}$ where $\mu \neq \hat{\mu} \in \sigma(L)$.*

Let $\mathcal{B}_\varepsilon$ denote open ball in H of radius ε centered at 0.

The coincidence degree of the pair of operators $(L, \mathcal{H})$ with respect to $\mathcal{B}_r$, denoted $d\,[(L, \mathcal{H}), \mathcal{B}_r]$, will be defined for any continuous operator $\mathcal{H} : H \to H$ which mapping bounded sets onto bounded sets, provided $Lu \neq \mathcal{H}(u)$ for $u \in \partial\mathcal{B}_r$ ([2, Ch. 3]). For convenience we shall introduce the following notation

$$d(L - \mathcal{H}, \mathcal{B}_r) = d[(L, \mathcal{H}),\, \mathcal{B}_r].$$

Let **B** denote the set of asymptotic bifurcation points of problem (1.1).

Theorem 2.3 *Let $\mu \in \sigma(L)$ is of odd multiplicity and condition* (2.4) *holds. Then* $\mathbf{B} \cap (I_\mu \times \{\infty\}) \neq \emptyset$.

Proof. Assume the contrary, i.e. suppose that $\mathbf{B} \cap (I_\mu \times \{\infty\}) = \emptyset$. Then there exists sufficiently $R_\mu > 0$ and sufficiently small $\delta_\mu > 0$ such that the problem (1.1) has no solution in $I_\mu(\delta_\mu) \times (E \backslash \mathcal{B}_{R_\mu})$, i.e.

$$Lu \neq \lambda u + F(\lambda, u) + G(\lambda, u), \;\; \lambda \in I_\mu(\delta_\mu), \;\; ||u|| \,\geq\, R_\mu. \tag{2.5}$$

where $I_\mu(\delta) = [\mu - M - \delta, \mu + M + \delta]$.

Let

$$\delta_{\mu,\,0} = \frac{\text{dist}\,(\mu\,;\,\sigma(L)\backslash\{\mu\}) - 2M}{2}\,. \tag{2.6}$$

Without loss of generality we can assume that

$$\delta_\mu < \delta_{\mu,\,0}. \tag{2.7}$$

It follows by (1.3) that there exists $R_{\mu,\,0} > R_\mu$ such that

$$\frac{||G(\lambda, u)||}{||u||} < \frac{\delta_\mu}{4}\,, \;\; \lambda \in I_\mu(\delta_\mu), \;\; ||u|| \,\geq\, R_{\mu,\,0}. \tag{2.8}$$

Set $\underline{\lambda} = \mu - M - \frac{\delta_\mu}{2}$ and $\overline{\lambda} = \mu + M + \frac{\delta_\mu}{2}$. It follows by (2.4), (2.6) and (2.7) that

$$\text{dist}\,(\underline{\lambda} : \sigma(L)) = \text{dist}\,\big(\overline{\lambda} : \sigma(L)\big) = M + \frac{\delta_\mu}{2}\;. \tag{2.9}$$

In view (2.9), from (2.5) we obtain the following relations

$$Lu \neq \underline{\lambda} u + F(\underline{\lambda}, u) + G(\underline{\lambda}, u), \;\; u \in \partial\mathcal{B}_{R_{\mu,\,0}}, \tag{2.10}$$

and

$$Lu \neq \overline{\lambda} u + F(\overline{\lambda}, u) + G(\overline{\lambda}, u), \;\; u \in \partial\mathcal{B}_{R_{\mu,\,0}}. \tag{2.11}$$

Then, it follows by homotopy invariance of the coincide degree that

$$d(L - \underline{\lambda} I - F(\,\underline{\lambda}\,,\,\cdot\,) - G(\,\underline{\lambda}\,,\,\cdot\,), \mathcal{B}_{\mu,0}) = d(L - \overline{\lambda} I - F(\,\overline{\lambda}\,,\,\cdot\,) - G(\,\overline{\lambda}\,,\,\cdot\,), \mathcal{B}_{\mu,0}). \tag{2.12}$$

On the other hand, by (1.2), (2.6)-(2.10) for any $t \in [0,1]$ and any $u \in \partial\mathcal{B}_R \cap D(L), \;\; R \geq R_{\mu,0}$, we obtain

$$||Lu - \underline{\lambda}\, u - \, t\, F(\underline{\lambda}\,,\, u) - t\, G(\underline{\lambda}\,,\, u)|| \,\geq\, ||Lu - \underline{\lambda}\, u|| - \, t||F(\underline{\lambda}\,,\, u)||$$

$$-t||G(\underline{\lambda}\,,\, u)|| \,\geq c(\underline{\lambda})||u|| - ||F(\underline{\lambda}\,,\, u)|| \, - ||G(\underline{\lambda}\,,\, u)|| \,\geq\, c(\underline{\lambda})||u||$$

$$-M||u|| - \tfrac{\delta_\mu}{2}||u|| > \tfrac{\delta_\mu}{2}||u|| = \tfrac{\delta_\mu R_{\mu,\,0}}{2} > 0.$$

Hence, for any $u \in D(L)$ with $R \geq R_{\mu,0}$ and for any $t \in [0,1]$ we have

$$Lu \neq \underline{\lambda}\, u - t\, F(\underline{\lambda}\,,\, u) - t\, G(\underline{\lambda}\,,\, u).$$

So, using the homotopy invariance of the coincide degree again, we obtain that

$$d(L - \underline{\lambda}\, I - \, F(\underline{\lambda}\,, \cdot) - \, G(\underline{\lambda}\,, \cdot), \mathcal{B}_R) = d(L - \underline{\lambda}\, I, \mathcal{B}_R).$$

Since $\underline{\lambda} \notin \sigma(L)$, then $d(L - \underline{\lambda}\, I, \mathcal{B}_r)$ does not depend on r ($r > 0$), and

$$d(L - \underline{\lambda}\, I, \mathcal{B}_r) = i(\underline{\lambda}) \text{ for any } r > 0$$

(see [2, Ch. 3; 3, Ch. 2]). Consequently,

$$d(L - \underline{\lambda}\, I - F(\underline{\lambda}\,, \cdot\,) - G(\underline{\lambda}\,, \cdot\,), \mathcal{B}_R) = i(\underline{\lambda}). \tag{2.13}$$

Similarly, we obtain

$$d(L - \overline{\lambda}\, I - F(\overline{\lambda}\,, \cdot\,) - G(\overline{\lambda}\,, \cdot\,), \mathcal{B}_R) = i(\overline{\lambda}). \tag{2.14}$$

Then by (2.13) and (2.14) it follows from (2.12) that

$$i(\underline{\lambda}) = i(\overline{\lambda}). \tag{2.15}$$

On the other hand, since μ is the only eigenvalue of L in $[\underline{\lambda}, \overline{\lambda}]$ and has odd multiplicity, then by Leray-Schauder formula [2, p. 501] we have

$$i(\underline{\lambda}) = -i(\overline{\lambda}),$$

which contradicts relation (2.15). The resulting contradiction completes the proof of Theorem 2.2.

Lemma 2.1 *If* $(\lambda, \infty) \in \mathbf{B}$*, then*

$$\text{dist}\,(\lambda\, :\, \sigma(L)) < M. \tag{2.16}$$

Proof. Let $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset \mathbb{R} \times E$ is a sequence of solutions of problem (1.1) such that $\lambda_n \to \lambda$ and $||u_n|| \to \infty$ as $n \to \infty$. Without loss of generality we assume that $\lambda \notin \sigma(L)$ and $||u_n|| > 1$. Then for sufficiently large n it follows by (1.1) that

$$u_n = (L - \lambda_n I)^{-1} \left(F(\lambda_n, u_n) + G(\lambda_n, u_n)\right). \tag{2.17}$$

For each $n \in \mathbb{N}$ let $v_n = \dfrac{u_n}{||u_n||} \in E$. Dividing (2.17) by $||u_n||$ shows that v_n satisfies the equation

$$v_n = (L - \lambda_n I)^{-1} \left(F_n + G_n\right), \tag{2.18}$$

where $F_n = \dfrac{F(\lambda_n, u_n)}{||u_n||}$ and $G_n = \dfrac{G(\lambda_n, u_n)}{||u_n||}$. Based on the conditions (1.2) and (1.3) we have

$$||F_n|| \le M, \ \lim_{n\to\infty} G_n = 0. \tag{2.19}$$

By the relation

$$(L - \lambda_n I)^{-1} = \Big(I - (\lambda_n - \lambda)\,(L - \lambda I)^{-1}\Big)^{-1} (L - \lambda I)^{-1}$$

it follows that

$$||\,(L - \lambda_n I)^{-1}\,|| \to ||\,(L - \lambda I)^{-1}\,|| \text{ as } n \to \infty. \tag{2.20}$$

In view (2.19) and (2.20), from we obtain

$$1 \le \ M\, ||\,(L - \lambda I)^{-1}\,||\,,$$

which by

$$||\,(L - \lambda I)^{-1}\,|| \ = \frac{1}{\text{dist}\,(\lambda : \sigma(L))}$$

implies (2.16). The proof of the Lemma 2.1 is complete.

Corollary 2.1 *Let* $\mu \in \sigma(L)$ *is of odd multiplicity and* $\text{dist}\,(\mu\,;\, \sigma(L)\backslash\{\mu\}) > 2M$. *Then* $\mathbf{B} \cap (I_\mu(\delta_{\mu,\,0})\backslash I_\mu) = \emptyset$.

3 Global bifurcation from infinity for problem (1.1)

Let $\mathcal{T}$ be the set of nontrivial solutions of (1.1) in $\mathbb{R} \times H$. We define the set $\mathcal{C}_\mu^* \in \mathcal{T}$ to be the union of all the components of $\mathcal{T}$ which meet $I_\mu \times \{\infty\}$ (by theorem 2.3, Lemma 2.1 and Corollary 2.1 this set is nonempty). Note that the set $\mathcal{C}_\mu^*$ may not be connected in $\mathbb{R} \times H$, but the set $\mathcal{C}_\mu = \mathcal{C}_\mu^* \cup (I_\mu \times \{\infty\})$ is connected.

For any set $A \subset \mathbb{R} \times H$ we let $P_R(A)$ denote the natural projection of A onto $\mathbb{R} \times \{0\}$.

It follows by theorem 2.1 that the set $\mathcal{C}_\mu$ meets $I_\mu \times \{\infty\}$ and unbounded in $\mathbb{R} \times H$. Moreover, we have a global bifurcation result which generalizes Theorem 1.6 of [7]

Theorem 3.1 *Let $\mu \in \sigma(L)$ is of odd multiplicity and assume that executed the condition* (2.4)*. Then at least one of the following holds:*

(i) $\mathcal{C}_\mu$ meets $I_{\hat{\mu}} \times \{\infty\}$ where $\mu \neq \hat{\mu} \in \sigma(L)$;

(ii) C_μ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$;

(iii) $P_R(C_\mu)$ is unbounded.

Proof . If $(\lambda, u) \in \mathcal{T}$ with $u \neq 0$, dividing (1.1) by $||u||^2$ and setting $v = \frac{u}{||u||^2}$ we have

$$Lv = \lambda v + \tilde{F}(\lambda, v) + \tilde{G}(\lambda, v), \tag{3.1}$$

where

$$\tilde{F}(\lambda, v) = ||v||^2 F(\lambda, \frac{v}{||v||^2}), \;\; \tilde{G}(\lambda, v) = ||v||^2 G(\lambda, \frac{v}{||v||^2}) \;\text{ for }\; v \neq 0. \tag{3.2}$$

Extend $\tilde{F}$ and $\tilde{G}$ to $v = 0$ by $\tilde{F}(\lambda, 0) = 0$ and $\tilde{G}(\lambda, 0) = 0$, respectively. The hypotheses made for $\tilde{F}$ and $\tilde{G}$ imply $\tilde{F} : \mathbb{R} \times H \to H$ and $\tilde{G} : \mathbb{R} \times H \to H$ are continuous. The inversion $(\lambda, u) \to T(\lambda, u) = (\lambda, v)$ was used in the papers [6],[8] and [9] turns a "bifurcation at infinity" problem into a "bifurcation at zero" problem.

By (1.2) and (1.3) it follows from (3.2) that

$$||\tilde{F}(\lambda, v)|| = ||v||^2 ||F(\lambda, \frac{v}{||v||^2})|| \leq M||v||, \;\forall \lambda \in \mathbb{R}, \;\forall v \in H, \;||v|| < 1; \tag{3.3}$$

for any bounded interval $\Lambda \subset \mathbb{R}$,

$$\tilde{G}(\lambda, v) = o(||v||) \;\text{ at }\; v = 0, \tag{3.4}$$

uniformly with respect to $\lambda \in \Lambda$.

Since $\mu \in \sigma(L)$ is of odd multiplicity and the condition (2.4) holds, by (3.3) and (3.4), the component $\mathcal{C}_\mu$ of nontrivial solutions of problem (3.1) satisfied the alternatives of Theorem 2.2. Under the inversion $(\lambda, v) \to T^{-1}(\lambda, v) = (\lambda, u)$, $\tilde{\mathcal{C}}_\mu \to \mathcal{C}_\mu$ satisfying (1.1). If $\tilde{\mathcal{C}}_\mu$ contains the another bifurcation interval $I_{\hat{\mu}} \times \{0\}$, then $\tilde{\mathcal{C}}_\mu$ contains also the another bifurcation interval $I_{\hat{\mu}} \times \{\infty\}$. If $\tilde{\mathcal{C}}_\mu$ is unbounded in $\mathbb{R} \times H$, then we have two possible cases: (a) $P_R(\tilde{C}_\mu)$ is bounded in $\mathbb{R}$; (b) $P_R(\tilde{C}_\mu)$ is unbounded in $\mathbb{R}$. Consequently, in the case (a) C_μ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, and in the case (b) $P_R(C_\mu)$ is unbounded in $\mathbb{R}$. The proof of Theorem 3.1 is complete.

Now we give one application to a nonlinear eigenvalue problem for ordinary differential equations of second order.

Consider the following problem

$$-y'' = \lambda y + y + 1 + \lambda, \;\; 0 < x < \pi, \tag{3.5}$$

$$y(0) = 0 = y(\pi). \tag{3.6}$$

Let $H = L_2(0, \pi)$. Define the operators $L : D(L) \subset H \to H, \;\; F : H \to H$ and $G : \mathbb{R} \times H \to H$ as follows:

$$D(L) = \left\{ y \in H \mid y \in W_2^2(0, \pi), \;-y'' \in L_2(0, \pi), \;y(0) = 0 = y(\pi) \right\},$$

$$Ly = -y'', \;\; F(\lambda, y) = y + 1, \;\; G(\lambda, y) = \lambda. \tag{3.7}$$

Then the problem (3.5)-(3.6) can be written as an operator equation in the form of (1.1), i.e.

$$Ly = \lambda y + F(\lambda, y) + G(\lambda, y). \tag{3.8}$$

It is known that L is a semi-bounded from below self-adjoint operator in H and possesses infinitely many eigenvalues $\mu_k = k^2$, $k = 1, 2, \ldots$, all of which are simple. By (3.7) it follows that the conditions (1.2) and (1.3) are satisfied, and $M = 2$. Accordingly, $I_k = I_{\mu_k} = [k^2 - 2, k^2 + 2]$ for our problem. Then by Theorem 3.1 for each $k \in \mathbb{N}$ connected component $\mathcal{C}_k \equiv C_{\mu_k}$ of solutions of problem (3.5)-(3.6), containing $I_k \times \{\infty\}$ is either (i) contains the interval $I_s \times \{\infty\}$, where $s \neq k$ or (ii) contains the point $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$ or (iii) $P_R(\mathcal{C}_k)$ is unbounded.

On the other hand, for $\lambda \neq k^2 - 1,\ k \in \mathbb{N}$, the solution of problem (3.5)-(3.6) is unique and given by

$$u(x) = -1 + \cos\sqrt{\lambda + 1}x + \frac{1 - \cos\sqrt{\lambda}\pi}{\sin\sqrt{\lambda}\pi}\sin\sqrt{\lambda + 1}x\,.$$

For k odd, $u_\lambda(x) \to \infty$ as $\lambda \to k^2 - 1$; for k^2 even, $u_\lambda(x) \to -1 + \cos kx \equiv u_{k^2-1}$, and in addition to the solution $(k^2 - 1, u_{k^2-1})$, (3.5)-(3.6) possesses the family of solutions $(k^2 - 1, u_{k^2-1} + \alpha \sin kx)$, $\alpha \in \mathbb{R}$. Thus

$$\mathcal{C}_1 = \{(\lambda, u_\lambda) : \lambda \in (-1, 0) \cup (0, 8)\} \cup \{(3, u_3 + \alpha \sin 2x) : \alpha \in \mathbb{R}\} \cup$$

$$\cup \{(-1, 0)\} \cup (I_1 \times \{\infty\}) \cup (I_2 \times \{\infty\}) \cup (I_3 \times \{\infty\}).$$

Note that in each interval $I_k \times \{\infty\}$ there is only one bifurcation point $(k^2 - 1, \infty)$. Note also that $\mathcal{C}_1$ meets $(-1, 0)$ and meets $I_2 \times \{\infty\}$ as well as $I_3 \times \{\infty\}$, i.e. $\mathcal{C}_1$ does not satisfy (iii). Hence it also is seen that if $k \geq 2$ then $\mathcal{C}_k$ does not satisfy (ii) and (iii).

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