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Generalized Gegenbauer shift and some problems of the theory of approximation of functions on the metric of $L_{2,\lambda}$

Vagif S. Guliyev · **Elman J. Ibrahimov**

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Abstract. *In this paper we consider some problems of the theory of approximation of functions on interval* $[0,\infty)$ *in the metric of* $L_{2,\lambda}$ *with weight* $sh^{2\lambda}x$*. The modulus of continuity used in those problems is constructed with the help of generalized Gegenbauer shift operator. The direct Jakson type theorems are proved. The function spaces of Nikolski-Besov type associated with Gegenbauer differential operator* D_λ *are introduced and their descriptions in terms of best approximations are obtained.*

Keywords. Approximation of functions, generalized Gegenbauer shift, Gegenbauer transformations, the boundedness of spectrum functions.

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1 Introduction. Statement of main results

In the classical theory of approximation of functions on $\mathbb{R} = (-\infty;\infty)$ the classical shift operator $f(x) \mapsto f(x+y)$, $x, y \in \mathbb{R}$ plays a central role. In the approximation theory shift operators are used in the construction of the modulus of continuity and smoothness, which are the basic elements of the direct and inverse theorems. Various generalizations of shift operators enable to obtain natural analogues of problems in approximation theory. Groups and semigroups of operators on Banach spaces are generalizations of the shift operator. Many problems of this type in approximation theory were considered in [1, 2, 5, 6, 43]. These operators may not form a group or semigroup, but the generalized module of smoothness defined in terms of them can be better adapted for the study of relations between the smoothness properties of functions and the best approximations of these functions in weighted function spaces. Some results on the approximation of functions with the use of generalized shift operators can be found in [20, 34-40, 47]

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V.S. Guliyev
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ1141, Baku, Azerbaijan.
Ahi Evran University, Department of Mathematics, Kırsehir, Turkey
E-mail: vagif@guliyev.com

E.J. Ibrahimov
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzade str., AZ1141, Baku, Azerbaijan.
E-mail: elmanibrahimov@yahoo.com

(and the references in these). Note that most of the papers on this topic deal with the approximation of functions by polynomials on a finite line segment.

For the half-line, most popular shift operators are the generalized Bessel and Dunkl shift operators (see example [3, 29-33, 44, 46]). Fourier-Bessel and Fourier-Dunkl harmonic analysis, which are deal with the Bessel and Dunkl integral transformations and their approximations, are closely connected with the generalized Bessel and Dunkl shifts. Moreover, generalized Bessel shift is widely used in the potential theory (see example [4, 9-11]). Obtained results are analogues of the results for generalized Bessel shift obtained the works [29, 30]. Similar questions constrained with generalized Gegenbauer shift are considered in [12-14, 16-19]. The file of constructions of theory generalized shift operators generalize in the theory of transformation operators (see for example [7]). The references of quoting works is far from completion yet, perhaps essential supplement. We use only these works which have at least some relation to this paper. In this paper we consider the generalized Gegenbauer shift and study some questions of approximation theory of functions in the interval $[0,\infty)$ with the metric of $L_{2,\lambda}$ and the weight $sh^{2\lambda}x$. We describe our results in more detail now. Let

$$\begin{aligned} D_\lambda &= (x^2-1)^{\frac{1}{2}-\lambda}\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx} \\ &= (x^2-1)\frac{d^2}{dx^2}+(2\lambda+1)x\frac{d}{dx}. \end{aligned} \tag{1.1}$$

be the Gegenbauer differential operator. The functions (see [8], p. 1934)

$$\begin{aligned} P_\alpha^\lambda(x) &= \frac{\Gamma\,(\alpha+2\lambda)\;cos\pi\lambda}{\Gamma(\lambda)\,\Gamma(\alpha+\lambda+1)\,2^{\alpha+2\lambda}}\;x^{-\alpha-2\lambda} \\ &\times {}_2F_1\left(\frac{\alpha}{2}+\lambda,\frac{\alpha}{2}+\lambda+\frac{1}{2};\alpha+\lambda+1;\frac{1}{x^2}\right) \end{aligned} \tag{1.2}$$

and ([8], formulas (2.3) and (2.8), also [15] p. 1045, formulas 8.936 (1) and 8.932 (1))

$$C_\alpha^\lambda(x) = \frac{\Gamma(\alpha+\lambda)}{\Gamma(\lambda)\,\Gamma(\alpha+1)}(2x)^\alpha{}_2F_1\left(-\frac{\alpha}{2},-\frac{\alpha}{2}+\frac{1}{2};1-\alpha-\lambda;\frac{1}{x^2}\right), \tag{1.3}$$

where $\alpha,x\in[1,\infty)$, and $\lambda\in(0,\frac{1}{2})$, are eigen functions of the operator D_λ, and ${}_2F_1\,(\alpha,\ \beta;\ \gamma;\ x)$ is Gausss hypergeometric function. Note that the functions $P_\alpha^\lambda(x),\ C_\alpha^\lambda(x)$ are linearly independent solutions of the equations

$$\{(x^2-1)\frac{d^2}{dx^2}+(2\lambda+1)x\frac{d}{dx}-\alpha(\alpha+2\lambda)\}y(x)=0, \tag{1.4}$$

and

$$sh^2x\,y''\,(chx)+(2\lambda+1)\;chx\,y'\,(chx)-\alpha\,(\alpha+2\lambda)\;y(chx)=0. \tag{1.5}$$

For the functions $P_\alpha^\lambda(chx),\ C_\alpha^\lambda(chx)$ the formulas ([8], c. 1939)

$$\begin{aligned} &P_\alpha^\lambda(chxcht-shxsht\cos\varphi) \\ &= \frac{\Gamma(2\lambda-1)}{\Gamma^2(\lambda)}\sum_{n=0}^{[\alpha]}(-1)^n\frac{4^n\Gamma(\alpha-n+1)\Gamma^2(\lambda+n)(2n+2\lambda-1)}{\Gamma(\alpha+2\lambda+n)} \\ &\times sh^nxsh^ntP_{\alpha-n}^{\lambda+n}(chx)C_{\alpha-n}^{\lambda+n}(cht)C_n^{\lambda-\frac{1}{2}}(\cos\varphi), \end{aligned} \tag{1.6}$$

$$\begin{aligned} &C_\alpha^\lambda(chxcht-shxsht\cos\varphi) \\ &= \frac{\Gamma(2\lambda-1)}{\Gamma^2(\lambda)}\sum_{n=0}^{[\alpha]}(-1)^n\frac{4^n\Gamma(\alpha-n+1)\Gamma^2(\lambda+n)(2n+2\lambda-1)}{\Gamma(\alpha+2\lambda+n)} \\ &\times sh^nxsh^ntC_{\alpha-n}^{\lambda+n}(chx)C_{\alpha-n}^{\lambda+n}(cht)C_n^{\lambda-\frac{1}{2}}(\cos\varphi) \end{aligned} \tag{1.7}$$

are valid. Taking into account the equation ([15], c. 844)

$$\int_0^{\pi} C_n^{\lambda-\frac{1}{2}}(\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi = \begin{cases} 0, & n \geq 1, \\ \frac{\Gamma(\lambda)\Gamma(\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})}, & n = 0, \end{cases}$$

from (1.6) and (1.7) we obtain:

$$\begin{aligned} A_{cht}^{\lambda} P_{\alpha}^{\lambda}(chx) &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda)\Gamma(\frac{1}{2})} \int_0^{\pi} P_{\alpha}^{\lambda}(chxcht - shxsht\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi \\ &= \frac{\Gamma(2\lambda)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} P_{\alpha}^{\lambda}(cht)C_{\alpha}^{\lambda}(chx) = P_{\alpha}^{\lambda}(cht)Q_{\alpha}^{\lambda}(chx), \end{aligned} \tag{1.8}$$

$$\begin{aligned} A_{cht}^{\lambda} C_{\alpha}^{\lambda}(chx) &= \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^{\pi} C_{\alpha}^{\lambda}(chxcht - shxsht\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi \\ &= \frac{\Gamma(2\lambda)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} C_{\alpha}^{\lambda}(cht)C_{\alpha}^{\lambda}(chx) = C_{\alpha}^{\lambda}(cht)Q_{\alpha}^{\lambda}(chx), \end{aligned} \tag{1.9}$$

where $Q_{\alpha}^{\lambda}(chx) = \frac{\Gamma(2\lambda)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} C_{\alpha}^{\lambda}(chx)$.

Let $a_{\lambda} = \Gamma(\lambda+1/2)/(\Gamma(\lambda)\Gamma(1/2))$. Here the functions

$$\begin{aligned} A_{cht}f(chx) &\equiv A_{cht}^{\lambda}f(chx) \\ &= a_{\lambda}\int_0^{\pi} f(chxcht - shxsht\cos\varphi)(\sin\varphi)^{2\lambda-1}d\varphi \end{aligned} \tag{1.10}$$

are generalized Gegenbauer shift operators for $f \in L_{p,\lambda}, 1 \leq p \leq \infty$ (see [16]). Their existence are a result of the $L_{p,\lambda}$- boundedness (see section 2, property 5). The generalized Gegenbauer shift possesses much analogous properties to the generalized Bessel shift from the work of Levitan [23]. These properties are proved in Section 2. Note that the results of Sections 1 and 2 are obtained by scheme used in work [29]. For proof of direct Jackson theorem in [30] S.S. Platonov essentially used Lemma 3.5 whose the analogue does not take place for Gegenbauer function. Therefore we had to look for a new approach which is distinct from the method of S. S. Platonov. For this we had to prove series of auxiliary results of Lemmas 3.3-3.12, which present independent interest. For example in the Lemma 3.9 the formula for resting term of Teylor-Delsart formulas, which is different from analogous of formulas for generalized Bessel shift that is obtained by the other method of B. M. Levitan ([24], p. 124). Moreover, for proof of Theorem 1.1 in Section 4 we had to prove Lemmas 4.1-4.4, i.e., we find the another approach. Here we used the scheme of the proof in [30] for the results of Section 5.

Let $\mathbb{R}_+ := [0,\infty)$. We denote $C(\mathbb{R}_+)$ and $C_c(\mathbb{R}_+)$ the set of all even continuous functions on $\mathbb{R}$ and the set of continuous functions on $\mathbb{R}$ with compact supports, respectively. Let $C^{(k)}(\mathbb{R}_+)$ be the set of k-times differentiable even functions on $\mathbb{R}_+$, $D(\mathbb{R}_+)$ be the set of infinitely differentiable even functions on $\mathbb{R}_+$ with compact supports, and $D'(\mathbb{R}_+)$ be the set of all generalized even functions that is continuous linear functional on $D(\mathbb{R}_+)$.

The value of $f \in D'(\mathbb{R}_+)$ at $\varphi \in D(\mathbb{R}_+)$ will be denoted by $< f, \varphi >$. By $L_{2,\lambda} \equiv L_{2,\lambda}(\mathbb{R}_+)$ we denote the Hilbert space of measurable function f by $\mathbb{R}_+$ (defined up to their values on a set of measure zero), such that the norm

$$\|f\|_{2,\lambda} = \left(\int_0^{\infty} |f(chx)|^2 \, sh^{2\lambda}x dx\right)^{\frac{1}{2}}$$

is finite.

Scalar production in $L_{2,\lambda}$ is determined by the formula

$$(f,g) = \int\limits_0^\infty f(chx)\, g(chx)\, sh^{2\lambda} x dx, \quad f,g \in L_{2,\lambda}.$$

The space $L_{2,\lambda}$ is invested in $D'(\mathbb{R}_+)$ if for $f \in L_{2,\lambda}$ and $\varphi \in D(\mathbb{R})$ put
$< f, \varphi > := \int\limits_0^\infty f(chx)\, \varphi(chx)\, sh^{2\lambda} x dx.$

For any function $f \in L_{2,\lambda}$ with the help of generalized Gegenbauer shift, we denote the differences

$$\varDelta^1_{cht} f(chx) := A_{cht} f(chx) - f(chx), \ldots, \varDelta^k_{cht} f(chx) := \varDelta^1_{cht}\left(\varDelta^{k-1}_{cht} f(chx)\right), k = 2,3,...$$

or

$$\varDelta^k_{cht} f(chx) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} A^i_{cht} f(chx).$$

For all natural k we define the generalized modulus of continuity of k-order in the metric of $L_{2,\lambda}$ by the formula

$$\omega_k(f;\delta)_{2,\lambda} := \sup_{0<t\le\delta} \|\varDelta^k_{cht} f\|_{2,\lambda}\ , \ \ \delta > 0.$$

The best approximation function $f \in L_{2,\lambda}$ by functions belonging to I_ν (the definition of classes I_ν see on the page 32) is defined as

$$E_\nu(f)_{2,\lambda} = \inf_{g\in I_\nu} \|f - g\|_{2,\lambda}.$$

We extend the action of the differential operator D_λ in a natural way to the space of generalized functions $D'(\mathbb{R}_+)$ by putting

$$< D_\lambda f, \varphi > = < f, D_\lambda \varphi >, \quad f \in D'(\mathbb{R}_+)\,, \quad \varphi \in D(\mathbb{R}_+).$$

In particular, for every function $f \in L_{2,\lambda}$ generalized functions $D_\lambda f,\ D^2_\lambda f, \ldots,$ belonging $D'(\mathbb{R}_+)$ are defined.

The following theorem is an analogue of Jackson's direct theorem in classical approximation theory.

Theorem 1.1. *Suppose that* $f,\ D_\lambda f, \ldots, D^s_\lambda f$ *belong to* $L_{2,\lambda}$. *Then*

$$E_\nu(f)_{2,\lambda} \le 2^{-s}\Big(sh\frac{1}{\nu}\Big)^{2s} \omega_n\Big(D^s_\lambda f, \frac{1}{\nu}\Big)_{2,\lambda} \sim 2^{-s}\nu^{-2s}\omega_n\Big(D^s_\lambda f, \frac{1}{\nu}\Big)_{2,\lambda}, \quad \nu \to \infty.$$

Let $r > 0$ be a real number and let k and s be arbitary non-negative numbers such that $2k > r - 2s > 0$. We denote by $H^r_{2,\lambda}$ the set of all $f \in L_{2,\lambda}$ for which $D_\lambda f,\ D^2_\lambda f,\ \ldots, D^s_\lambda f \in L_{2,\lambda}$ and the inequality is valid.

$$\omega_k\left(D^s_\lambda f, \delta\right)_{2,\lambda} \le A_f \delta^{r-2s}, \ \ \delta > 0 \tag{1.11}$$

for some $A_f > 0$. For $f \in H^r_{2,\lambda}$ we define the seminorm $h^r_{2,\lambda}(f)$ as

$$h^r_{2,\lambda}(f) = \sup_{\delta>0} \frac{\omega_k\left(D^s_\lambda f, \delta\right)_{2,\lambda}}{\delta^{r-2s}}. \tag{1.12}$$

$H^r_{2,\lambda}$ is a Banach space with the norm (see the section 5)

$$\|f\|_{H^r_{2,\lambda}} := \|f\|_{2,\lambda} + h^r_{2,\lambda}(f).$$

In the following theorem we describe the space $H^r_{2,\lambda}$ in terms of the best approximation by functions belonging to I_ν. This theorem implies that the $L^r_{2,\lambda}$ does not depend on k and s.

We denote by $C_1,\ C_2,\ldots$ positive constants that do not depend on f but they can depend on $k,\ r,\ s,\ \lambda$.

Theorem 1.2. If $f\in H^r_{2,\lambda}$, then for $\nu\geq 1$ the inequality

$$E_\nu(f)_{2,\lambda}\leq C_{\lambda,k,s}\frac{h^r_{2,\lambda}(f)}{\nu^r} \tag{1.13}$$

is valid. Conversely, if $f\in L_{2,\lambda}$ and $\nu\geq 1$

$$E_\nu(f)_{2,\lambda}\leq\frac{A_f}{\nu^r}, \tag{1.14}$$

where A_f is a constant that does not depend on ν (but depends on f) then $f\in H^r_{2,\lambda}$ and

$$\|f\|_{H^r_{2,\lambda}}\leq C_1\left(\|f\|_{2,\lambda}+A_f\right). \tag{1.15}$$

Let $1\leq q\leq\infty,\ r>0$, and let k,s be non-negative integers such that $2k>r-2s>0$. As in [30] we say that a function f belongs to the Nikolskii-Besov class $B^r_{2,\,q,\,\lambda}$ associated with the Gegenbauer differential operator D_λ if $D_\lambda f,\ \ldots,D^s_\lambda f\in L_{2,\lambda}$ and the norm

$$b^r_{2,\,q,\,\lambda}(f)=\begin{cases}\left(\int\limits_0^\infty\frac{\left(\omega_k(D^s_\lambda f,\delta)_{2,\lambda}\right)^q}{\delta^{(r-2s)q}}\frac{d\delta}{\delta}\right)^{\frac{1}{q}} & q<\infty\ ,\\ \sup\limits_{\delta>0}\frac{\omega_k(D^s_\lambda f,\delta)_{2,\lambda}}{\delta^{r-2s}}, & \text{for}\quad q=\infty\,.\end{cases}$$

is finite.

The class $B^r_{2,\,q,\,\lambda}$ is a Banach space with the norm

$$\|f\|_{B^r_{2,\,q,\,\lambda}}:=\|f\|_{2,\lambda}+b^r_{2,\,q,\,\lambda}. \tag{1.16}$$

Note that $B^r_{2,\,\infty,\,\lambda}=H^r_{2,\lambda}$.

Theorem 1.3. *Let $a>1$ be an arbitary number (we can take, for example, $a=2$). Then $f\in L_{2,\,\lambda}$ belongs to $B^r_{2,\,q,\,\lambda}$ if and only if the seminorm*

$$\tilde{b}^r_{2,\,q,\,\lambda}(f):=\begin{cases}\left(\sum\limits_{n=0}^\infty a^{nrq}\left(E_{a^n}(f)_{2,\lambda}\right)^q\right)^{\frac{1}{q}}, & q<\infty,\\ \sup\limits_{n\in Z_+}a^{nr}E_{a^n}(f)_{2,\lambda}, & q=\infty,\end{cases}$$

is finite, where $Z=\{0,1,2,\ldots\}$. In this case the norm (1.16) in $B^r_{2,\,q,\,\lambda}$is equivalent to the norm

$$\|f\|_{2,\lambda}+\tilde{b}^r_{2,\,q,\,\lambda}(f)\,.$$

2 Transformations and generalized Gegenbauer shift

Here we reduce some information on Gegenbauer transformations and generalized Gegenbauer shift. The Gegenbauer transformations of the functions $P^\lambda_\alpha(cht)$ and $Q^\lambda_\alpha(cht)$ are direct P-transformation called the following integrals transformations [16]:

$$F_P:f(cht)\mapsto\widehat{f}_P(\alpha)=\int\limits_0^\infty f(cht)P^\lambda_\alpha(cht)sh^{2\lambda}tdt, \tag{2.1}$$

inverse P-transformation

$$F_P^{-1}:\widehat{f}_P(\alpha)\mapsto f(chx)=C_\lambda^*\int\limits_1^\infty \widehat{f}_P(\alpha)Q_\alpha^\lambda(chx)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}}d\alpha\,, \tag{2.2}$$

and direct Q-transformation

$$F_Q: f(cht)\mapsto \widehat{f}_Q(\alpha)=\int\limits_0^\infty f(cht)Q_\alpha^\lambda(cht)sh^{2\lambda}tdt,$$

inverse Q-transformation

$$F_Q^{-1}:\widehat{f}_Q(\alpha)\mapsto f(chx)=C_\lambda^*\int\limits_1^\infty \widehat{f}_Q(\alpha)P_\alpha^\lambda(chx)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}}d\alpha\,, \tag{2.3}$$

where $Q_\alpha^\lambda(chx)=\frac{\Gamma(2\lambda)\,\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)}C_\alpha^\lambda(chx)$ and

$$C_\lambda^*=\frac{2^{\frac{3}{2}-\lambda}\Gamma(\frac{1}{2})\Gamma(\lambda+1)\Gamma\left(\frac{1}{2}-\lambda\right)\Gamma(\frac{3+2\lambda}{4})\Big(\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{5-2\lambda}{4})cos\pi\lambda\Big)^{-1}}{{}_2F_1(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1}{2})-{}_2F_1(1,\frac{1}{2}-\lambda;\frac{5-2\lambda}{4};\frac{1-2\lambda}{2})}.$$

For $f\in D(\mathbb{R}_+)$ the transformations (2.1) and (2.2) is defined.

We note that, if direct and inverse Bessel transformations differ only by a numerical factor (see [29]), then the transformation (2.1) and (2.2) have different constructions. But they are natural and dictate of formula (1.6) (ground see in [16]).

In [16] (Lemma 8) it is proved that if $f\in L_{1,\lambda}\bigcap L_{2,\lambda}$, then $\widehat{f}_P(\alpha)\widehat{f}_Q(\alpha)\in L_{1,\lambda}$ and the equality

$$\int_0^\infty f^2(chx)\,sh^{2\lambda}x\,dx=C_\lambda^*\int_1^\infty \widehat{f}_P(\alpha)\widehat{f}_Q(\alpha)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}}d\alpha \tag{2.4}$$

is valid.

For the formula (1.10) the operator A_{cht} is spread for even continuous functions, in particular on the functions in $C_c\,[0,\infty)$. We have the following

$$|A_{cht}f(cht)|^2\le A_{chx}\left(|f(chx)|^2\right). \tag{2.5}$$

By Cauchy-Bunyakowski inequality we have

$$|A_{cht}f(chx)|^2=\left|a_\lambda\int_0^\pi f\left(chx\,cht-shx\,sht\cos\varphi\right)\,(\sin\varphi)^{2\lambda-1}\,d\varphi\right|^2$$

$$\le a_\lambda\int_0^\pi\left|f\left(chx\,cht-shx\,sht\cos\varphi\right)\right|^2(\sin\varphi)^{2\lambda-1}\,d\varphi\cdot a_\lambda\int_0^\pi(\sin\varphi)^{2\lambda-1}\,d\varphi$$

$$=a_\lambda\int_0^\pi\left|f\left(chx\,cht-shx\,sht\cos\varphi\right)\right|^2(\sin\varphi)^{2\lambda-1}\,d\varphi=A_{cht}\left(|f(chx)|^2\right).$$

We remind that the operator A_{cht} is self-adjoint (see property 8). In particular, if $f,g\in C(\mathbb{R}_+)$, moreover $f\in L_{1,\lambda}$ and g is bounded, then (2.9) takes place. The equality (2.9) takes place also, if $f\in C(\mathbb{R}_+)$ and $g\in C_c(\mathbb{R}_+)$. For this, it is enough to make the sequence of the functions $f_n\in C(\mathbb{R}_+)$, which is convergence uniformly to f on every segment and in the equality

$$\int_0^\infty A_{cht}f_n(chx)g(cht)sh^{2\lambda}tdt=\int_0^\infty f_n(cht)\,A_{cht}g(chx)sh^{2\lambda}tdt$$

the limit is taken for $n\to\infty$.

We note that

$$\|A_{cht}f\|_{2,\lambda} \le \|f\|_{2,\lambda} \tag{2.6}$$

for $f \in C_c\ [0,\infty)$.

In fact, in (2.4) and (2.6) for $g(chx) \equiv 1$ we have

$$\|A_{cht}f\|^2_{2,\lambda} = \int_0^\infty |A_{cht}f(chx)|^2\, sh^{2\lambda}xdx$$
$$= \int_0^\infty \left(A_{cht}\,|f(chx)|^2\right) sh^{2\lambda}xdx \le \int_0^\infty |f(chx)|^2\, sh^{2\lambda}xdx = \|f\|^2_{2,\lambda}\,.$$

From inequality (2.6) it follows that the operator A_{cht} is continuous on $C_c\ [0,\infty)$ and is a bounded operator on $L_{2,\lambda}$. Lasting operator we will also denote A_{cht} and for the inequality (2.6) remains true. The generalized Gegenbauer shift has the following properties.

1) Linearity:

$$A_{cht}\{af(chx) + bg(chx)\} = aA_{cht}f(chx) + bA_{cht}g(chx),$$

which follows from the integral property.

2) Positivity: $A_{cht}f(chx) \ge 0$, if $f(chx) \ge 0$, which is obvious.

3) $A_{cht}1 \equiv 1$.

4) If $f(chx) \equiv 0$ for $x \ge a$, then $A^\lambda_{cht}f(chx) \equiv 0$ for $|x-t| \ge a$.

In fact, $chxcht - shxsht\cos\varphi \ge ch(x-t) \ge |x-t|$, from this it follows the property 4).

For every $1 \le p < \infty$ we denote by $L_{p,\lambda}$ the space of measurable functions on R_+ (defined up to their on a set of measure zero) such that the norm $\|f\|_{p,\lambda} = \left(\int\limits_0^\infty |f(chx)|^p sh^{2\lambda}xdx\right)^{\frac{1}{p}}$ is finite. For $p = \infty$ we denote by $L_{\infty,\lambda}$ the set of all functions f that are uniformly continuous and bounded on $[0,\infty)$. The norm in $L_{\infty,\lambda}$ is defined by the formula $\|f\|_{\infty,\lambda} := \sup\limits_{x\ge 0}|f(chx)|$.

5) $L_{p,\lambda}$ boundedness of the operator A^λ_{cht} : For any $f \in L_{p,\lambda}, 1 \le p \le \infty$ the following inequality is valid

$$\|A_{cht}f\|_{p,\lambda} \le \|f\|_{p,\lambda}.$$

Corollary. *The operator A^λ_{cht} is continuous on $L_{p,\lambda}$. This follows from the properties 1) and 5).*

Let f_n be an arbitrary sequence, such that $\|f_n - f\|_{p,\lambda} \to 0$ as $n \to \infty$, ($f \in L_{p,\lambda}, 1 \le p \le \infty$), then

$$\|A_{cht}f_n - A_{cht}f\|_{p,\lambda} = \|A_{cht}(f_n - f)\|_{p,\lambda} \le \|f_n - f\|_{p,\lambda} \to 0$$

as $n \to \infty$, i.e., the operator A_{cht} is continuous on $L_{p,\lambda}$

6) Symmetry of the operator A_{cht}:

$$A_{cht}f(chx) = A_{chx}f(cht)$$

is obvious.

7) Commutativity of the operator A_{cht}: For every continuous functions $f(chx)$, $x \in [0,\infty)$ and $y, t \ge 0$ the following equality is valid

$$A_{chy}A_{cht}f(chx) = A_{cht}A_{chy}f(chx). \tag{2.7}$$

In fact, let $f(chx) = C^\lambda_\alpha(chx)$, from (1.9) and property 6) we have

$$A_{cht}A_{chy}C^\lambda_\alpha(chx) = A_{cht}C^\lambda_\alpha(chy)Q^\lambda_\alpha(chx)$$

$$= A^\lambda_{chy}C^\lambda_\alpha(cht)Q^\lambda_\alpha(chx) = A^\lambda_{chy}A^\lambda_{cht}C^\lambda_\alpha(chx).$$

For

$$\sigma_n(chx) = \sum_{k=1}^{n} a_k C_k^{\lambda}(chx) \tag{2.8}$$

the equality (2.6) follows from the property 1) of the operatorA_{cht}. If we take into account (1.3) and (2.7) and suppose $chx = u$, then sum (2.7) passes to be an $n-$ order algebraic polynomial. But then according to the Weierstrass theorem (see for example [26], p. 19) every continuous functions one can be approximated uniformly on any segment of sums (2.7), i.e.

$$\lim_{n\to\infty} \sigma_n(chx) = \sigma(chx).$$

From (2.6) and (2.7) it follows that

$$A_{chy}A_{cht}\sigma_n(chx) = \sum_{k=1}^{n} a_k A_{chy}A_{cht}C_k^{\lambda}(chx)$$

$$= \sum_{k=1}^{n} a_k A_{cht}A_{chy}C_k^{\lambda}(chx).$$

From this we have

$$\lim_{n\to\infty} A_{chy}A_{cht}\sigma_n(chx) = A_{chy}A_{cht}\sigma(chx)$$

$$= \sum_{k=1}^{\infty} a_k A_{chy}A_{cht}C_k^{\lambda}(chx) = A_{cht}A_{chy}\sigma(chx).$$

The property 7) is proved.

8) The operator A_{cht}^{λ} is self-adjoind (see [16], lemma 3). For $f, g \in L_{1,\lambda}$ the following equality is valid

$$\int_0^{\infty} A_{cht}f(chx)g(cht)sh^{2\lambda}tdt = \int_0^{\infty} A_{cht}g(chx)f(cht)sh^{2\lambda}tdt \tag{2.9}$$

for almost all $x \in [0, \infty)$.

Lemma 2.1. *([16], lemma 1). Let $f \in L_{1,\lambda}$, then*

$$\left(\widehat{A_{cht}f}\right)_P(\alpha) = \widehat{f}_P(\alpha)\, Q_{\alpha}^{\lambda}(cht).$$

The convolution of functions $f, g \in L_{1,\lambda}$ on $[0, \infty)$ is defined by the relation

$$(f * g)\ (chx) = \int_0^{\infty} g(cht)A_{cht}f(chx)sh^{2\lambda}tdt\ . \tag{2.10}$$

The convolution exists for almost all $x \in [0, \infty)$, moreover $f * g \in L_{1,\lambda}$ and in particular, if $f, g \in C_c\,[0, \infty)$, then the convolution $f * g \in C_c\,[0, \infty)$.

Lemma 2.2. *For $f, g \in C_c(R_+)$ the following equalities are valid*

$$a)\,(f * g)\ (chx) = (g * f)\,(chx),$$

$$b)\left(\widehat{f * g}\right)_P (chx) = \widehat{f}_P(chx)\widehat{g}_Q(chx). \tag{2.11}$$

Proof. The property a) immediately follows from (2.8). We will prove b). From (2.1) and (2.10) we have

$$\left(\widehat{f*g}\right)_P(\alpha) = \int_0^\infty (f*g)(chx)P_\alpha^\lambda(chx)sh^{2\lambda}xdx$$
$$= \int_0^\infty \left(\int_0^\infty g(chu)A_{chx}f(chu)sh^{2\lambda}udu\right)P_\alpha^\lambda(chx)sh^{2\lambda}xdx.$$

By changing the order of integration we obtain

$$\left(\widehat{f*g}\right)_P(\alpha) = \int_0^\infty \left(\int_0^\infty A_{chx}f(chu)P_\alpha^\lambda(chx)sh^{2\lambda}xdx\right)g(chu)sh^{2\lambda}udu$$
$$= \int_0^\infty (\widehat{A_{chx}f})_P(\alpha)g(chu)sh^{2\lambda}udu.$$

Taking into account Lemma 2.1, we have

$$\left(\widehat{f*g}\right)_P(\alpha) = \hat{f}_P(\alpha)\int_0^\infty g(chu)Q_\alpha^\lambda(chu)sh^{2\lambda}udu = \hat{f}_P(\alpha)\hat{g}_Q(\alpha).$$

Lemma 2.2 is proved.

3 The functions with bounded spectrum and their properties

Definition. *We call the function $f \in L_{2,\lambda}$ with bounded spectrum of order ν, if $\widehat{f}_P(\alpha) = 0$ for $\alpha > \nu$. The class of such functions we denote I_ν.*

The functions of class I_ν will be used as approximation tools. We consider some properties of these functions.

For any functions $f \in L_{2,\lambda}$ and $g \in C_c[0,\infty)$ we define the convolution $f*g$ and moreover

$$\|f*g\|_{2,\lambda} \le \|f\|_{2,\lambda}\|g\|_{1,\lambda},$$

in particular $f*g \in L_{2,\lambda}$. In fact, using the generalized Minkovsky inequality and the property (2.5), we obtain

$$\|f*g\|_{2,\lambda} \le \int_0^\infty \|A_{cht}f\|_{2,\lambda}|g(cht)|\ sh^{2\lambda}tdt$$

$$\le \|f\|_{2,\lambda}\int_0^\infty |g(cht)|\ sh^{2\lambda}tdt = \|f\|_{2,\lambda}\|g\|_{1,\lambda}.$$

Lemma 3.1. *For every $f \in L_{2,\lambda}[1,\infty)$ the inequality is valid.*

$$|\hat{f}_P(\alpha)| \le C_\lambda\alpha^{\lambda-3/2}\|f\|_{2,\lambda}.$$

We denote by C_λ a positive constant, which depend only on copied out indexes generally speaking are different in different formulas.

Proof. Since the appointed values of the parameters ([15], p. 1054), the Gauss's hypergeometric function ${}_2F_1(\frac{\alpha}{2}+\lambda, \frac{\alpha}{2}+\lambda+\frac{1}{2}; \alpha+\lambda+1; x^{-2}) > 0$ is convergences uniformly for all $x \in [1, \infty)$, then from (1.2) it follows that $\lim\limits_{x\to\infty} P_\alpha^\lambda(x) = 0$ and consequently, $P_\alpha^\lambda(x)$ accepts greatest value in the point $x = 1$. But taking into account the equality ([15], c. 1056)

$$ {}_2F_1(\frac{\alpha}{2}+\lambda, \frac{\alpha}{2}+\lambda+\frac{1}{2}; \alpha+\lambda+1; 1) = \frac{\Gamma(\alpha+\lambda+1)\Gamma(\frac{1}{2}-\lambda)}{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac{1}{2})}, $$

and from (1.2) we obtain

$$ \max_{x\in[1,\infty)} P_\alpha^\lambda(x) = P_\alpha^\lambda(1) = \frac{\Gamma(\alpha+2\lambda)\Gamma(\frac{1}{2}-\lambda)\cos\pi\lambda}{2^{\alpha+2\lambda}\Gamma(\lambda)\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac{1}{2})}. $$

Taking into account the doubling formula ([15], p. 952)

$$ \Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma\Big(x+\frac{1}{2}\Big) \,/\, \Gamma\Big(\frac{1}{2}\Big), $$

we obtain

$$ P_\alpha^\lambda(1) = \frac{\Gamma(\frac{1}{2}-\lambda)\Gamma(\frac{\alpha}{2}+\lambda)\Gamma(\frac{\alpha}{2}+\lambda+\frac{1}{2})\cos\pi\lambda}{2\Gamma(\lambda)\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{1}{2})}. $$

Using the relation ([15], p. 951)

$$ \lim_{\alpha\to\infty} \frac{\Gamma(\alpha+\lambda)}{\alpha^\lambda\Gamma(\alpha)} = 1 $$

we will have

$$ \lim_{\alpha\to\infty} P_\alpha^\lambda(1)\alpha^{1-2\alpha} = \frac{\Gamma(\frac{1}{2}-\lambda)\cos\pi\lambda}{4^\lambda\Gamma(\lambda)\Gamma(\frac{1}{2})}, $$

consequently

$$ P_\alpha^\lambda(chx) \le C_\lambda \alpha^{2\lambda-1}, x \in [0,\infty). \tag{3.1} $$

On the other hand from (1.2) we have

$$ P_\alpha^\lambda(chx) \le C_\lambda \alpha^{\lambda-1}\,(chx)^{-\alpha-2\lambda}, x \in (0,\infty). \tag{3.2} $$

By Hölder inequality we have

$$ \begin{aligned} |\hat{f}_P(\alpha)| &\le \int_1^\infty |f(x)||P_\alpha^\lambda(x)|(x^2-1)^{\lambda-\frac{1}{2}}\,dx \le \|f\|_{2,\lambda}\Big(\int_0^\infty (P_\alpha^\lambda(chx))^2 sh^{2\lambda}x\,dx\Big)^{\frac{1}{2}} \\ &= \|f\|_{2,\lambda}\Big(\int_0^{1/\alpha}(P_\alpha^\lambda(chx))^2 sh^{2\lambda}x\,dx + \int_{1/\alpha}^\infty (P_\alpha^\lambda(chx))^2 sh^{2\lambda}x\,dx\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.3} $$

Using (3.1) we obtain

$$ \begin{aligned} \int_0^{1/\alpha} (P_\alpha^\lambda(chx))^2 sh^{2\lambda}x\,dx &\le C_\lambda\alpha^{4\lambda-2}\int_0^{1/\alpha} sh^{2\lambda}x\,dx \\ &\le C_\lambda\alpha^{4\lambda-3}sh^{2\lambda}\frac{1}{\alpha} \le C_\lambda\alpha^{2\lambda-3}sh^{2\lambda}1 = C_\lambda\alpha^{2\lambda-3}. \end{aligned} \tag{3.4} $$

By using (3.2) we have

$$\int\limits_{1/\alpha}^{\infty} (P_\alpha^\lambda(chx))^2 sh^{2\lambda}x\,dx \le C_\lambda \alpha^{2\lambda-2} \int\limits_{1/\alpha}^{\infty} \frac{sh^{2\lambda}x\,dx}{(chx)^{2\alpha+4\lambda}}$$

$$\le C_\lambda \alpha^{2\lambda-2} \int\limits_{1/\alpha}^{\infty} \frac{dx}{(chx)^{2\alpha+2\lambda}} \le C_\lambda \alpha^{2\lambda-2} \int\limits_{1/\alpha}^{\infty} e^{-2(\alpha+\lambda)x} dx$$

$$= C_\lambda \frac{\alpha^{2\lambda-2}}{2(\alpha+\lambda)} e^{-2(\alpha+\lambda)x} \mid_{\infty}^{1/\alpha} \le C_\lambda \alpha^{2\lambda-3}. \tag{3.5}$$

Using (3.4) and (3.5) in (3.3), we obtain the assertion of Lemma 3.1.

Lemma 3.2. *Let $f \in L_{2,\lambda} \cap L_{1,\lambda}$. In order the reflection $g \mapsto f * g$ on $C_c[0,\infty)$ to extend the continuous reflection from $L_{2,\lambda}$ into $L_{2,\lambda}$ necessary and sufficient, that the functions $\widehat{f}_Q(x)$ and $\widehat{f}_P(x)$ are essentially bounded on a $[1,\infty)$, i.e. $\widehat{f}_Q$, $\widehat{f}_P \in L_\infty[1,\infty)$.*

Proof. Since $C_c[0,\infty)$ is dense in the space $L_{2,\lambda}$ it follows that the equality b) and (2.10) is valid and for $f \in L_{2,\lambda}$, $g \in C_c[0,\infty)$. From equalities (2.4) and (2.10) it follows that for $f, g \in L_{2,\lambda} \cap L_{1,\lambda}$

$$\|f * g\|_{2,\lambda}^2 = \int_1^\infty \widehat{f}_P(\alpha)\widehat{g}_P(\alpha)\widehat{f}_Q(\alpha)\widehat{g}_Q(\alpha) \left(\alpha^2 - 1\right)^{\lambda-\frac{1}{2}} d\alpha. \tag{3.6}$$

Suppose that the reflection $g \mapsto f * g$ from $C_c[0,\infty)$ into $L_{2,\lambda}$ continues to continuous reflection from $L_{2,\lambda}$ into $L_{2,\lambda}$, which denote $g \mapsto f * g$ $\left(g \in L_{2,\lambda}\right)$. We check that $\widehat{(f * g)}_P(x) = \widehat{f}_P(x)\widehat{g}_Q(x)$ for all f, $g \in L_{2,\lambda} \cap L_{1,\lambda}$.

For every $g \in L_{2,\lambda}$ there is a sequence $g_n \in C_c[0,\infty)$ which converges to g in $L_{2,\lambda}$. Then $f * g_n \to f * g$ in $L_{2,\lambda}$, according to Young inequality we have

$$\|f * g_n - f * g\|_{2,\lambda} = \|f * (g_n - g)\|_{2,\lambda} \le \|g_n - g\|_{2,\lambda} \|f\|_{1,\lambda} \to 0 \text{ as } n \to \infty.$$

Let $f \in L_{1,\lambda} \cap L_{2,\lambda}$, $g \in L_{2,\lambda}$, $g_n \in C_c[0,\infty)$. We will show that $\left(\widehat{f * g_n}\right)_P \to \left(\widehat{f * g}\right)_P$ in $L_{2,\lambda}$. By Lemma 3.1 and Young inequality we have

$$\left\|\left(\widehat{f * g_n}\right)_P - \left(\widehat{f * g}\right)_P\right\|_{2,\lambda}^2 = \left\|\left(f * \widehat{(g_n - g)}\right)_P\right\|_{2,\lambda}^2$$

$$= \int_0^\infty (f * \widehat{(g_n - g)})_P^2(\alpha)\, sh^{2\lambda}\alpha\, d\alpha = C_\lambda \| f * (g_n - g) \|_{2,\lambda}^2 \int_1^\infty \frac{\left(\alpha^2 - 1\right)^{\lambda-\frac{1}{2}}}{\alpha^{3-2\lambda}} d\alpha$$

$$\le C_\lambda \| g_n - g \|_{2,\lambda}^2 \|f\|_{2,\lambda}^2 \int_1^\infty \frac{\left(\alpha^2 - 1\right)^{\lambda-\frac{1}{2}}}{\alpha^{3-2\lambda}} d\alpha$$

$$= C_\lambda \| g_n - g \|_{2,\lambda}^2 \|f\|_{2,\lambda}^2 \int_0^\infty \frac{\alpha^{\lambda-\frac{1}{2}} d\alpha}{(\alpha+1)^{2-\lambda}}$$

$$= C_\lambda \| g_n - g \|_{2,\lambda}^2 \|f\|_{2,\lambda}^2 \frac{\Gamma\left(\lambda + \frac{1}{2}\right) \Gamma\left(2 - 2\lambda\right)}{\Gamma\left(\frac{5}{2} - \lambda\right)}.$$

From this, it follows that

$$\left\|\left(\widehat{f * g_n}\right)_P - \left(\widehat{f * g}\right)_P\right\|_{2,\lambda} \le C_\lambda \|g_n - g\|_{2,\lambda} \|f\|_{1,\lambda} \to 0 \text{ for } n \to \infty.$$

Further, on (2.10)

$$\left(\widehat{f * g_n}\right)_P (\alpha) = (\widehat{g}_n)_P(\alpha) \widehat{f}_Q(\alpha).$$

We will show that $\widehat{(g_n)}_P(\alpha) \to \widehat{g}_P(\alpha)$ as $n \to \infty$. By Lemma 3.1

$$\left|\widehat{(g_n)}_P(\alpha) - \widehat{g}_P(\alpha)\right| \le C_\lambda \alpha^{\lambda - \frac{3}{2}} \|g_n - g\|_{2,\lambda} \to 0 \text{ for } n \to \infty.$$

From this, it follows that $\widehat{f}_Q(\alpha)\,\widehat{(g_n)}_P(\alpha) \to \widehat{f}_Q(\alpha)\,\widehat{g}_P(\alpha)$ and consequently $\widehat{(f*g)}_P(\alpha) = \widehat{f}_Q(\alpha)\,\widehat{g}_P(\alpha)$. Therefore the operator multiplication by function $\widehat{f}_Q(\alpha)$ is a continuous operator on a $L_{2,\lambda}$, for this it is necessary that $\widehat{f}_P \in L_\infty\,[1,\infty)$, $\widehat{f}_Q \in L_\infty\,[1,\infty)$. Then (3.6) has the form

$$\|f * g\|_{2,\lambda}^2 = \int_1^\infty \widehat{(f*g)}_P^{\,2}(\alpha)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}} d\alpha\ .$$

Conversely, if $\widehat{f}_P\,,\ \widehat{f}_Q \in L_\infty\,[1,\infty)$, from (3.6) and equality (2.4) it follows that for $g \in C_c\,[0,\infty)$

$$\begin{aligned}
\|f * g\|_{2,\lambda}^2 &= \int_1^\infty \widehat{f}_P(\alpha)\,\widehat{g}_P(\alpha)\,\widehat{f}_Q(\alpha)\widehat{g}_P(\alpha)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}} d\alpha \\
&\le \|\widehat{f}_P\|\|\widehat{f}_Q\| \int_1^\infty \widehat{g}_P(\alpha)\,\widehat{g}_P(\alpha)\,\left(\alpha^2-1\right)^{\lambda-\frac{1}{2}} d\alpha \\
&= \|\widehat{f}_P\|_\infty\|\widehat{f}_Q\|_\infty \int_0^\infty g^2(chx) sh^{2\lambda}x dx = \|\widehat{f}_P\|_\infty\|\widehat{f}_Q\|_\infty\|g\|_{2,\lambda}^2,
\end{aligned}$$

from this it follows that

$$\|f*g\|_{2,\lambda} \le \sqrt{\|\widehat{f}_P\|_\infty\|\widehat{f}_Q\|_\infty}\|g\|_{2,\lambda}, \tag{3.7}$$

where

$$\|\widehat{f}_P\|_\infty = \operatorname*{ess\,sup}_{\alpha\in[1,\infty)} |\widehat{f}_P(\alpha)|.$$

From (3.7) it follows that the operator $g \mapsto f*g$ continues to the continuous operator from $L_{2,\lambda}$ into $L_{2,\lambda}$. We note that equality (2.11) remains true for any $g \in L_{2,\lambda}$. Lemma 3.2 is proved.

We consider the function

$$C_k(chx) = -\int_1^{chx} \theta\,(chx,\sigma)\,C_{k-1}\,(\sigma)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma\ ,\ C_0 = 1,\ \ k = 1,2,\ldots$$

where

$$\theta\,(chx,\sigma) = \begin{cases} -\int_\sigma^{chx}\left(u^2-1\right)^{-\lambda-\frac{1}{2}}\,du\,, & 1<\sigma<chx\,, \\ 0, & \sigma \ge chx\,, \end{cases}$$

$$R_1\,(chs)\,f(chx) = \int_1^{chs} \theta\,(chs,\sigma)\,\left(A_\sigma D_\lambda f\right)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma, \tag{3.8}$$

and

$$R_k\,(chs)\,f(chx) = \int_1^{chs} \theta\,(chs,\sigma)\,\left(R_{k-1}\,(\sigma)\,D_\lambda f\right)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma\,,\ k = 2,3,\ldots \tag{3.9}$$

Denote by $D_\lambda^{(k)}(0,\infty)$ the class of functions which k-time the operator D_λ is applied.

Lemma 3.3. [12] *If $f \in D_\lambda^{(n-1)}(0,\infty)$, then the Taylor-Delsartes formula is valid*

$$R_n\,(chs)\,f(chx) = A_{cht}f(chx) - \sum_{k=0}^{n-1} C_k\,(chs)\;D_\lambda^k f(chx)\ .$$

From this formula we will construct the function (approximating aggregate) $P_\nu^{0,1} f(chx) = A_{ch\frac{1}{\nu}} f(chx)$,

$$P_\nu^{s,n} f(chx) = A_{ch\frac{1}{\nu}}^{n+s} f(chx) - \sum_{k=1}^{n+s-1} C_k \left(ch\frac{1}{\nu}\right) \left(A_{ch\frac{1}{\nu}}^{n+s-1} D_\lambda^k f\right)(chx) , \quad n = 1, 2, \ldots , \tag{3.10}$$

where

$$A_{cht}^0 f(chx) = f(chx), \; A_{cht}^n f(chx) = A_{cht} \left(A_{cht}^{n-1} f\right)(chx) , \quad n = 1, 2, \ldots .$$

Further we show that the function (3.10) is a function with bounded spectrum attached to certain conditions applying on the function f. For this we need some auxiliary approvals.

Lemma 3.4. *Let $f \in L_{2,\lambda}$. Then the following equality is valid.*

$$\left(\widehat{A_{cht}^k f}\right)_P (\alpha) = \widehat{f}_P(\alpha) \left(Q_\alpha^\lambda(cht)\right)^k , \quad k = 1, 2, \ldots .$$

In fact, according to Lemma 2.1 we can write the next chain of equalities

$$\left(\widehat{A_{cht}^k f}\right)_P (\alpha) = \left(A_{cht} \widehat{\left(A_{cht}^{k-1} f\right)}\right)_P (\alpha) = \left(\widehat{A_{cht}^{k-1} f}\right)_P (\alpha) \, Q_\alpha^\lambda(cht)$$

$$= \left(\widehat{A_{cht}^{k-2} f}\right)_P (\alpha) \left(Q_\alpha^\lambda(cht)\right)^2 = \ldots = \widehat{f}_P(\alpha) \left(Q_\alpha^\lambda(cht)\right)^k .$$

Lemma 3.5. *Let f and $D_\lambda^k f$ belong to $L_{2,\lambda}$, then*

$$\left(\widehat{D_\lambda^k f}\right)_P (\alpha) = (\alpha\,(\alpha + 2\lambda))^k \widehat{f}_P(\alpha) , \quad k = 1, 2, \ldots$$

Proof. Using the symmetry of the operator D_λ, from (1.5) we obtain

$$\widehat{(D_\lambda f)}_P(\alpha) = \int_0^\infty P_\alpha^\lambda(chx) D_\lambda f(chx) \, sh^{2\lambda} x \, dx = \int_0^\infty f(chx) \left(D_\lambda P_\alpha^\lambda(chx)\right) sh^{2\lambda} x \, dx$$

$$= \alpha\,(\alpha + 2\lambda) \int_0^\infty f(chx) \, P_\alpha^\lambda(chx) \, sh^{2\lambda} x \, dx = \alpha\,(\alpha + 2\lambda) \, \widehat{f}_P(\alpha).$$

For $k = 1$ our approval is proved, the generalized case is proved by the induction.

Lemma 3.6. *Let $f \in I_\nu$. If $D_\lambda^k f \in L_{2,\lambda}$, $k = 1, 2, \ldots, n + s - 1$,then and $P_\nu^{s,n} f \in I_\nu$.*

Proof. From (3.10) by Minkowsky inequality we have

$$\begin{aligned} \left\|P_\nu^{s,n} f\right\|_{2,\lambda} &= \left\| A_{ch\frac{1}{\nu}}^{n+s} f - \sum_{k=1}^{n+s-1} C_k \left(ch\frac{1}{\nu}\right) \left(A_{ch\frac{1}{\nu}}^k D_\lambda^k f\right) \right\|_{2,\lambda} \\ &\le \left\| A_{ch\frac{1}{\nu}}^{n+s} f \right\|_{2,\lambda} + \sum_{k=1}^{n+s-1} C_k \left(ch\frac{1}{\nu}\right) \left\| A_{ch\frac{1}{\nu}}^k D_\lambda^k f \right\|_{2,\lambda} \\ &\le \|f\|_{2,\lambda} + \sum_{k=1}^{n+s-1} C_k \left(ch\frac{1}{\nu}\right) \left\| D_\lambda^k f \right\|_{2,\lambda} , \end{aligned} \tag{3.11}$$

at the end we used the inequality (2.6).

Further taking into account Lemmas 3.3 and 3.4 we can write

$$\widehat{(P_\nu^{s,n} f)}_P(\alpha) = \left(\widehat{A_{ch\frac{1}{\nu}}^{n+s} f}\right)_P (\alpha) - \sum_{k=1}^{n+s-1} C_k \left(ch\frac{1}{\nu}\right) \left(\widehat{A_{ch\frac{1}{\nu}}^k D_\lambda^k f}\right)_P (\alpha)$$

$$= \widehat{f}_P(\alpha)\,\left(Q_\alpha^\lambda\left(ch\frac{1}{\nu}\right)\right)^{n+s} - \widehat{f}_P(\alpha)\sum_{k=1}^{n+s-1} C_k\left(ch\frac{1}{\nu}\right)\,(\alpha\,(\alpha+2\lambda))^k\,\left(Q_\alpha^\lambda\left(ch\frac{1}{\nu}\right)\right)^k$$

$$= \widehat{f}_P(\alpha)\left[\left(Q_\alpha^\lambda\left(ch\frac{1}{\nu}\right)\right)^{n+s} - \sum_{k=1}^{n+s-1} C_k\left(ch\frac{1}{\nu}\right)\,(\alpha\,(\alpha+2\lambda))^k\,\left(Q_\alpha^\lambda\left(ch\frac{1}{\nu}\right)\right)^k\right]. \tag{3.12}$$

The assertion of Lemma 3.6 follows from (3.11) and (3.12).

Lemma 3.7. *If $f \in D_\lambda^r(0,\infty)$, then for almost every $x \in (0,\infty)$ the equality is valid.*

$$\left(A_{cht}^k D_\lambda^r f\right)(chx) = \left(D_\lambda^r A_{cht}^k f\right)(chx), \quad k, r = 1, 2, ...$$

Proof. Let $k = 1$. In (2.2) we can write

$$\left(A_{cht} D_\lambda^r f\right)(chx) = C_\lambda^* \int_0^\infty \left(\widehat{A_{cht} D_\lambda^r} f\right)_P(\alpha)\, Q_\alpha^\lambda(chx)\, sh^{2\lambda}\alpha d\alpha, \tag{3.13}$$

$$\left(D_\lambda^r A_{cht} f\right)(chx) = C_\lambda^* \int_0^\infty \left(\widehat{D_\lambda^r A_{cht}} f\right)_P(\alpha)\, Q_\alpha^\lambda(chx)\, sh^{2\lambda}\alpha d\alpha. \tag{3.14}$$

From Lemmas 3.4 and 3.5 we get

$$\left(\widehat{D_\lambda^r A_{cht}} f\right)_P(\alpha) = (\alpha\,(\alpha+2\lambda))^r \left(\widehat{A_{cht}} f\right)_P(\alpha) = (\alpha\,(\alpha+2\lambda))^r\, Q_\alpha^\lambda(chx) \widehat{f}_P(\alpha). \tag{3.15}$$

On the other hand

$$\left(\widehat{A_{cht} D_\lambda^r} f\right)_P(\alpha) = \left(\widehat{D_\lambda^r f}\right)_P(\alpha)\, Q_\alpha^\lambda(chx) = (\alpha\,(\alpha+2\lambda))^r\, Q_\alpha^\lambda(chx) \widehat{f}_P(\alpha). \tag{3.16}$$

From (3.15) and (3.16) it follows that

$$\left(\widehat{D_\lambda^r A_{cht}} f\right)_P(\alpha) = \left(\widehat{A_{cht} D_\lambda^r} f\right)_P(\alpha).$$

But, from (3.13) and (3.14) it follows that

$$\left(A_{cht} D_\lambda^r f\right)(chx) = \left(D_\lambda^r A_{cht} f\right)(chx) \quad \text{(a.e.)}.$$

For $k = 1$ Lemma 3.6 is proved. The generalized case is proved by the induction:

$$\left(A_{cht}^k D_\lambda^r f\right)\,(chx) = A_{cht}\left(A_{cht}^{k-1} D_\lambda^r f\right)\,(chx)$$

$$= A_{cht}\left(D_\lambda^r A_{cht}^{k-1} f\right)\,(chx) = \left(D_\lambda^r A_{cht}^k f\right)\,(chx)\,.$$

Lemma 3.8. *The following equality is valid.*

$$A_{cht}^k R_n\,(chs)\, f(chx) = R_n\,(chs)\left(A_{chs}^k f\right)\,(chx)\,,\;\; k = 1, 2, \ldots\,.$$

Proof. By Lemmas 3.3 and 3.7 we have

$$R_n\,(chs)\,\left(A_{chs}^k f\right)\,(chx) = A_{chs}^{k+1} f(chx) - \sum_{\nu=0}^{n-1} C_\nu\,(chs)\, D_\lambda^\nu \left(A_{chs}^k f\right)\,(chx)$$

$$= A_{chs}^{k+1} f\,(chx) - \sum_{\nu=0}^{n-1} C_\nu\,(chs)\, A_{chs}^k \left(D_\lambda^\nu f\right)\,(chx)\;, \tag{3.17}$$

$$A_{chs}^k R_n\,(chs)\, f(chx) = A_{chs}^{k+1} f(chx) - \sum_{\nu=0}^{n-1} C_\nu\,(chs)\, A_{chs}^k \left(D_\lambda^\nu f\right)\,(chx). \tag{3.18}$$

The assertion of Lemma 3.8 follows from (3.17) and (3.18).

Lemma 3.9 *The following equality is valid.*

$$R_n\,(chs)\,f(chx) = C_n\,(chs)\,A_{chs}D_\lambda^n f(chx), \ \ n = 1, 2, \cdots .$$

Proof. Using Lemma 3.3 and the formula (3.9), we have

$$R_n\,(chs)\,f(chx) = A_{chs}f(chx) - \sum_{\mathrm{k}=0}^{\mathrm{n}} \mathrm{C}_{\mathrm{k}}\,(chs)\ D_\lambda^k f\,(chx) + C_n\,(chs)\,D_\lambda^n f(chx)$$

$$= \int\limits_1^{chs} \theta\,(chs,\sigma)\,(D_\lambda R_n\,(\sigma))\,f(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma + C_n\,(chs)\,D_\lambda^n f(chx).$$

Taking into account Lemma 3.8 we obtain

$$\begin{aligned}
&A_{chs}R_n\,(chs)\,f(chx) = R_n\,(chs)\,(A_{chs}f)\,(chx)\\
&= \int_1^{chs} \theta\,(chs,\sigma)\,(R_n\,(\sigma)\,D_\lambda A_\sigma f)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma + C_n\,(chs)\,A_{chs}D_\lambda^n f(chx)\\
&= \int_1^{chs} \theta\,(chs,\sigma)\,(R_n A_\sigma D_\lambda f)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma + C_n\,(chs)\,A_{chs}D_\lambda^n f(chx)\\
&= \int_1^{chs} \theta\,(chs,\sigma)\,(A_\sigma D_\lambda R_n f)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma + C_n\,(chs)\,A_{chs}D_\lambda^n f(chx)\\
&= A_{chs}R_n\,(chs)\,f\,(chx) - R_n\,(chs)\,f\,(chx) + C_n\,(chs)\,A_{chs}D_\lambda^n f(chx)\ ,
\end{aligned}$$

from this it follows the assertion of Lemma 3.9.

Lemma 3.10. *If $f \in D_\lambda^{(k)}(0,\infty)$, then the following equality is valid.*

$$\Delta_{chs}^k f(chx) = R_k\,(chs)\left(A_{chs}^{k-1}f\right)\,(chx)\ ,\ \ k = 1, 2, \ldots\ .$$

Proof. For $k = 1$ we have

$$R_1\,(chs)\,f(chx) = \int_1^{chs} \theta\,(chs,\sigma)\,(A_\sigma D_\lambda f)\,(chx)\,\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} d\sigma$$

$$= \int_1^{chs} \theta\,(chs,\sigma)\ d\left[\left(\sigma^2-1\right)^{\lambda-\frac{1}{2}} \frac{d}{d\sigma}A_\sigma f(chx)\right]$$

$$= \theta\,(chs,\sigma)\,\left(\sigma^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{d\sigma}A_\sigma f(chx)\ |_1^{chs} + \int_1^{chs} \frac{d}{d\sigma}A_\sigma f(chs)\,d\sigma$$

$$= A_{chs}f(chx) - f(chx) = \Delta_{chs}^1 f(chx).$$

Let

$$\Delta_{chs}^k f(chx) = R_k\,(chs)\ \left(A_{chs}^{k-1}f\right)\,(chx)\ .$$

Then we have

$$
\begin{aligned}
\varDelta_{chs}^{k+1} f(chx) &= \varDelta_{chs}\left(\varDelta_{chs}^{k} f\right)(chx) = R_1(chs)\left(\varDelta_{chs}^{k} f\right)(chx) \\
&= \int_1^{chs} \theta(chs,\sigma)\ A_\sigma D_\lambda R_k\left(A_\sigma^{k-1} f\right)(chx)\left(\sigma^2-1\right)^{\lambda-\frac12} d\sigma \\
&= \int_1^{chs} \theta(chs,\sigma)\ R_k\left(A_\sigma^{k-1}\right) D_\lambda\left(A_\sigma - f + f\right)(chx)\left(\sigma^2-1\right)^{\lambda-\frac12} d\sigma \\
&= \int_1^{chs} \theta(chs,\sigma)\ R_k\left(A_\sigma^{k-1} D_\lambda R_1 f\right)(chx)\left(\sigma^2-1\right)^{\lambda-\frac12} d\sigma \\
&+ \int_1^{chs} \theta(chs,\sigma)\ R_k\left(A_\sigma^{k-1}\right) D_\lambda f(chx)\left(\sigma^2-1\right)^{\lambda-\frac12} d\sigma \\
&= R_{k+1}\left(A_{chs}^{k-1} R_1 f\right)(chx) + R_{k+1}\left(A_{chs}^{k-1} f\right)(chx) \\
&= R_k\left(A_{chs}^{k-1} R_1 f\right)(chx) - C_k(chs)\ D_\lambda^k\left(A_{chs}^{k-1} R_1 f\right)(chx) \\
&+ R_k\left(A_{chs}^{k-1} f\right)(chx) - C_k(chs)\ D_\lambda^k\left(A_{chs}^{k-1} f\right)(chx) \\
&= R_k\left(A_{chs}^{k} f - A_{chs}^{k-1}\right) f(chx) - C_k(chs)\ D_\lambda^k\left(A_{chs}^{k-1} f - A_{chs}^{k-1} f\right)(chx) \\
&+ R_k\left(A_{chs}^{k-1} f\right)(chx) - C_k(chs)\ D_\lambda^k\left(A_{chs}^{k-1} f\right)(chx) \\
&= R_k\left(A_{chs}^{k} f\right) f(chx) - R_k\left(A_{chs}^{k-1} f\right)(chs) - C_k(chs)\, D_\lambda^k\left(A_{chs}^{k}\right)(chx) \\
&+ C_k(chs)\, D_\lambda^k\left(A_{chs}^{k-1} f\right)(chx) + R_k\left(A_{chs}^{k-1} f\right)(chx) \\
&- C_k(chs)\, D_\lambda^k\left(A_{chs}^{k-1} f\right)(chx) = R_k\left(A_{chs}^{k} f\right)(chs) \\
&- C_k(chs)\, D_\lambda^k\left(A_{chs}^{k} f\right)(chx) = R_{k+1}\left(A_{chs}^{k} f\right)(chx)\ .
\end{aligned}
$$

By the Principle of Mathematical Induction it follows the assertion of Lemma 3.10.

Lemma 3.11. *If* $f \in D_\lambda^{(k)}(0,\infty)$, *then the following equality is valid.*

$$\varDelta_{chs}^{k} f(chx) = C_k(chs)\ A_{chs}^{k}\left(D_\lambda^k f\right)(chx)\ ,\ k = 1, 2, \dots\ . \tag{3.19}$$

The assertion of lemma at once follows from Lemmas 3.9 and 3.10. From Lemmas 3.10 and 3.3 it follows that

$$
\begin{aligned}
\varDelta_{cht}^{n} f(chx) &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} A_{cht}^{k} f(chx) \\
&= A_{cht}^{n} f(chx) - \sum_{k=0}^{n-1} C_k(cht) A_{cht}^{n-1} D_\lambda^k f(chx).
\end{aligned} \tag{3.20}
$$

Lemma 3.12. *If* $f \in D_\lambda^{(k)}(0,\infty)$, *then the following equality is valid.*

$$\varDelta_{cht}^{m} D_\lambda^k f(chx) = D_\lambda^k \varDelta_{cht}^{m} f(chx)\ ,\ k = 1, 2, \dots\ ;\quad m = 1, 2, \dots\ .$$

Proof. From Lemma 3.6 at $m = 1$ we obtain

$$
\begin{aligned}
D_\lambda^k \varDelta_{cht} f(chx) &= D_\lambda^k\left(A_{cht} f - f\right)(chx) = D_\lambda^k \varDelta_{cht} f(chx) - D_\lambda^k f(chx) \\
&= A_{cht}\left(D_\lambda^k f\right)(chx) - D_\lambda^k f(chx) = \varDelta_{cht} D_\lambda^k f(chx).
\end{aligned}
$$

From this we have the chain of the following equalities

$$D_\lambda^k \Delta_{ht}^m f(chx) = D_\lambda^k \left(\Delta_{cht}\left(\Delta_{cht}^{m-1} f\right)\right)(chx) = \Delta_{cht}\left(D_\lambda^k\left(\Delta_{cht}^{m-1} f\right)\right)(chx)$$

$$= \Delta_{cht}\left(\Delta_{cht} D_\lambda^k\left(\Delta_{cht}^{m-2} f\right)\right)(chx)$$

$$= \Delta_{cht}^2\left(D_\lambda^k\left(\Delta_{cht}^{m-2} f\right)\right)(chx) = \Delta_{cht}^m D_\lambda^k f(chx).$$

Thus Lemma 3.12 is proved.

From (3.10) it follows that

$$P_\nu^{s,n} f(chx) - f(chx) = A_{ch\frac{1}{\nu}}^{n+s} f(chx) - \sum_{k=0}^{n+s-1} C_k\left(ch\frac{1}{\nu}\right) A_{ch\frac{1}{\nu}}^{n+s-1} D_\lambda^k f(chx)\ . \tag{3.21}$$

Remind ([25], Ch. VIII) that the line operator on a Hilbert space H with the off dense field of definition $D = D(A) \subset H$ is called an essential self-adjoint, if its closure $\bar{A}$ is the self-adjoint operator. Also we note that for the self-adjoint on an essential operator of the equality $\bar{A} = A^*$ is satisfied. (i.e. the closure operator A agrees with the self-adjoint operator). The operator A is called positive, if $(A\varphi, \varphi) \geq c(\varphi, \varphi)$ for all $\varphi \in D$ and some $c > 0$.

Lemma 3.13. *The Gegenbauer operator D_λ with the field of definition $D = D(\mathbb{R}_+)$ on an essentially self-adjoint.*

Proof. In fact, integration by parts, from (1.1) we obtain

$$\int_0^\infty (D_\lambda f)\ (chx) g(chx)\, sh^{2\lambda} x\, dx = \int_1^\infty (D_\lambda f) g(x)(x^2-1)^{\lambda-\frac{1}{2}} dx$$

$$= \int_1^\infty \Big[\frac{d}{dx}(x^2-1)^{\lambda+\frac{1}{2}} \frac{df(x)}{dx}\Big] g(x) dx = \int_1^\infty g(x) d\Big[(x^2-1)^{\lambda+\frac{1}{2}} \frac{df(x)}{dx}\Big]$$

$$= (x^2-1)^{\lambda+\frac{1}{2}} g(x)\ \frac{df(x)}{dx}\Big|_1^\infty - \int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}} \frac{df(x)}{dx}\frac{dg(x)}{dx} dx$$

$$= -\int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}} \frac{dg(x)}{dx} df(x) = -(x^2-1)^{\lambda+\frac{1}{2}}\ f(x)\frac{dg(x)}{dx}\ \Big|_1^\infty$$

$$+ \int_1^\infty f(x)\frac{d}{dx}\Big[(x^2-1)^{\lambda+\frac{1}{2}} \frac{dg(x)}{dx}\Big] dx = \int_1^\infty f(x)(D_\lambda g)(x)(x^2-1)^{\lambda-\frac{1}{2}} dx \tag{3.22}$$

$$= \int_0^\infty f(chx)(D_\lambda g)(chx) sh^{2\lambda} x dx,$$

for every $f, g \in D(\mathbb{R}_+)$, i.e., D_λ is the self-adjoint operator.

(3.22) is valid for every $f \in D_c(\mathbb{R}_+)$ and $g \in C^{(2)}(\mathbb{R}_+)$ (see also [12]), and for the self-adjoint operator its closure is a self-adjoint operator (see for examply [41] p. 355).

We note that the operator $(-D_\lambda)$ is positive. In fact, from the proof of Lemma 3.13 we see that

$$(D_\lambda f, f) = -\int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}} \Big(\frac{df(x)}{dx}\Big)^2 dx.$$

From this it follows that $(-D_\lambda)$ is positive. The operator $(-D_\lambda + \mu)$ is strictly positive for any $\mu > 0$, since, $((-D_\lambda + \mu), f) \geq \mu(f, f)$ for every $f \in D_c$.

In fact,

$$((-D_\lambda + \mu), f) - \mu(f,f) = (-D_\lambda f, f) + \mu(f,f) - \mu(f,f)$$

$$= (-D_\lambda f, f) \geq 0 \Leftrightarrow ((-D_\lambda + \mu), f) \geq \mu(f,f).$$

It is clear that the operator D_λ is self-adjoint in essential if and only if the operator $(-D_\lambda + \mu)$ is self-adjoint in essential.

For strictly positive symmetric operator A the criterion of self-adjoint exists (see [41], Theorem 26): A is self-adjoint in essential if and only if $KerA^* = \{0\}$. To prove that the operator $(-D_\lambda + \mu)$ an essential is self-adjoint it's enough to show that $Ker\,(-D_\lambda + \mu)^* = \{0\}$.

Let

$$(-D_\lambda + \mu)^* u = 0\,, \quad u \in D\left(D_\lambda^*\right)\,. \tag{3.23}$$

So far as the field of definition of operator D_λ equals $D(\mathbb{R}_+)$ then the equality (3.32) is equivalent to the following equality

$$(-D_\lambda + \mu)\ u = 0\,, \quad u \in L_{2,\lambda}, \tag{3.24}$$

where the product is in the sense of theory of generalized functions, i.e., for any function $f \in D(\mathbb{R}_+)$ probably the equality is carry out:

$$\int_0^\infty u\,(chx)\ (-D_\lambda +\ \mu)\ f(chx)\,sh^{2\lambda}x\,dx = 0$$

or

$$\int_0^\infty u\,(chx)\ (D_\lambda f)\ (chx)\,sh^{2\lambda}x\,dx = \mu\int_0^\infty u\,(chx)\,f\,(chx)\,sh^{2\lambda}x\,dx. \tag{3.25}$$

Since, the differential operator $(-D_\lambda + \mu)$ is elliptic on the interval $[0,\infty)$, from theorems of regularity it follows that the function $u\,(chx)$ is probably smooth (of classes C^∞) even function on $\mathbb{R}\setminus\{0\}$ and is the solution of equation $D_\lambda\ u = \mu\,u$ in classical means. On the interval $(0,\infty)$ the aggregate of all solutions of the equation $D_\lambda\ u = \mu\,u$ set of functions $u\,(chx) = c_1u_1(chx) + c_2u_2(chx)$, where $u_1\,(chx)$ and $u_2\,(chx)$ are fundamental systems of solutions, but c_1 and c_2 are arbitary constans. As in the capacity of fundamental systems of solutions one may take the functions $P_\alpha^\lambda(chx)$ and $Q_\alpha^\lambda(chx)$ defined by formulas (1.2) and (1.3). From (1.3) it follows that $u_1\,(chx) = Q_\alpha^\lambda(chx)$ is even smooth function on the substantial line $\mathbb{R}$. We show that $u_1\,(chx)$ doesn't provide of equality (3.25).

In fact, from (1.5) it follows that

$$D_\lambda\ Q_\alpha^\lambda(chx) = \alpha\,(\alpha + 2\lambda)\,Q_\alpha^\lambda(chx). \tag{3.26}$$

On the other hand ([8], c. 1935)

$$\frac{d}{dx}Q_\alpha^\lambda(x) = 2\lambda Q_{\alpha-1}^{\lambda+1}(x). \tag{3.27}$$

We take any function $f \in D(\mathbb{R}_+)$.

Let $supp\ f \in [1, a]$. Then, taking into account (3.26) and (3.27) we obtain

$$\int_1^\infty Q_\alpha^\lambda(x)(D_\lambda f)(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$
$$= \int_1^\infty Q_\alpha^\lambda(x)\frac{d}{dx}\Big[(x^2-1)^{\lambda+\frac{1}{2}}\frac{df(x)}{dx}\Big]dx$$
$$= (x^2-1)^{\lambda+\frac{1}{2}}Q_\alpha^\lambda(x)\frac{d}{dx}f(x)\Big|_1^\infty - \int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}Q_\alpha^\lambda(x)\frac{d}{dx}f(x)dx$$
$$= -\int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}Q_\alpha^\lambda(x)df(x)$$
$$= -(x^2-1)^{\lambda+\frac{1}{2}}f(x)(2\lambda\, Q_{\alpha-1}^{\lambda+1}(x))\Big|_1^a + \int_1^\infty f(x)D_\lambda Q_\alpha^\lambda(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$
$$= -2\lambda(\alpha^2-1)^{\lambda+\frac{1}{2}}f(a)Q_{\alpha-1}^{\lambda+1}(a) + \alpha(\alpha+2\lambda)\int_1^\infty Q_\alpha^\lambda(x)f(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$
$$= c_1(\alpha,\lambda,a) + \alpha(\alpha+2\lambda)\int_0^\infty Q_\alpha^\lambda(x)f(x)(x^2-1)^{\lambda-\frac{1}{2}}dx,$$

where

$$c_1(\alpha,\lambda,a) = -2\lambda(\alpha^2-1)^{\lambda+\frac{1}{2}}f(a)Q_{\alpha-1}^{\lambda+1}(a).$$

In this way,

$$\int_0^\infty Q_\alpha^\lambda(chx)(D_\lambda f)(chx)sh^{2\lambda}xdx = c_1(\alpha,\lambda,a) + \alpha(\alpha+2\lambda)\int_0^\infty Q_a^\lambda(chx)f(chx)sh^{2\lambda}xdx.$$

Consequently, (3.25) non fulfills, if $c_1 \neq 0$ but therefore $u_1(chx) = c_1Q_\alpha^\lambda(chx) = 0$ for $c_1 = 0$.

By analogy it is proved that $u_2(x)$ non fulfills (3.25). Twice integrating of parts we obtain

$$\int_1^\infty u_2(x)\,(D_\lambda f)\,(x)(x^2-1)^{\lambda-\frac{1}{2}}dx = \int_1^\infty u_2(x)\,d\left[(x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}f(x)\right]$$
$$= (x^2-1)^{\lambda+\frac{1}{2}}u_2(x)\frac{d}{dx}f(x)\Big|_1^\infty - \int_1^\infty (x^2-1)^{\lambda+\frac{1}{2}}\frac{d}{dx}f(x)\frac{d}{dx}u_2(x)dx$$
$$= -(x^2-1)^{\lambda+\frac{1}{2}}f(x)u_2(x)\Big|_1^a + \int_1^\infty f(x)\,(D_\lambda u_2)\,(x)(x^2-1)^{\lambda-\frac{1}{2}}dx$$
$$= -(a^2-1)^{\lambda+\frac{1}{2}}f(a)\frac{d}{dx}u_2(a) + \int_1^\infty f(x)\,(D_\lambda u_2(x))\,(x^2-1)^{\lambda-\frac{1}{2}}dx.$$

The formula (3.27) is valid and for $P_\alpha^\lambda(x)$ and that's why from (1.2) we have

$$\frac{d}{dx}u_2(a) = \frac{d}{dx}P_\alpha^\lambda(a) = 2\lambda P_{\alpha-1}^{\lambda+1}(a).$$

Thus

$$\int_0^\infty P_\alpha^\lambda(chx)(D_\lambda f)(chx)sh^{2\lambda}xdx$$
$$= c_2(\alpha,\lambda,a) + \alpha(\alpha+2\lambda)\int_0^\infty f(chx)P_\alpha^\lambda(chx)sh^{2\lambda}xdx,$$

where

$$c_2(\alpha,\lambda,a) = -2\lambda(\alpha^2-1)^{\lambda+\frac{1}{2}} f(a) P^{\lambda+1}_{\alpha-1}.$$

Consequently, (3.25) non fulfills, if $c_2 \neq 0$, but therefore $u_2(chx) = c_2 P^{\lambda}_{\alpha}(chx) = 0$ for $c_2 = 0$.

Thus, we prove that

$$u(chx) \equiv 0,$$

from where the assertion of Lemma 3.13 follows.

Corollary 3.4. *If the functions f and $D_\lambda f \in L_{2,\lambda}$, then there exists $f_n \in D(\mathbb{R}_+)$, such that $f_n \to f$ and $D_\lambda f_n \to D_\lambda f$ in $L_{2,\lambda}$.*

Really, from the definition of the couplinged operator D^*_λ it follows that $f \in D(D^*_\lambda)$ and $g = D^*_\lambda f$ if and only if $g = D_\lambda f$ in meaning of the theory of generalized functions. It remains to reproduce locking operator D_λ is agree to D^*_λ.

4 Direct theorems of Jackson type

In this section we will prove the Theorem 1.1. Firstly we will prove a particular case of this theorem.

Theorem 4.1. *For $f \in L_{2,\lambda}$ the following inequality is valid*

$$E_\nu(f)_{2,\lambda} \leq \omega_k \left(f; \frac{1}{\nu}\right)_{2,\lambda}.$$

Proof. From (3.20) and (3.21) for $s = 0$ we have

$$\left\|P^{0,n}_\nu f - f\right\|^2_{2,\lambda} = \int_0^\infty \left| A^k_{ch\frac{1}{\nu}} f(chx) - \sum_{i=0}^{k-1} C_i \left(ch\frac{1}{\nu}\right) A^{k-1}_{ch\frac{1}{\nu}} D^i_\lambda f(chx)\right|^2 sh^{2\lambda} dx$$

$$= \left\| \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} A^i_{ch\frac{1}{\nu}} \right\|^2 \leq \left(\omega_k \left(f; \frac{1}{\nu}\right)_{2,\lambda}\right)^2.$$

Hence the assertion of Theorem 4.1 follows.

Denote through $W^{(m)}_\lambda L_{2,\lambda}$ the class of functions which is m-time applied the operator D_λ, moreover $D^m_\lambda f \in L_{2,\lambda}$, $m = 1, 2, \ldots$.

Lemma 4.1. *If $f \in W^{(m)}_\lambda L_{2,\lambda}$, then the following inequality is valid*

$$\omega_m \left(f; \frac{1}{\nu}\right)_{2,\lambda} \leq \left(\frac{ch\frac{1}{\nu} - 1}{2\lambda+1}\right)^m \left\|D^m_\lambda f\right\|_{2,\lambda}.$$

Proof. From Lemma 5 in [12] we have

$$\|R_k(cht)f\|_{2,\lambda} \leq \left(\frac{cht-1}{2\lambda+1}\right)^k \left\|D^k_\lambda f\right\|_{2,\lambda}, \quad k = 1, 2, \ldots .$$

Hence, for $k = 1$ we get

$$\left\|\Delta^1_{cht} f\right\|_{2,\lambda} = \|R_1(cht)f\|_{2,\lambda} \leq \left(\frac{cht-1}{2\lambda+1}\right) \left\|D^k_\lambda f\right\|_{2,\lambda}.$$

Now,

$$\left\|\Delta^m_{cht} f\right\|_{2,\lambda} = \left\|\Delta^1_{cht}\left(\Delta^{m-1}_{cht} f\right)\right\|_{2,\lambda} \leq \left(\frac{cht-1}{2\lambda+1}\right) \left\|\Delta^{m-1}_{cht}(D_\lambda f)\right\|_{2,\lambda}$$

$$\leq \left(\frac{cht-1}{2\lambda+1}\right)^2 \left\|\Delta_{cht}^{m-2}(D_\lambda^2 f)\right\|_{2,\lambda} \leq \ldots \leq \left(\frac{cht-1}{2\lambda+1}\right)^m \left\|D_\lambda^m f\right\|_{2,\lambda}, \tag{4.1}$$

hence it follows that

$$\omega_m\left(f;\frac{1}{\nu}\right)_{2,\lambda} \leq \left(\frac{ch\frac{1}{\nu}-1}{2\lambda+1}\right)^m \left\|D_\lambda^m f\right\|_{2,\lambda}.$$

Thus Lemma 4.1 is proved.

This lemma admittances the following amplification.

Lemma 4.2. *If $f \in W_\lambda^{(m)} L_{2,\lambda}$, then the following inequality is valid*

$$\omega_m\left(f;\frac{1}{\nu}\right)_{2,\lambda} \leq \left(\frac{\left(ch\frac{1}{\nu}-1\right)^m}{m!\,(2\lambda+1)\ldots(2\lambda+2m-1)}\right) \left\|D_\lambda^m f\right\|_{2,\lambda}.$$

Proof. In (3.19), taking into account (2.6) and Lemma 5 in [12] we obtain ,

$$\left\|\Delta_{cht}^m f\right\|_{2,\lambda} \leq C_m(cht) \left\|\Delta_{cht}^1\left(A_{cht}^m D_\lambda^m f\right)\right\|_{2,\lambda} \leq C_m(cht)\left\|D_\lambda^m f\right\|_{2,\lambda}$$

$$\leq \left(\frac{(cht-1)^m}{m!\,(2\lambda+1)\ldots(2\lambda+2m-1)}\right)\left\|D_\lambda^m f\right\|_{2,\lambda},$$

hence the assertion of Lemma 4.2 follows.

Lemma 4.3. *If $f \in W_\lambda^{(m)} L_{2,\lambda}$, then*

$$\omega_{n+k}\left(f;\frac{1}{\nu}\right)_{2,\lambda} \leq \left(\frac{ch\frac{1}{\nu}-1}{2\lambda+1}\right)^k \omega_n\left(D_\lambda^k f;\frac{1}{\nu}\right)_{2,\lambda}.$$

Proof. From (4.1) and, Lemmas 3.11 and 3.12 we have

$$\left\|\Delta_{cht}^{n+k} f\right\|_{2,\lambda} = \left\|\Delta_{cht}^k\left(\Delta_{cht}^n f\right)\right\|_{2,\lambda} \leq \left(\frac{cht-1}{2\lambda+1}\right)^k \left\|D_\lambda^k(\Delta_{cht}^n f)\right\|_{2,\lambda}$$

$$= \left(\frac{cht-1}{2\lambda+1}\right)^k \left\|\Delta_{cht}^n(D_\lambda^k f)\right\|_{2,\lambda},$$

hence the assertion of Lemma 4.3 follows.

Lemma 4.4. *If $f \in W_\lambda^{(n)} L_{2,\lambda}$, then*

$$\omega_{n+k}\left(f;\frac{1}{\nu}\right)_{2,\lambda} \leq \left(\frac{\left(ch\frac{1}{\nu}-1\right)^k}{k!\,(2\lambda+1)\ldots(2\lambda+2m-1)}\right) \omega_n\left(D_\lambda^k f;\frac{1}{\nu}\right)_{2,\lambda}.$$

In fact, from (2.6), Lemmas 3.10 and 3.11, and also from Lemma 5 in [12] we have

$$\left\|\Delta_{cht}^{n+k} f\right\|_{2,\lambda} = \left\|\Delta_{cht}^k\left(\Delta_{cht}^n f\right)\right\|_{2,\lambda} \leq C_k(cht)\left\|A_{cht}^k D_\lambda^k(\Delta_{cht}^n f)\right\|_{2,\lambda}$$

$$\leq \left(\frac{(cht-1)^k}{k!\,(2\lambda+1)\ldots(2\lambda+2k-1)}\right)\left\|\Delta_{cht}^n(D_\lambda^k f)\right\|_{2,\lambda},$$

hence the assertion of Lemma 4.4 follows.

Proof of Theorem 1.1. From (3.20) and (3.21) we have

$$\left\|P_\nu^{s,n} f - f\right\|_{2,\lambda} = \left\|\sum_{k=0}^{n+s}(-1)^{n+s-k}\binom{n+s}{k} A_{ch\frac{1}{\nu}}^k f\right\| \leq \omega_{n+s}\left(f;\frac{1}{\nu}\right)_{2,\lambda}.$$

From this, taking into account Lemmas 4.3, we obtain

$$E_\nu(f)_{2,\lambda} \le \left(\frac{ch\frac{1}{\nu}-1}{2\lambda+1}\right)^s \omega_n\left(D_\lambda^s f; \frac{1}{\nu}\right)_{2,\lambda}$$
$$\le \left(2sh^2\frac{1}{2\nu}\right)^s \omega_n\left(D_\lambda^s f; \frac{1}{\nu}\right)_{2,\lambda} \le 2^{-s}(sh\frac{1}{\nu})^{2s}\omega_n(D_\lambda^s, \frac{1}{\nu})_{2,\lambda}$$

as, $sh\frac{t}{a} \le \frac{1}{a}sht$ for $a \ge 1$, (see further (5.32)).

5 Nikolski-Besov Space associated with the Gegenbauer operator and their description approximation

For proof of inverse theorems of approximation theory the inequalities of Bernsteint's type are used.

Lemma 5.1. *(The inequality of Bernsteint's type). For every $f \in I_\nu$ the following equality is valid*

$$\|D_\lambda^k f\|_{2,\lambda} \le (\nu(\nu+2\lambda))^k \|f\|_{2,\lambda}\,.$$

Proof. Using the equality (2.3) and Lemma 3.4, we obtain

$$\|D_\lambda^k f\|_{2,\lambda}^2 \int_0^\infty |(D_\lambda^k f)\,(chx)|^2 sh^{2\lambda}x dx$$
$$= \int_0^\infty \widehat{(D_\lambda^k f)}_P(\alpha)\,\widehat{(D_\lambda^k f)}_Q(\alpha)\;(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$= \int_1^\infty (\alpha(\alpha+2\lambda))^{2k}\widehat{f}_P(\alpha)\widehat{f}_Q(\alpha)\;(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$= \int_1^\nu (\alpha(\alpha+2\lambda))^{2k}\widehat{f}_P(\alpha)\widehat{f}_Q(\alpha)\;(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$\le (\nu(\nu+2\lambda))^{2k}\int_1^\infty \widehat{f}_P(\alpha)\widehat{f}_Q(\alpha)\;(\alpha^2-1)^{\lambda-\frac{1}{2}}d\alpha$$
$$= (\nu(\nu+2\lambda))^{2k}\int_0^\infty f^2(chx)sh^{2\lambda}x\,dx = (\nu(\nu+2\lambda))^{2k}\|f\|_{2,\lambda}^2\,,$$

hence the assertion of Lemma 5.1 follows.

Lemma 5.2. *For $\Phi \in I_\nu$ and $t > 0$ the following equality is valid*

$$\left\|\Delta_{cht}^k\Phi\right\|_{2,\lambda} \le \Big(\nu\,(\nu+2\lambda)^k\,(cht-1)\Big)^k\,\|\Phi\|_{2,\lambda}\;.$$

Proof. From Lemma 3.11, Lemmas 2 and 5 in [12] we have

$$\left\|\Delta_{cht}^k\Phi\right\|_{2,\lambda} = C_k\,(chs)\left\|A_{cht}^k\left(D_\lambda^k\Phi\right)\right\|_{2,\lambda}$$
$$\le C_k\,(chs)\left\|\left(D_\lambda^k\Phi\right)\right\|_{2,\lambda} \le C_k\,(chs)\,(\nu\,(\nu+2\lambda))^k\,\|\Phi\|_{2,\lambda}$$
$$\le (\nu\,(\nu+2\lambda))^k\;(cht-1)\,\|\Phi\|_{2,\lambda}\,.$$

Thus Lemma 5.2 is proved.

We have defined the spaces $H_{2,\lambda}^{r}$ and $B_{2,q,\lambda}$ in Section 1. Now we will show that these spaces are Banach spaces. For this we need to prove some auxiliary results. The following lemma and the corollary are the analogues of the classical Boas inequality ([45], p. 266).

Lemma 5.3. *Let $\nu > 0$ be an arbitary number and $f \in W_{2,\lambda}^{m}$. For every number δ and t such that $0 < \delta < t < \frac{1}{\nu}$ the following inequality is valid*

$$\left(sh\frac{\delta}{2}\right)^{-2m} \left\|\Delta_{ch\delta}^{m} f\right\|_{2,\lambda} \leq c\left(m,\lambda\right)\left(sh\frac{t}{2}\right)^{-2m} \left\|\Delta_{cht}^{m} f\right\|_{2,\lambda}. \tag{5.1}$$

Proof. Using (3.19) we obtain

$$\left\|\Delta_{ch\delta}^{m} f\right\|_{2,\lambda} = C_m\left(ch\delta\right)\left\|A_{ch\delta}^{m}\left(D_{\lambda}^{m} f\right)\right\|_{2,\lambda}, \quad m = 1, 2, ...,$$

$$\left\|\Delta_{cht}^{m} f\right\|_{2,\lambda} = C_m\left(cht\right)\left\|A_{cht}^{m}\left(D_{\lambda}^{m} f\right)\right\|_{2,\lambda}, \quad m = 1, 2, \dots .$$

From continuity of the operator A_{cht} in $W_{2,\lambda}^{m}$ (see [12], Corollary 1 and Lemma 7) we will have

$$\lim_{t\to 0}\frac{\|\Delta_{cht}^{m} f\|_{2,\lambda}}{C_m\left(cht\right)} = \lim_{\delta\to 0}\frac{\|\Delta_{ch\delta}^{m} f\|_{2,\lambda}}{C_m\left(ch\delta\right)} = \left\|D_{\lambda}^{m} f\right\|_{2,\lambda}.$$

So far as $t \to 0$ involves $\delta \to 0$, then

$$\frac{\|\Delta_{cht}^{m} f\|_{2,\lambda}}{C_m\left(cht\right)} \sim \frac{\|\Delta_{ch\delta}^{m} f\|_{2,\lambda}}{C_m\left(ch\delta\right)}, \; t \to 0 \;\; (\delta \to 0),$$

from this it follows that

$$C_m\left(cht\right)\left\|\Delta_{ch\delta}^{m} f\right\|_{2,\lambda} \sim C_m\left(ch\delta\right)\left\|\Delta_{cht}^{m} f\right\|_{2,\lambda}, \; t \to 0 \;\; (\delta \to 0).$$

Taking into account this the following correlation (see [12], Lemma 5) is satisfied.

$$c_1\left(m,\lambda\right)\left(sh\frac{t}{2}\right)^{2m} \leq C_m\left(sht\right) \leq c_2\left(m,\lambda\right)\left(sh\frac{t}{2}\right)^{2m},$$

where $c_1\left(m,\lambda\right)$ and $c_2\left(m,\lambda\right)$ are some constants depending on m and λ. We obtain the following confirmation of Lemma 5.3.

Corollary from Lemma 5.3. Let $f \in W_{2,\lambda}^{m}, \; m \in N$. For every $t \in \left(0; \frac{1}{\nu}\right]$ the following inequality is valid

$$\left\|D_{\lambda}^{m} f\right\|_{2,\lambda} \leq c\left(m,\lambda\right)\left(sh\frac{t}{2}\right)^{-2m} \left\|\Delta_{cht}^{m} f\right\|_{2,\lambda}. \tag{5.2}$$

Taking into account the continuity of the operator A_{cht} in $W_{2,\lambda}^{m}$, and also the correlation (see [12], Lemma 5) $C_m\left(cht\right) \sim c\left(m,\lambda\right)\left(cht-1\right)^{m}$ for $t \to 0$, where $c\left(m,\lambda\right) = \frac{1}{m!(2\lambda+1)(2\lambda+3)\dots(2\lambda+2m-1)}$, we obtain

$$\lim_{\delta\to 0}\frac{\|\Delta_{ch\delta}^{m}\|_{2,\lambda}}{C_m\left(ch\delta\right)} = \lim_{\delta\to 0}\left\|\Delta_{ch\delta}^{m}\left(D_{\lambda}^{m} f\right)\right\|_{2,\lambda} = \left\|D_{\lambda}^{m} f\right\|_{2,\lambda}.$$

If we take limit as $\delta \to 0$ in (5.1), we obtain the inequality (5.2), which is the analogue of the Nikolski-Stechkin classical inequality (see [28, 42]).

Taking into account the obvious inequality

$$\left\|\Delta_{cht}^{m} f\right\|_{2,\lambda} \leq 2^{m}\left\|f\right\|_{2,\lambda} \tag{5.3}$$

which follows from (2.6) to the chain

$$\left\|\Delta_{cht}^{m} f\right\|_{2,\lambda} \leq 2\left\|\Delta_{cht}^{m-1} f\right\|_{2,\lambda} \leq \dots \leq 2^{m}\left\|f\right\|_{2,\lambda},$$

or also the inequality $sh\frac{t}{2} \geq \frac{t}{2}$ in (5.2), we have

$$\left\| D_\lambda^m f \right\|_{2,\lambda} \leq c\,(m,\lambda)\, 2^{3m} t^{-2m}\, \|f\|_{2,\lambda}\,.$$

Putting $t = \frac{1}{\nu}$, we obtain

$$\left\| D_\lambda^m f \right\|_{2,\lambda} \leq c\,(m,\lambda,\nu)\, \|f\|_{2,\lambda}\,, (\nu > 0).$$

From here for every $f_n, f_m \in H_{2,\lambda}^r$ we have

$$\left\| D_\lambda^s f_n - D_\lambda^s f_m \right\|_{2,\lambda} = \left\| D_\lambda^s \left(f_n - f_m\right) \right\|_{2,\lambda} \leq c\,(s,\lambda,\nu)\, \|f_n - f_m\|_{2,\lambda}\,. \tag{5.4}$$

We show that $H_{2,\lambda}^r$ is a Banach space with the norm

$$\|f\|_{H_{2,\lambda}^r} = \|f\|_{2,\lambda} + \sup_{\delta>0} \frac{\omega_k\left(D_\lambda^s f, \delta\right)_{2,\lambda}}{\delta^{r-2s}}. \tag{5.5}$$

For the class $H_{2,\lambda}^r$ we are able to write

$$\|f_n - f_m\|_{H_{2,\lambda}^r} = \|f_n - f_m\|_{2,\lambda} + \sup_{\delta>0} \frac{\omega_k\left(D_\lambda^s f_n - D_\lambda^s f_m, \delta\right)_{2,\lambda}}{\delta^{r-2s}}. \tag{5.6}$$

From (5.3) it follows that

$$\omega_k\left(D_\lambda^s f, \delta\right)_{2,\lambda} \leq 2^k \left\| D_\lambda^s f \right\|_{2,\lambda}, \tag{5.7}$$

but then

$$\frac{\omega_k\left(D_\lambda^s f, \delta\right)_{2,\lambda}}{\delta^{r-2s}} \leq \left\| D_\lambda^s f \right\|_{2,\lambda}. \tag{5.8}$$

For $\delta \geq 1$ the inequality (5.8) is obvious. But for $0 < \delta < 1$ it is submitted to condition $2^{m-1} < \frac{1}{\delta} < 2^m$ (evidently, $m \geq 1$). Taking (5.4) and (5.8) in (5.6), we obtain

$$\|f_n - f_m\|_{H_{2,\lambda}^r} \leq \|f_n - f_m\|_{2,\lambda} + \left\| D_\lambda^s f_n - D_\lambda^s f_m \right\|_{2,\lambda} \leq \|f_n - f_m\|_{2,\lambda}\,. \tag{5.9}$$

Let $\|f_n - f_m\|_{2,\lambda} < \varepsilon$, for $n, m > N$ and for all $\varepsilon > 0$. Then from (5.9) it follows that

$$\|f_n - f_m\|_{H_{2,\lambda}^r} < \varepsilon, \;\; n, m > N.$$

Because of completeness of the space $L_{2,\lambda}$ we are able to write

$$\|f_n - f\|_{H_{2,\lambda}^r} \leq \|f_n - f\|_{2,\lambda} < \varepsilon, \;\; n > N,$$

from this the completeness of the space $H_{2,\lambda}^r$ follows.

Now we show that $B_{2,q,\lambda}^r$ is a Banach space. For every $\varepsilon > 0$ let

$$\|f_n - f_m\|_{B_{2,q,\lambda}^r} = \|f_n - f_m\|_{2,\lambda} + \left(\int_0^\infty \frac{\omega_k\left(D_\lambda^s f_n - D_\lambda^s f_m, \delta\right)_{2,\lambda}^q}{\delta^{(r-2s)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} < \varepsilon,$$

$$(n, m > N, \;\; k \geq 1; \; s = 1, 2, ...).$$

From the Lebesgue dominated convergence theorem for$m \to \infty$, we obtain

$$\|f_n - f\|_{B_{2,q,\lambda}^r} = \|f_n - f\|_{2,\lambda} + \left(\int_0^\infty \frac{\omega_k\left(D_\lambda^s f_n - D_\lambda^s f, \delta\right)_{2,\lambda}}{\delta^{(r-2s)q}} \frac{d\delta}{\delta}\right)^{\frac{1}{q}} < \varepsilon$$

for $n > N$, that means the completeness of the space $B_{2,q,\lambda}^r$.

In Theorems 1.2 and 1.3 the description of these spaces through the best approximation of the functions on I_ν are reduced.

Proof of Theorem 1.2. If $f \in H^r_{2,\lambda}$, then

$$\omega_k\left(D^s_\lambda f,\delta\right)_{2,\lambda} \le h^r_{2,\lambda}(f)\delta^{r-2s}$$

and from Theorem 1.1 it follows that

$$E_\nu(f)_{2,\lambda} \le 2^{-s}\frac{\omega_k\left(D^s_\lambda f,\frac{1}{\nu}\right)_{2,\lambda}}{\nu^{2s}} \le 2^{-s}\frac{h^r_{2,\lambda}(f)}{\nu^r}.$$

For the proof of the reverse inequality, the method is coming back to Bernstein (see [27], p. 236). Assume that inequality (1.14) holds. Consider a sequence of functions $\psi_n \in I_{2^n}\ \ (n=0,1,2\ldots)$ such that

$$\|f-\psi_n\|_{2,\lambda} \le A\cdot 2^{-n\,r}.$$

Let $\varphi_0=\psi_0$ and $\varphi_n=\psi_n-\psi_{n-1}$ for $n\ge 1$.

Then the series

$$f=\sum_{n=0}^{\infty}\varphi_n \tag{5.10}$$

converges in $L_{2,\lambda}$ and $\varphi_n \in I_{2^n}$. We obtain upper bounds for the norm of the terms of (5.10) as follows:

$$\|\varphi_0\|_{2,\lambda}=\|\psi_0\|_{2,\lambda} \le \|\psi_0-f\|_{2,\lambda}+\|f\|_{2,\lambda} \le \|f\|_{2,\lambda}+A\,. \tag{5.11}$$

$$\|\varphi_n\|_{2,\lambda} \le \|f-\psi_n\|_{2,\lambda}+\|f-\psi_{n-1}\|_{2,\lambda} \le A\left(2^{-nr}+2^{-(n-1)r}\right)=A\left(1+2^r\right)\,2^{-nr}. \tag{5.12}$$

Combining (5.11) and (5.12) we write

$$\|\varphi_n\|_{2,\lambda} \le c_1\,2^{-nr}\left(\|f\|_{2,\lambda}+A\right)\ ,\ (n=0,1,2,\ldots)\ . \tag{5.13}$$

Let ℓ be one of the numbers $1,2,\ldots,s$. From Lemma 5.1 it follows that

$$\left\|D^\ell_\lambda\varphi_n\right\|_{2,\lambda} \le \left(2^n\right)^\ell\left(2^n+2\lambda\right)^\ell\|\varphi_n\|_{2,\lambda}$$

$$=\left(2^{2n}+2\lambda\cdot 2^n\right)^\ell\|\varphi_n\|_{2,\lambda}\,. \tag{5.14}$$

From (5.13), (5.14) and the fact that $r-2\ell>0$ it follows that the series

$$\sum_{n=0}^{\infty}D^\ell_\lambda\varphi_n$$

converges in $L_{2,\lambda}$. In fact, if $0<\lambda<\frac{1}{2}$, then from (5.14) it follows that

$$\left\|D^\ell_\lambda\varphi_n\right\|_{2,\lambda} \le 2^\ell\cdot 2^{2n\ell}\,\|\varphi_n\|_{2,\lambda}\,.$$

But then

$$\left\|\sum_{n=0}^{\infty}D^\ell_\lambda\varphi_n\right\|_{2,\lambda} \le \left\|2^\ell\cdot\sum_{n=0}^{\infty}2^{2n\ell}\,\|\varphi_n\|_{2,\lambda}\right\|_{2,\lambda} \le \left\|2^\ell\cdot c_3\left(\|f\|_{2,\lambda}+A\right)\sum_{n=0}^{\infty}\frac{1}{2^{n(r-2\ell)}}\right\|_{2,\lambda}$$

$$=\left\|2^\ell\cdot c_3\left(\|f\|_{2,\lambda}+A\right)\frac{1}{1-\frac{1}{2^{r-2\ell}}}\right\|_{2,\lambda}=\left\|2^{\ell+1}\cdot c_3\left(\|f\|_{2,\lambda}+A\right)\frac{2^{r-2\ell}}{2^{r-2\ell}-1}\right\|_{2,\lambda} \le M<\infty.$$

Since the operator D_λ is closed, then

$$D^\ell_\lambda f=\sum_{n=0}^{\infty}D^\ell_\lambda\varphi_n \in L_{2,\lambda}\,.$$

Let $\Phi_n = D^s_\lambda \varphi_n$, then

$$g = \sum_{n=0}^{\infty} \Phi_n \ , \ \Phi_n \in I_{2^n} \ , \|\Phi_n\|_{2,\lambda} \le \frac{c_2}{2^{n(r-2s)}} \Big(\|f\|_{2,\lambda} + A \Big) \ . \tag{5.15}$$

Take an arbitrary number $t > 0$. From the continuity of the difference operator Δ^k_{cht} it follows that

$$\Delta^k_{cht}\, g \ = \sum_{n=0}^{\infty} \Delta^k_{cht} \Phi_n \, .$$

Take a non-negative integer N such that

$$2^{-N} \le t < 2^{-(N-1)} \tag{5.16}$$

(if $t \ge 1$, then (5.16) contains only the left-hand inequality). Then

$$\Delta^k_{cht} \ g = \sum_{n=0}^{N-1} \Delta^k_{cht} \Phi_n + \sum_{n=N}^{\infty} \Delta^k_{cht} \Phi_n. \tag{5.17}$$

(if $N = 0$, then (5.17) contains only the second sum). Estimate the terms in (5.17). For $n \le N-1$ by using Lemma 5.2 and inequalities (5.15) and (5.16), we obtain

$$\left\| \Delta^k_{cht} \Phi_n \right\|_{2,\lambda} \le \left(2^n t\right)^{2k} \|\Phi_n\|_{2,\lambda} \le c_3 \Big(\|f\|_{2,\lambda} + A \Big)\, 2^{n(2s+2k-r)} \cdot 2^{-2(N-1)k}.$$

By using (5.16) we have

$$\left\| \sum_{n=0}^{N-1} \Delta^k_{cht} \Phi_n \right\|_{2,\lambda} \le \frac{c_4 \Big(\|f\|_{2,\lambda} + A \Big)}{2^{2k(N-1)}} \sum_{n=0}^{N-1} 2^{(2s+2k-r)n}$$

$$= \frac{c_4 \Big(\|f\|_{2,\lambda} + A \Big)}{2^{2k(N-1)}} \cdot \frac{2^{(2s+2k-r)N} - 1}{2^{(2s+2k-r)} - 1} \le c_5 \Big(\|f\|_{2,\lambda} + A \Big)\, t^{r-2s}. \tag{5.18}$$

For $n \ge N$ we use the inequality (5.3).

Then from (5.15) we have

$$\left\| \sum_{n=N}^{\infty} \Delta^k_{cht} \Phi_n \right\|_{2,\lambda} \le 2^k \cdot c_2 \Big(\|f\|_{2,\lambda} + A \Big) \sum_{n=N}^{\infty} 2^{-(r-2s)n}$$

$$= 2^k \cdot c_2 \Big(\|f\|_{2,\lambda} + A \Big)\, 2^{-N(r-2s)} \Big(1 - 2^{(2s-r)} \Big)^{-1} \le c_6 \Big(\|f\|_{2,\lambda} + A \Big)\, t^{r-2s}. \tag{5.19}$$

From (5.18) and (5.19) it follows that

$$\left\| \Delta^k_{cht}\, g \ \right\|_{2,\lambda} \le c_7 t^{r-2s} \Big(\|f\|_{2,\lambda} + A \Big),$$

from this

$$\omega_k \, (\ g \, , \delta)_{2,\lambda} \le c_7 \Big(\|f\|_{2,\lambda} + A \Big)\, \delta^{r-2s} \, , \ \delta > 0$$

and

$$h^r_{2,\lambda}(f) \le c_7 \Big(\|f\|_{2,\lambda} + A \Big).$$

Hence, $f \in H^r_{2,\lambda}$ and inequality (1.13) holds.

Let

$$\tilde{h}^r_{2,\lambda}(f) := \sup_{\nu \ge 1} \nu^r E_\nu(f)_{2,\lambda}.$$

It follows from Theorem 1.2 that $f \in L_{2,\lambda}$ belongs to $H^r_{2,\lambda}$ if and only if $\tilde{h}^r_{2,\lambda}(f) < \infty$, and the norm in $H^r_{2,\lambda}$ is equivalent to the norm

$$^1\|f\|_{H^r_{2,\lambda}} := \|f\|_{2,\lambda} + \tilde{h}^r_{2,\lambda}(f).$$

In particular, if k and s are as $2k > r - 2s > 0$, then the spaces $H^r_{2,\lambda}$ coincide and their norms are equivalent.

In the following theorem the various equivalent norms in the spaces $B^r_{2,q,\lambda}$ will be obtain. In particular the Theorem 1.3 will follow. As in section 1, $r > 0$ and $a > 1$ are real numbers, and k and s are arbitrary non-negative integers such that $2k > r - 2s > 0$. We shall say that a function f belongs to the space ${}^jB^r_{2,q,\lambda}$, $j = 1,2,3,4$ if $f \in L_{2,\lambda}$ and the seminorm ${}^jb^r_{2,q,\lambda}$ is finite, where ${}^1b^r_{2,q,\lambda} := b^r_{2,q,\lambda}$ (the definition of $b^r_{2,q,\lambda}$ can be found in Section 1)

$$^2b^r_{2,q,\lambda}(f) := \begin{cases} \left(\int\limits_0^a \frac{\left(\omega_k(D^s_\lambda f,\delta)_{2,\lambda}\right)^q}{\delta^{(r-2s)q}} \frac{d\delta}{\delta} \right)^{\frac{1}{q}} & 1 \le q < \infty \ , \\ \sup\limits_{0<\delta\le a} \delta^{-(r-2s)} \omega_k\left(D^s_\lambda f,\delta\right)_{2,\lambda} & q = \infty \ ; \end{cases}$$

$$^3b^r_{2,q,\lambda}(f) := \begin{cases} \left(\sum\limits_{j=0}^{\infty} a^{jrq} \left(E_{a^j}(f)_{2,\lambda}\right)^q \right)^{\frac{1}{q}} & 1 \le q < \infty \ , \\ \sup\limits_{j\in Z_+} a^{jr} E_{a^j}(f)_{2,\lambda} & q = \infty \ ; \end{cases}$$

$$^4b^r_{2,q,\lambda}(f) := \begin{cases} \inf \left(\sum\limits_{j=0}^{\infty} a^{jrq} \|Q_{a^j}\|^q_{2,\lambda} \right)^{\frac{1}{q}} & 1 \le q < \infty \ , \\ \inf \sup\limits_{j\in Z_+} a^{jr} \|Q_{a^j}\|_{2,\lambda} & q = \infty \ ; \end{cases}$$

where the infimum is taken over all representations series of the form

$$f(x) = \sum_{j=0}^{\infty} Q_{a^j}(x) \ , \ Q_{a^j}(x) \in I_{a^j}$$

which is convergent in $L_{2,\lambda}$ from functions with bounded spectrum. The spaces ${}^jB^r_{2,q,\lambda}$ are Banach spaces concerning to the norms

$$\|f\|_{{}^jB^r_{2,q,\lambda}} := \|f\|_{2,\lambda} + {}^j b^r_{2,q,\lambda} \ . \tag{5.20}$$

Theorem 5.1. *The spaces* ${}^jb^r_{2,q,\lambda}$, $j = 1,2,3,4$ *coincide and their norms (5.20) are equivalent.*

Note that from the equivalency of the Banach spaces ${}^1B^r_{2,q,\lambda}$ and ${}^3B^r_{2,q,\lambda}$ Theorem 1.3 follows. For brevity we use the notation ${}^jB^r := {}^jB^r_{2,q,\lambda}$, ${}^jb := {}^jb^r_{2,q,\lambda}$, $E_N(f) := E_N(f)_{2,q,\lambda}$, $\|f\| := \|f\|_{2,\lambda}$ and so on. The expression $V_1 \hookrightarrow V_2$ means that the Banach space V_1 is embedded in the Banach space V_2.

Proof. The general scheme conforms to the scheme of the proof of analogues theorems in [27] for usual modulus of continuity. We will assume everywhere that $q < \infty$.

1^0. The embedding ${}^1B \to {}^2B$ is obvious. We prove that ${}^2B \hookrightarrow {}^3B$. Let $f \in {}^2B$, then

$$\left({}^2b(f)\right)^q = \int\limits_0^a \left(\omega_k\left(D^s_\lambda f,\delta\right)\right)^q_{2,\lambda} \delta^{(2s-r)q-1} d\delta = \sum_{j=0}^{\infty} \int_{a^{-j}}^{a^{1-j}} \left(\omega_k\left(D^s_\lambda f,\delta\right)\right)^q_{2,\lambda} \delta^{(2s-r)q-1} d\delta. \tag{5.21}$$

Using the monotony of the modulus of continuity $\omega_k\left(f,\delta\right)_{2,\lambda}$ and δ by Theorem 1.1 we obtain that

$$\int_{a^{-j}}^{a^{1-j}} \left(\omega_k\left(D_\lambda^s f,\delta\right)\right)_{2,\lambda}^q \delta^{(2s-r)q-1} d\delta$$

$$\geq \left(\omega_k\left(D_\lambda^s f,a^{-j}\right)\right)_{2,\lambda}^q \left(a^{1-j}\right)^{(2s-r)q-1} \left(a^{1-j}-a^{-j}\right) \geq c_1 a^{jrq}\left(E_{a^j}(f)\right)_{2,\lambda}^q , \tag{5.22}$$

where the constant c_1 is independent of f and j. From (5.21) and (5.22) it follows that

$$\left({}^2b(f)\right)^q \geq c_1\left({}^3b(f)\right)^q ,$$

from this the inequality ${}^3b(f) \leq c_2{}^2b(f)$ and the embedding ${}^2B \hookrightarrow {}^3B$ follow.

3^0. Prove that ${}^3B \hookrightarrow {}^4B$. Let $f \in {}^3B$. For every $j \in Z_+$ take the function $g_{a^j} \in I_{a^j}$, satisfied the condition

$$\|f - g_{a^j}\| \leq 2E_{a^j}(f).$$

Let $Q_{a^0} = g_{a^0}$, $Q_{a^j} = g_{a^j} - g_{a^{j-1}}$ for $j \geq 1$. Then the series $f = \sum\limits_{j=0}^{\infty} Q_{a^j}$ converges in $L_{2,\lambda}$, since $E_{a^j}(f) \to 0$ as $j \to \infty$.

We note that

$$\|Q_{a^0}\| \leq \|f\| + \|f - g_{a^0}\| \leq \|f\| + 2E_{a^0}(f) \leq 3\|f\|,$$

$$\|Q_{a^j}\| \leq \|g_{a^j} - f\| + \|f - g_{a^{j-1}}\| \leq 4E_{a^{j-1}}(f), \ j \geq 1.$$

Using these inequalities we obtain that

$$\left({}^4b(f)\right)^q \leq \sum_{j=1}^{\infty} a^{iqr}\|Q_{a^j}\|^q \leq 3^q\|f\|^q + \sum_{j=1}^{\infty} 4^q a^{jqr}\left(E_{a^{j-1}}(f)\right)^q ,$$

and then it follows that

$${}^4b(f) \leq c_3\left(\|f\| + \left(\sum_{j=0}^{\infty} a^{jrq}\left(E_{a^j}(f)\right)^q\right)^{\frac{1}{q}}\right) = c_3\|f\|_{{}^3B} .$$

From the last inequality the embedding ${}^3B \hookrightarrow {}^4B$ follows.

4^0. Prove that ${}^4B \hookrightarrow {}^1B$. Let $f \in {}^4B$, $\varepsilon > 0$, then one can present f as the sum

$$f = \sum_{j=0}^{\infty} Q_{a^j} , Q_{a^j} \in I_{a^j} ,$$

moreover

$$\left(\sum_{j=0}^{\infty} a^{jqr}\|Q_{a^j}\|^q\right)^{\frac{1}{q}} \leq {}^4b(f) + \varepsilon. \tag{5.23}$$

We check that the series $\sum\limits_{j=0}^{\infty} D_\lambda^s Q_{a^j}$ converges in $L_{2,\lambda}$. According to Lemma 5.1 we can write

$$\begin{aligned}\left\|D_\lambda^s Q_{a^j}\right\| &\leq a^{js}(a^j+2\lambda)^s\|Q_{a^j}\| \\ &= a^{-(r-2s)j}a^{jr}\left(1+\frac{2\lambda}{a^j}\right)^s\|Q_{a^j}\| \leq 2^s a^{jr}\cdot a^{-(r-2s)j}\|Q_{a^j}\|.\end{aligned}$$

By the Hölder inequality we obtain

$$\sum_{j=0}^{\infty}\left\|D_\lambda^s Q_{a^j}\right\| \leq 2^s \sum_{j=0}^{\infty} a^{-(r-2s)j}a^{jr}\|Q_{a^j}\|$$

$$\leq 2^s \left(\sum_{j=0}^{\infty} a^{-(r-2s)p}\right)^{\frac{1}{p}} \left(\sum_{j=0}^{\infty} a^{irq} \|Q_{a^j}\|^q\right)^{\frac{1}{q}} \leq c_4({}^4b(f)+\varepsilon). \tag{5.24}$$

Consequently, the series $\sum\limits_{j=0}^{\infty} D_\lambda^s Q_{a^j}$ converges in $L_{2,\lambda}$. From closeness of the operator D_λ it implies that

$$D_\lambda^s f = \sum_{j=0}^{\infty} D_\lambda^s Q_a j \in L_{2,\lambda}. \tag{5.25}$$

We note also that from (5.24) and (5.25) it follows that

$$\left\|D_\lambda^s f\right\| \leq c_4({}^4b(f)+\varepsilon). \tag{5.26}$$

From (5.7) and (5.26) we have

$$\int_1^{\infty} (\omega_k(D_\lambda^s f,\ \delta)_{2,\lambda})^q \delta^{-(r-2s)q-1} d\delta \ \leq\ 2^{kq} \left\|D_\lambda^s f\right\|^q \int_1^{\infty} \delta^{-(r-2s)q-1} d\delta$$

$$= \frac{2^{kq}}{(r-2s)q} \left\|D_\lambda^s f\right\|^q \leq c_5({}^4b(f)+\varepsilon). \tag{5.27}$$

For every naturally N we can write the equality

$$\varDelta_{cht}^k(D_\lambda^s f) = \sum_{j=0}^{N} \varDelta_{cht}^k(D_\lambda^s Q_{a^j}) + \sum_{j=N+1}^{\infty} \varDelta_{cht}^k(D_\lambda^s Q_{a^j}).$$

Using Lemmas 5.1 and 5.2 we obtain that

$$\left\|\varDelta_{cht}^k(D_\lambda^s f)\right\| \leq (cht-1)^k \sum_{j=0}^{N} (a^j(a^j+2\lambda))^{k+s} \|Q_{a^j}\|$$

$$+2^k \sum_{j=N+1}^{\infty} (a^j(a^j+2\lambda))^s \ \|Q_{a^j}\|. \tag{5.28}$$

Taking into account the inequality $(a+b)^s \leq 2^s(a^s+b^s)$, for $s \geq 1$ we have $(0<\lambda<\frac{1}{2})$

$$a^{js}(a^j+2\lambda)^s \leq 2^s a^{js}(a^{js}+1) \leq 2^{s+1} \cdot a^{2js}$$

and

$$a^{j(k+s)}\,(a^j+2\lambda)^{k+s} \leq 2^{k+s+1}\ a^{2j(k+s)}.$$

Taking into account these inequalities in (5.28) we obtain

$$\left\|\varDelta_{cht}^k(D_\lambda^s f)\right\| \leq 2^{s+1+k} \left(2sh^2\frac{t}{2}\right)^k \sum_{j=0}^{N} a^{2j(k+s)} \|Q_{a^j}\| + 2^{k+s+1} \sum_{j=N+1}^{\infty} a^{2js} \|Q_{a^j}\|.$$

Then we get

$$\omega_k(D_\lambda^s f,\, a^{-N}) = \sup_{0<t<a^{-N}} \left\|\varDelta_{cht}^k(D_\lambda^s f)\right\|$$

$$\leq 2^{2k+s+1} \ \left(sh\frac{1}{2a^N}\right)^{2k} \sum_{j=0}^{N} a^{2j(k+s)} \|Q_{a^j}\| + 2^{k+s+1} \sum_{j=N+1}^{\infty} a^{2js} \|Q_{a^j}\|. \tag{5.29}$$

Making the substitution $\delta = a^{-u}$ we have

$$\int_0^1 (\omega_k(D_\lambda^s f,\delta))^q \delta^{-(r-2s)q-1} d\delta = \log a \int_0^{\infty} (\omega_k(D_\lambda^s f, a^{-u}))^q a^{q(r-2s)u} du$$

$$= \log a \sum_{N=0}^{\infty} \int_{N}^{N+1} a^{q(r-2s)u} (\omega_k(D_\lambda^s f, a^{-u}))^q du$$

$$\leq \log a \sum_{N=0}^{\infty} (\omega_k(D_\lambda^s f, a^{-N}))^q a^{q(r-2s)(N+1)} \leq c_6\tau_1 + c_7\tau_2 \,, \tag{5.30}$$

where

$$\tau_1 = \sum_{N=0}^{\infty} a^{q(r-2s-2k)N} \cdot a^{2kqN} \left(sh\frac{1}{2a^N}\right)^{2kq} \left(\sum_{j=0}^{N} a^{2j(k+s)} \left\|Q_{a^j}\right\|\right)^q ,$$

$$\tau_2 = \sum_{N=0}^{\infty} a^{q(r-2s)N} \left(\sum_{j=N+1}^{\infty} a^{2js} \left\|Q_{a^j}\right\|\right)^q . \tag{5.31}$$

We prove the inequality

$$sh\frac{t}{a} \;\leq\; \frac{1}{a}sht \quad a \geq 1. \tag{5.32}$$

Consider the function $\varphi(t) = \frac{1}{a}sht - sh\frac{t}{a}$. From this

$$\varphi'(t) = \frac{1}{a}cht - \frac{1}{a}ch\frac{t}{a} = \frac{1}{a}(cht - ch\frac{t}{a}) \geq 0.$$

Consequently the function $\varphi(t)$ increases and takes the least meaning for $t = 0$ and $\varphi(0) = 0$, that is $\varphi(t) \geq 0$, from this it follows that (5.32) is satisfied.

Applying (5.32) we will have

$$a^{2kqN} \left(sh\frac{1}{2a^N}\right)^{2kq} \leq \frac{a^{2kqN}}{(2a^N)^{2kq}}(sh1)^{2kq} = \frac{(sh1)^{2kq}}{4^{kq}} = \left(\frac{e-e^{-1}}{4}\right)^{2kq} < 1.$$

From this it follows that

$$\tau_1 \leq \sum_{N=0}^{\infty} a^{q(r-2s-2k)N} \left(\sum_{j=0}^{N} a^{2j(k+s)} \left\|Q_{a^j}\right\|\right)^q . \tag{5.33}$$

For expressions (5.31) and (5.33) (in [27] see p. 260 the formulas (17),(18)) the estimates

$$\tau_1 \leq c_8 \sum_{j=0}^{\infty} a^{jrq} \left\|Q_{a^j}\right\|^q , \tag{5.34}$$

$$\tau_2 \leq c_9 \sum_{j=0}^{\infty} a^{jrq} \left\|Q_{a^j}\right\|^q . \tag{5.35}$$

are obtained.

Finally from (5.27), (5.30) and (5.35) it follows that

$$\int_0^{\infty} (\omega_k(D_\lambda^s f, \delta)_{2,\lambda})^q \delta^{-(r-2s)q-1} d\delta \leq C_{10}({}^4b(f) + \varepsilon)^q,$$

but from this we get

$${}^1b(f) \leq C_{10}^4 b(f),$$

that proves the embedding ${}^4B \to {}^1B$. As a result of these the chain of embedding is obtained

$${}^1B \hookrightarrow {}^2B \hookrightarrow {}^3B \hookrightarrow {}^4B \hookrightarrow {}^1B,$$

that complete the proof of the Theorem 5.1 for $1 \leq q < \infty$.

Now consider the case when $q = \infty$.

a) The embedding ${}^1B \hookrightarrow {}^2B$ is obvious. We prove that ${}^2B \hookrightarrow {}^3B$. Let $f \in {}^2B$, then

$$\sup_{0<\delta\le a} \frac{\omega_k(D_\lambda^s f, \delta)}{\delta^{r-2s}} \ge \sup_{\frac{1}{2}a^{-j}<\delta\le a^{-j}} \frac{\omega_k(D_\lambda^s f, \delta)}{\delta^{r-2s}}$$

$$\ge \frac{\omega_k(D_\lambda^s f, \frac{1}{2}a^{-j})}{a^{-j(r-2s)}} \ge 2^s (2a^j)^{2s} a^{j(r-2s)} E_{2a^j}(f)$$

$$= 2^{3s} a^{jr} E_{2a^j}(f) = 2^{3s-r}(2a^j)^r E_{2a^j}(f), \qquad (5.36)$$

here we use the monotony of $\omega_k(f,\delta)$ and Theorem 1.1. From (5.36) it follows that ${}^3b(f) \le c_1 {}^1b(f)$ and the embedding ${}^2B \hookrightarrow {}^3B$ is valid.

b) We prove that ${}^3B \hookrightarrow {}^4B$. Let $f \in {}^3B$. For every $j \in \mathbb{Z}_+$ take the function $g_{a^j} \in I_{a^j}$ such that the condition

$$\|f - g_{a^j}\| \le 2E_{a^j}(f)$$

is satisfied.

Let $Q_0 = g_0$. $Q_{a^j} = g_{a^j} - g_{a^{j-1}}$, $j \ge 1$. Then $f = \sum\limits_{j=0}^{\infty} Q_{a^j}$, the series converges in $L_{2,\lambda}$, so $E_{a^j}(f) \to 0$ as $j \to \infty$.

Note that

$$\|Q_{a^0}\| \le \|g_{a^0} - f\| + \|f\| \le \|f\| + 2E_{a^0}(f) \le 3\|f\|,$$

$$\|Q_{a^j}\| \le \|g_{a^j} - f\| + \|f - g_{a^{j-1}}\| \le 4E_{a^{j-1}}(f),\ j \ge 1.$$

From these inequalities it follows that

$$\begin{aligned} {}^4b(f) &\le 3\|f\| + 4a^{jr}E_{a^{j-1}}(f), \quad j = 1,2,... \\ &\le c_{11}\Big(\|f\| + a^{jr}E_{a^j}(f)\Big) = c_{12}\|f\|_{{}^3B}, \quad j = 0,1,... \end{aligned}$$

from this the embedding ${}^3B \hookrightarrow {}^4B$ follows.

Now we prove that ${}^4B \hookrightarrow {}^1B$. Let $f \in {}^4B$, $\forall\varepsilon > 0$, then f can be present as the form

$$f = \sum_{j=0}^{\infty} Q_{a^j}, \quad Q_{a^j} \in I_{a^j},$$

moreover for every $\varepsilon > 0$

$$a^{jr}\|Q_{a^j}\| \le {}^4b(f) + \varepsilon. \qquad (5.37)$$

We verify that the series $\sum\limits_{j=0}^{\infty} D_\lambda^s Q_{a^j}$ converges in $L_{2,\lambda}$. For Lemma 5.1 we can write

$$\begin{aligned} &\|D_\lambda^s Q_{a^j}\| \le a^{js}(a^j + 2\lambda)^s\|Q_{a^j}\| \\ &= a^{-(r-2s)j}a^{jr}\Big(1 + \frac{2\lambda}{a^j}\Big)^s\|Q_{a^j}\| \le a^{-(r-2s)j}a^{jr}\|Q_{a^j}\|. \end{aligned}$$

From here, taking into account (5.37) we obtain

$$\sum_{j=0}^{\infty}\|D_\lambda^s Q_{a^j}\| \le \sum_{j=0}^{\infty} a^{-(r-2s)j}a^{jr}\|Q_{a^j}\|$$

$$\leq \left({}^4b(t) + \varepsilon\right) \sum_{j=0}^{\infty} a^{-(r-2s)j} \leq c_{13}\left({}^4b(f) + \varepsilon\right), \tag{5.38}$$

consequently the series $\sum\limits_{j=0}^{\infty} D_\lambda^s Q_{a^j}$ converges in $L_{2,\lambda}$.

From closeness of the operator D_λ it implies that

$$D_\lambda^s f = \sum_{j=0}^{\infty} D_\lambda^s Q_{a^j} \in L_{2,\lambda}. \tag{5.39}$$

Now note that from (5.38) and (5.39) the inequality is valid.

$$\|D_\lambda^s f\| \leq c_{14}\, {}^4b(f). \tag{5.40}$$

From (5.8) and (5.40) it follows that

$${}^1b(f) \leq c_{14}{}^4b(f),$$

hence the embedding ${}^4B \hookrightarrow {}^1B$ is proved.

Thus for $q = \infty$ the chain of the embedding is also obtained,

$${}^1B \hookrightarrow {}^2B \hookrightarrow {}^3B \hookrightarrow {}^4B \hookrightarrow {}^1B,$$

that completes the proof of Theorem 5.1.

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