Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 52-60 (2015).
Series of Physical-Technical and Mathematical Sciences.

Null controllability of heat equation with switching pointwise controls

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Received: 07.06.2015 / Accepted: 06.11.2015

Abstract. *In [1], the author analyzed the problem of two switching pointwise controls for null controllability of the 1-d heat equation with Dirichlet's boundary conditions and obtained sufficient conditions for null controls satisfying switching conditions. In this article, we consider the 1-d heat equation endowed with arbitrary number (finite) of pointwise controls and under suitable conditions on the placement of actuators, we show that our approach allows building switching controls.*

Keywords. heat equation, variational approach, snitching controls

Mathematics Subject Classification (2010): 93B05, 93C20.

1 Introduction

Control problems for PDEs/ODEs arise in many different contexts and ways. A classical problem is that of controllability. Roughly speaking, it consists in observing whether the solution of the PDE/ODE can be driven to a given final target by means of a control applied on a sub-domain of the domain or on the boundary. More precisely, the controllability problem may be characterized as follows. Consider an evolution system either described in terms of partial or ordinary differential equations. Given a time interval $t \in (0, T)$, and initial and final states we have to find a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$. This is a type of exact controllability problem. There are different type of controllability problems: when the final target is achieved to zero, then the system is null controllable or when the set of reachable states (set of final targets) is dense in the space where the evolution system is satisfied, then the system is approximate controllable. However, in finite dimensions, these apparently weaker notions often coincide with the exact controllability one. For instance, when dealing with the problem of approximate controllability, as we know the system is said to be approximately controllable when the set of reachable states is dense in $\mathbb{R}$. But, in $\mathbb{R}$ the only close affine dense subspace is the whole space itself. Thus, in finite-dimension, approximate controllability and exact controllability are equivalent notions. But this is no longer the case in the context of PDE. Indeed, in infinite-dimensional spaces there are strict dense subspaces, while in finite-dimension they do not exist. These are classical problems in control theory and we recommend for instance, the book by Lee and Marcus [6] for an introduction in the context of finite-dimensional systems and the book of Lions [5] for an introduction to the controllability of PDE, also referred to as Distributed Parameter Systems.
In this paper, our aim is to build suitable switching pointwise controls for the controllability problem of 1-d heat equation. To do this we first introduce a new functional based on the adjoint system whose minimizers yield the switching controls. We show that, due to the time analyticity of solutions, under

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suitable conditions on the location of the controllers, switching control strategies exist in the 1-d heat equation.

2 Switching Pointwise Controls

In this part, we would consider the problem of null controllability of the 1-d heat equation with pointwise controls and obtain sufficient conditions for switching controls. But firstly we consider the problem in which three controllers act at three different points in the interval (0, 1) and then we generalize the obtained result. Consider the case in which three pointwise controllers act at three different points a, b, c of the space interval (0, 1) where the equation is satisfied:

$$\begin{cases} y_t - y_{xx} = u_a(t)\delta_a + u_b(t)\delta_b + u_c(t)\delta_c, & 0 < x < 1, 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.1}$$

Here, δ_a, δ_b and δ_c are Dirac delta functions located at points a, b and c respectively. We consider the problem of null controllability. More precisely, given an initial datum $y^0 \in L^2(0,1)$ we look for controls $u_a(t), u_b(t), u_c(t) \in L^2(0,T)$ such that $y(x,T) = 0$ and the switching condition satisfies:

$$u_a(t)u_b(t) = 0, \;\; u_a(t)u_c(t) = 0, \;\; u_b(t)u_c(t) = 0, \;\; \forall i \neq j, \quad \text{a.e. } t \in (0,T). \tag{2.2}$$

We know that whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system (see e.g., [2], [3]).
For φ^0 in $L^2(0,1)$, we consider the solution $\varphi : [0,1] \times [0,T] \to C([0,T], L^2(0,1))$, of the following backward Cauchy linear problem:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, & 0 < x < 1, 0 < t < T, \\ \varphi(0,t) = \varphi(1,t) = 0, & 0 < t < T, \\ \varphi(x,T) = \varphi^0(x), & 0 < x < 1. \end{cases} \tag{2.3}$$

This linear system is called the adjoint system corresponding to the 1-d heat equation with Dirichlet's boundary condition (see, e.g. [1], [2]). Let φ^0 has the Fourier expansion

$$\varphi^0 = \sum_{k \geq 1} \beta_k \omega_k(x), \quad \text{where } \omega_k(x) = \sqrt{2}\sin(k\pi x)$$

then the solution φ of adjoint system is of the form

$$\varphi(x,t) = \sum_{k \geq 1} \beta_k e^{\pi^2 k^2 (t-T)} \omega_k(x). \tag{2.4}$$

As we know that the null control of 1-d heat equation could be computed by minimizing the following quadratic functional (see, e.g. [1])

$$J(\varphi^0) = \frac{1}{2}\int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\Big] dt - \int_0^1 y^0(x)\varphi(x,0)dx$$

over the class $\mathcal{H}$ of initial data given by

$$\mathcal{H} = \Big\{\varphi^0 : \int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2 dt < \infty\Big\},$$

where $\varphi(x,t)$ is the solution of the adjoint system (3) associated to the final target φ^0. We consider $\mathcal{H}$ space endowed with the canonical norm

$$||\varphi^0||_{\mathcal{H}} = \Big[\int_0^T |\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2 dt\Big]^{\frac{1}{2}}$$

constitutes a Hilbert space (see, e.g. [1]). It is easy to see that the functional J is continuous in $\mathcal{H}$, and strictly convex. Now, we give very important lemma on families of real exponentials. This lemma is known as estimates on families of real exponentials (see e.g., [1], [3], [4]).

Lemma 2.1 *In our case, it is guaranteed that*

$$\int_0^T \Big|\sum_{k\geq 1}\beta_k e^{\pi^2k^2(t-T)}\Big|^2 dt \geq c_1 \sum_{k\geq 1} e^{-2\pi^2k^2T}\beta_k^2$$

for a suitable positive constants $c_1 > 0$ which is independent from $\{\beta_k\}_{k\geq 1}$.

By using this lemma, we obtain the following weighted observability inequality:

$$||\varphi^0||_{\mathcal{H}}^2 \geq c_1 \sum_{k\geq 1} e^{-2\pi^2k^2T}\Big[|\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2\Big]\beta_k^2. \tag{2.5}$$

As we know that null controllability in time T implies approximate controllability in time T. This comes form the fact that all the range of the semigroup generated by the heat equation is reachable (see e.g., [2]). Therefore, we first prove the approximate controllability of the heat system in time T under some conditions. For this, we will consider new functional very similar with previous one: for any $\epsilon > 0$ and any $y^1 \in L^2(0,1)$

$$J_\epsilon(\varphi^0) = \frac{1}{2}\int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\Big]dt + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)}$$
$$+ \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx,$$

where E is finite dimensional subspace of $L^2(0,1)$ and π_E denotes the ortogonal projection from $L^2(0,1)$ over E.

Our aim is to build approximate pointwise control satisfying Dirichlet's boundary condition. In other words, given $\epsilon > 0$ we try to find (finite) approximate controls $u_a^\epsilon, u_b^\epsilon, u_c^\epsilon$ such that the solution y_ϵ of heat equation satisfies the following condition

$$||y_\epsilon(x,T) - y^1||_{L^2(0,1)} \leq \epsilon. \tag{2.6}$$

For this to be true, the following property suffices (see e.g., [2])

If $\forall t \in (0,T)$ we have $\varphi(a,t) = \varphi(b,t) = \varphi(c,t) = 0$, then $\varphi(x,t) \equiv 0$

which is unique continuation property of the adjoint system.

Lemma 2.2 *Assume that the following unique continuation property*

$$\forall t \in (0,T),\ \varphi(a,t) = \varphi(b,t) = \varphi(c,t) = 0 \implies \varphi(x,t) \equiv 0 \tag{2.7}$$

holds, then the heat equation (1) is approximate controllable.

Proof. For obtaining approximate controllability of (1), we should minimize J_ϵ over $\mathcal{H}$. We have already proved that J_ϵ is convex and continuous in $\mathcal{H}$. On the other hand, in view of (7) above, one can prove that

$$\lim_{||\varphi^0||_{L^2(0,1)}\to\infty} \frac{J_\epsilon(\varphi^0)}{||\varphi^0||_{L^2(0,1)}} \geq \epsilon. \tag{2.8}$$

Let us give the proof of this coercivity property. In order to prove above inequality, let $\{\varphi_j^0\} \subset L^2(0,1)$ be sequence of initial data for the adjoint sytem with $||\varphi_j^0||_{L^2(0,1)} \to \infty$ Now normalize them by

$$\tilde{\varphi_j^0} = \frac{\varphi_j^0}{||\varphi_j^0||_{L^2(0,1)}}$$

so that $||\tilde{\varphi_j^0}||_{L^2(0,1)} = 1$. On the other hand, let $\tilde{\varphi}_j$ be the solution of adjoint system with initial data $\tilde{\varphi_j^0}$. Then we would have

$$J_\epsilon(\varphi_j^0)/||\varphi_j^0||_{L^2(0,1)} = \frac{1}{2}||\varphi_j^0||_{L^2(0,1)}\int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big]dt$$
$$+ \epsilon||(I-\pi_E)\tilde{\varphi}_j^0||_{L^2(0,1)} + \int_0^1 \tilde{\varphi}_j^0 y^1 dx - \int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx.$$

The following two cases may occur:

1. $\liminf_{j\to\infty} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big] dt > 0$. In this case we have

$$J_\epsilon(\varphi_j^0)/||\varphi_j^0||_{L^2(0,1)} \to \infty.$$

2. $\liminf_{j\to\infty} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\tilde{\varphi}_j(c,t)|^2\Big] dt = 0$.

For the last case, since $\tilde{\varphi}_j^0$ is bounded in $L^2(0,1)$, by extracting a subsequence we can guarantee that $\tilde{\varphi}_j^0 \rightharpoonup \psi^0$ weakly in $L^2(0,1)$ and so weakly in $\mathcal{H}$, moreover

$$\varphi^0 \longmapsto \int_0^T \Big[|\varphi(a,t)|^2 + |\varphi(b,t)|^2 + |\varphi(c,t)|^2\Big] dt$$

is lower semi-continuous in the weak topology of $\mathcal{H}$.
Therefore we would obtain

$$\int_0^T \Big[|\psi(a,t)|^2 + |\psi(b,t)|^2 + |\psi(c,t)|^2\Big] dt \le \liminf_{j\to\infty} \int_0^T \Big[|\tilde{\varphi}_j(a,t)|^2 + |\tilde{\varphi}_j(b,t)|^2 + |\psi(c,t)|^2\Big] dt,$$

where ψ is the solution of adjoint system with given initial data ψ^0. Therefore we obtain the following condition

$$\forall t \in (0,T), \psi(a,t) = \psi(b,t) = \psi(c,t) = 0.$$

From (7) we conclude that $\psi \equiv 0$. Therefore $\psi^0 = 0$ and $\tilde{\varphi}_j^0 \rightharpoonup 0$ weakly in $L^2(0,1)$ and consequently

$$\int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx \to 0,$$

and

$$\int_0^1 \tilde{\varphi}_j^0 y^1 dx \to 0.$$

Furthermore, E is being finite-dimensional, π_E would be compact operator and then $\pi_E\tilde{\varphi}_j^0 \to 0$ strongly in $L^2(0,1)$. Consequently,

$$||(I-\pi_E)\tilde{\varphi}_j^0||_{L^2(0,1)} \to 1$$

as $j\to\infty$. At the end, we obtain our coercivity property

$$\liminf_{j\to\infty} \frac{J_\epsilon(\varphi_j^0)}{||\varphi_j^0||} \ge \liminf_{j\to\infty} \Big[\epsilon + \int_0^1 \tilde{\varphi}_j^0 y^1 dx - \int_0^1 y^0(x)\tilde{\varphi}_j(x,0)dx\Big] = \epsilon.$$

Therefore J_ϵ admits an unique minimizer $\hat{\varphi}^0 \in \mathcal{H}$. That means, for any $\psi^0 \in L^2(0,1)$ and $h \in \mathbb{R}$ we have $J_\epsilon(\hat{\varphi}^0) \le J_\epsilon(\hat{\varphi}^0 + h\psi^0)$. Namely,

$$\begin{aligned} J_\epsilon(\hat{\varphi}^0 + h\psi^0) - J_\epsilon(\hat{\varphi}^0) = & \int_0^T h\Big[\hat{\varphi}(a,t)\psi(a,t) + \hat{\varphi}(b,t)\psi(b,t) + \hat{\varphi}(c,t)\psi(c,t)\Big]dt \\ & + \int_0^T h^2\Big[\psi(a,t)^2 + \psi(b,t)^2 + \psi(c,t)^2\Big]dt \\ & + \epsilon\Big[||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)}\Big] \\ & + \int_0^1 h\psi^0 y^1 dx - \int_0^1 h\psi(x,0)y^0(x)dx \ge 0. \end{aligned}$$

We know from triangular inequality that

$$\Big[||(I-\pi_E)(\hat{\varphi}^0 + h\psi^0)||_{L^2(0,1)} - ||(I-\pi_E)\hat{\varphi}^0||_{L^2(0,1)}\Big] \le |h|||((I-\pi_E)\psi^0||_{L^2(0,1)}. \tag{2.9}$$

Now, let us define

$$\mathcal{A} \overset{\text{def}}{=} \int_0^T \Big[\hat{\varphi}(a,t)\psi(a,t) + \hat{\varphi}(b,t)\psi(b,t) + \hat{\varphi}(c,t)\psi(c,t)\Big] dt.$$

Then using above inequality and after considering the cases: $h > 0$, $h < 0$ and taking $h \to 0$ at the end, we would get the following inequality:

$$\Big|\mathcal{A} + \int_0^1 \psi^0 y^1 dx - \int_0^1 \psi(x,0) y^0(x) dx\Big| \leq \epsilon\Big[||((I - \pi_E)\psi^0||_{L^2(0,1)}\Big]. \tag{2.10}$$

Now, if we take $u_a(t) = -\hat{\varphi}(a,t)$, $u_b(t) = -\hat{\varphi}(b,t)$, $u_c(t) = -\hat{\varphi}(c,t)$, and multiplying the heat equation (2.1) with initial data $y^0(x) \in L^2(0,1)$ by ψ which is the solution of adjoint system (2.3) with initial data ψ^0 and integrating by parts we finally get

$$\mathcal{A} = \int_0^1 \psi(x,0) y^0(x) dx - \int_0^1 \psi^0 y(x,T) dx.$$

By combining these two and letting $E = 0$,[1] we finally get

$$\Big| \int_0^1 \psi^0 (y(x,T) - y^1) dx\Big| \leq \epsilon ||\psi^0||_{L^2(0,1)}$$

for every $\psi^0 \in L^2(0,1)$ which is equivalent to

$$||y(x,T) - y^1||_{L^2(0,1)} \leq \epsilon. \tag{2.11}$$

Therefore for every $\epsilon > 0$, by using variational approach, we obtain the following approximate controls

$$\begin{cases} u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t), \\ u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t), \\ u_c^\epsilon(t) = -\hat{\varphi}_\epsilon(c,t). \end{cases} \tag{2.12}$$

Now, to get null controls, we should prove that $u_a^\epsilon(t), u_b^\epsilon(t), u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$. We know that the space of null controllable initial data is the dual one $\mathcal{H}'$. Therefore, to get null controllability of (1), we should put some conditions on the Fourier coefficients $\{y_k^0\}_{k\geq 1}$ of initial datum y^0.

Lemma 2.3 *Assume that the Fourier coefficients $\{y_k^0\}_{k\geq 1}$ of initial datum y^0 of (1) satisfy the finiteness property*

$$\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{|\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2} |y_k^0|^2 < \infty. \tag{2.13}$$

Then, $y^0 \in \mathcal{H}'$ which is the dual one of $\mathcal{H}$ and our approximate controls $u_a^\epsilon(t), u_b^\epsilon(t), u_c^\epsilon(t)$ would be uniformly bounded in $L^2(0,T)$.

Proof. Using (5) and Cauchy-Schwarz (CS) inequality (see e.g., [8]), we would get the following

$$\begin{aligned} \frac{\Big|\sum_{k\geq 1} y_k^0 \beta_k\Big|}{||\varphi^0||_{\mathcal{H}}} &\leq C \frac{\Big|\sum_{k\geq 1} y_k^0 \upsilon_k \beta_k \upsilon_k^{-1}\Big|}{\Big|\sum_{k\geq 1} (\beta_k \upsilon_k^{-1})^2\Big|^{\frac{1}{2}}} \overset{\text{CS}}{\leq} C \frac{\Big|\sum_{k\geq 1} (y_k^0 \upsilon_k)^2\Big|^{\frac{1}{2}} \Big|\sum_{k\geq 1} (\beta_k \upsilon_k^{-1})^2\Big|^{\frac{1}{2}}}{\Big|\sum_{k\geq 1} (\beta_k \upsilon_k^{-1})^2\Big|^{\frac{1}{2}}} \\ &\leq C\Big[\sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{|\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2} |y_k^0|^2\Big]^{\frac{1}{2}} < \infty, \end{aligned}$$

[1] In this case, finite approximate controllability turns out to be approximate controllability of (1)

where

$$\upsilon_k = \Big|\frac{e^{2\pi^2 k^2 T}}{|\omega_k(a)|^2 + |\omega_k(b)|^2 + |\omega_k(c)|^2}\Big|^{\frac{1}{2}}.$$

As a result we get $y^0 \in \mathcal{H}^{'}$.

Now, we will prove that our approximate controls $u_a^\epsilon(t)$, $u_b^\epsilon(t)$ and $u_c^\epsilon(t)$ satisfy uniform boundedness in $L^2(0,T)$. Note that $u_a^\epsilon(t) = -\hat{\varphi}_\epsilon(a,t)$, $u_b^\epsilon(t) = -\hat{\varphi}_\epsilon(b,t)$, $u_c^\epsilon(t) = -\hat{\varphi}_\epsilon(c,t)$ where $\hat{\varphi}_\epsilon(x,t)$ solves adjoint system (3) with initial data $\hat{\varphi}_\epsilon^0$ at time $t = T$ obtained by minimizing the functional J_ϵ when $E = 0$ and $y^1 = 0$. At the minimizer $\hat{\varphi}_\epsilon^0$, we have $J_\epsilon(\hat{\varphi}_\epsilon^0) \leq J_\epsilon(0) = 0$. This implies that

$$\frac{1}{2}\int_0^T \Big[|\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2\Big]dt \leq \Big|\int_0^1 y^0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|.$$

From (5), we have

$$\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt \leq \frac{\hat{C}\Big|\int_0^1 y^0(x)\hat{\varphi}_\epsilon(x,0)dx\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2 \upsilon_k^{-2}\Big|}$$

for suitable $\hat{C} > 0$ which is independent from $\{\beta_k\}_{k\geq 1}$.
Since $\{\omega_k(x)\}_{k\geq 1}$ form orthogonal basis in $L^2(0,1)$ after some simplification, we have

$$\Big|\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt\Big| \leq \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2\upsilon_k^{-2}\Big|}.$$

But applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\beta_k\Big|^2}{\Big|\sum_{k\geq 1}\beta_k^2\upsilon_k^{-2}\Big|} &= \frac{\hat{C}\Big|\sum_{k\geq 1} y_k^0\upsilon_k\beta_k\upsilon_k^{-1}\Big|^2}{\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|}\\ &\overset{\text{CS}}{\leq} \frac{\hat{C}\Big|\sum_{k\geq 1}(y_k^0\upsilon_k)^2\Big|\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|}{\Big|\sum_{k\geq 1}(\beta_k\upsilon_k^{-1})^2\Big|} = \hat{C}\sum_{k\geq 1}(y_k^0\upsilon_k)^2.\end{aligned}$$

Hence, at the end we get the following result

$$\Big|\int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt\Big| \leq \tilde{C}\sum_{k\geq 1}|\upsilon_k|^2|y_k^0|^2 < \infty, \tag{2.14}$$

where $\tilde{C} > 0$ is independent from $\{\beta_k\}_{k\geq 1}$. Since we know that

$$||u_a^\epsilon(t)||^2_{L^2(0,T)} \leq \int_0^T |\hat{\varphi}_\epsilon(a,t)|^2 + |\hat{\varphi}_\epsilon(b,t)|^2 + |\hat{\varphi}_\epsilon(c,t)|^2 dt,$$

and similarly, the above inequality is valid for u_b^ϵ and u_c^ϵ.

Now, using (14) we conclude that $\forall\epsilon > 0$, $u_a^\epsilon(t), u_b^\epsilon(t)$ and $u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$.

Therefore, by using Lemma 2.3, we conclude that under the assumption of finiteness property, $u_a^\epsilon(t)$, $u_b^\epsilon(t)$ and $u_c^\epsilon(t)$ are uniformly bounded in $L^2(0,T)$ and so, by extracting subsequences, we have $u_a^\epsilon \rightharpoonup u_a$, $u_b^\epsilon \rightharpoonup u_b$ and $u_c^\epsilon \rightharpoonup u_c$ weakly in $L^2(0,T)$. Using the continuous dependence of the solution of the heat equation, we can show that $y_\epsilon(x,T)$ converges to $y(x,T)$ weakly in $L^2(0,T)$ which implies that $y(x,T) = 0$ (easily seen by letting $y^1 = 0$ in (6)), i.e., the limit controls u_a, u_b and u_c fulfil the null controllability requirement.

However, we check that our null controls do not fulfil the switching condition (2) in general (see e.g., [1]). Therefore, we realize that minimizing J over $\mathcal{H}$ just solves the problem of null controllability of heat system, but we still have no switching controls. For getting switching controls, we will consider the following functional J_s, which is a variant of our functional J, with the same coercivity properties, allows building switching controllers:

$$J_s(\varphi^0) = \frac{1}{2}\int_0^T \max\left\{|\varphi(a,t)|^2, |\varphi(b,t)|^2, |\varphi(c,t)|^2\right\}dt - \int_0^T y^0(x)\varphi(x,0)dx. \tag{2.15}$$

The functional $J_s : \mathcal{H} \longrightarrow \mathbb{R}$ is well defined, continuous thanks to well-poseness of adjoint system (3) and convexity comes from the following inequality: for given $a_1, a_2, b_1, b_2 \in \mathbb{R}$,

$$\max((a_1+a_2)^2, (b_1+b_2)^2) \le \max(a_1^2, b_1^2) + 2\max(a_1a_2, b_1b_2) + \max(a_2^2, b_2^2). \tag{2.16}$$

Same as before, we consider the problem of approximate controllability, i.e., for all $\epsilon > 0$ we could find (finite) approximate controls $u_a^\epsilon(t), u_b^\epsilon(t)$ and $u_c^\epsilon(t)$ such that the solution y_ϵ of heat equation satisfies (6). For obtaining approximate controls, we should consider the following new functional very similar with (15): for any $\epsilon > 0$ and any $y^1 \in L^2(0,1)$

$$\begin{aligned} J_s^\epsilon(\varphi^0) = \frac{1}{2}\int_0^T \max\left\{|\varphi(a,t)|^2, |\varphi(b,t)|^2, |\varphi(c,t)|^2\right\}dt + \epsilon||(I-\pi_E)\varphi^0||_{L^2(0,1)} \\ + \int_0^1 \varphi^0 y^1 dx - \int_0^1 y^0(x)\varphi(x,0)dx, \end{aligned}$$

where E is finite dimensional subspace of $L^2(0,1)$ and π_E denotes the orthogonal projection from $L^2(0,1)$ over E. Observe that when $y^1 = 0$ and $E = 0$ we would obtain our previous functional J_s.

Lemma 2.4 *Assume that the following unique continuation property*

$$\mu\{t \in (0,T) : |\varphi(a,t)| = |\varphi(b,t)| = |\varphi(c,t)|\} > 0 \Rightarrow \varphi \equiv 0 \tag{2.17}$$

holds. Then, the heat equation (1) is approximate controllable.

Proof. We skip the proof of that lemma which is closely related with Lemma 2.2 (see e.g., [1]).

Hence, from Lemma 2.4, we know that for getting approximate controllability of (1), we need to have (17). Since we know that

$$\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} \subset \{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)|\},$$
$$\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} \subset \{t \in (0,T) : |\psi(b,t)| = |\psi(c,t)|\},$$
$$\{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)| = |\psi(c,t)|\} \subset \{t \in (0,T) : |\psi(c,t)| = |\psi(a,t)|\}.$$

Therefore, we obtain that

$$\begin{aligned} I_a^b &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(a,t)| = |\psi(b,t)|\}, \\ I_b^c &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(b,t)| = |\psi(c,t)|\}, \\ I_c^a &\stackrel{\text{def}}{=} \{t \in (0,T) : |\psi(c,t)| = |\psi(a,t)|\}, \end{aligned}$$

are of positive measure. Now using again the Fourier representation of solution of (3) we have

$$\varphi(a,t) \pm \varphi(b,t) = \sum_{k \ge 1} \beta_k e^{\pi^2 k^2 (t-T)}(\omega_k(a) \pm \omega_k(b)).$$

The function $\varphi(a,t) \pm \varphi(b,t)$ are time analytic for $t \le T$ (see e.g., [1]). Consequently, if they vanish for a set of time instants of positive measure, then they vanish for all $t \le T$. It is then easy to see, by multiplying above identity by the real exponentials $e^{-\eta^2(t-T)}$ successively, starting from $\eta = 1$ and taking limits as $t \to -\infty$, that

$$\beta_k(\omega_k(a) \pm \omega_k(b)) = 0, \qquad \forall k \ge 1.$$

To conclude that $\beta_k = 0$ for all $k \ge 1$, it is sufficient to show that

$$\omega_k(a) \pm \omega_k(b) = \sin(k\pi a) \pm \sin(k\pi b) \ne 0 \qquad \forall k \ge 1.$$

This holds if and only if

$$a \pm b \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.18}$$

Similarly, we have

$$b \pm c \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.19}$$

$$c \pm a \neq m/k, \qquad \forall k \geq 1, \quad m \in \mathbb{Z}. \tag{2.20}$$

As a result, under irrationality conditions (18), (19), and (20) we have the unique continuation property. Observe that if the unique continuation property satisfies, then for $\varphi^0 \neq 0$, we have I_a^b, I_b^c, I_c^a and I are of measure zero. Now define

$$S_a \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(a,t)| > \max(|\varphi(b,t)|, |\varphi(c,t)|)\}$$

$$S_b \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(b,t)| > \max(|\varphi(a,t)|, |\varphi(c,t)|)\}$$

$$S_c \stackrel{\text{def}}{=} \{t \in (0,T) : |\varphi(c,t)| > \max(|\varphi(b,t)|, |\varphi(a,t)|)\}.$$

We know that J_s^ϵ admits an unique minimizer $\hat{\varphi}^0 \in \mathcal{H}$. Namely, for any $\psi^0 \in L^2(0,1)$ and $h \in \mathbb{R}$ sufficiently small, we would have $J_s^\epsilon(\hat{\varphi}^0) \leq J_s^\epsilon(\hat{\varphi}^0 + h\psi^0)$. Hence by using variational approach, for all $\epsilon > 0$, we obtain the following approximate switching controls

$$u_a^\epsilon(t) = -\chi_{S_a}\hat{\varphi}_\epsilon(a,t), \qquad u_b^\epsilon(t) = -\chi_{S_b}\hat{\varphi}_\epsilon(b,t), \qquad u_c^\epsilon(t) = -\chi_{S_c}\hat{\varphi}_\epsilon(c,t),$$

where χ_{S_i} is the characteristic function defined on the set S_i which gets 1 in S_i and 0 otherwise for $i = a, b, c$. Using Lemma 2.3, as $\epsilon \to 0$, we obtain the null switching controls. In conclusion, we have the following result

Theorem 2.1 *Assume that points a, b, c in the interval (0,1) are such that the irrationality conditions hold (i.e., $a \pm b, b \pm c, c \pm a \neq m/k$). Let the initial datum y^0 be in $\mathcal{H}'$ which is the dual one of the space $\mathcal{H}$. More precisely, assume that Fourier coefficients of y^0 satisfying (13). Then, for all $T > 0$, there exist switching controls*

$$u_a(t) = -\hat{\varphi}(a,t)\chi_{S_a}, \qquad u_b(t) = -\hat{\varphi}(b,t)\chi_{S_b}, \qquad u_c(t) = -\hat{\varphi}(c,t)\chi_{S_c},$$

satisfying switching condition (2) and that the solution of heat equation (1) satisfies

$$y(x,T) = 0,$$

i.e, null controllability are satisfied. These switching controls obtained by minimizing the functional (15) over $\mathcal{H}$.

In general, we could examine the case in which $n \in \mathbb{N}$, pointwise controllers act at n different points $(a_i)_{i=1}^{i=n}$ of the space interval $(0,1)$.

Consider the heat system

$$\begin{cases} y_t - y_{xx} = \sum_{i=1}^n u_{a_i}(t)\delta_{a_i}(x), & 0 < x < 1, 0 < t < T, \\ y(0,t) = y(1,t) = 0, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1. \end{cases} \tag{2.21}$$

Here now, given an initial datum $y^0 \in L^2(0,1)$ we are looking for controls $\{u_{a_i}(t)\}_{i=1}^{i=n} \in L^2(0,T)$ such that null controllability of heat equation holds, i.e, $y(x,T) = 0$ and switching condition satisfies:

$$u_{a_i}(t)u_{a_j}(t) = 0, \;\; \forall i \neq j, \quad \text{a.e. } t \in (0,T). \tag{2.22}$$

At first, we consider the approximate controllability problem. To obtain approximate switching controls, one should minimize an appropriate quadratic functional over suitable Hilbert space, and under some conditions on the Fourier coefficients of y^0, we will get our desired null switching controls satisfying switching property. Consequently, we obtain following general new result for switching controls:

Theorem 2.2 *Assume that points* $\{a_i\}_{i=0}^{i=n}$ *in the interval (0,1) are such that the irrationality conditions hold (i.e,* $a_i \pm a_j$*, are irrationals* $\forall i \neq j$ *). Let the initial datum* y^0 *be in* H_n' *which is the dual space of class of initial data of adjoint system (3)*

$$H_n = \{\varphi^0 : \int_0^T \sum_{i=1}^n |\varphi(a_i,t)|^2 dt < \infty\}.$$

More precisely, let y^0 *be of the form*

$$y^0 = \sum_{k\geq 1} y_k^0 \omega_k(x) \quad \textit{with} \quad \sum_{k\geq 1} \frac{e^{2\pi^2 k^2 T}}{\sum_{i=1}^{i=n} |\omega_k(a_i)|^2} |y_k^0|^2 < \infty.$$

Then, for all $T > 0$*, there exist switching controls* $\{u_{a_i}(t)\}_{i=1}^{i=n} \in L^2(0,T)$ *satisfying (22) and that the solution of heat equation satisfies*

$$y(x,T) = 0,$$

i.e, null controllability are satisfied. These switching controls are

$$u_{a_i}(t) = -\hat{\varphi}(a_i,t), \;\; u_{a_j}(t) = 0 \;\; \textit{for } j \neq i, \quad \textit{in } S_{a_i} \quad \forall i \in \{1,2,...,n\},$$

where

$$S_{a_i} = \Big\{ t \in (0,T) : |\varphi(a_i,t)| > \max_{\substack{1\leq j\leq n \\ j\neq i}} \{|\varphi(a_j,t)|\} \Big\},$$

and $\hat{\varphi}^0 = \hat{\varphi}(x,T)$ *is the minimizer of the functional*

$$J_s^n(\varphi^0) = \frac{1}{2}\int_0^T \max_{1\leq i\leq n} \{|\varphi(a_i,t)|^2\} dt - \int_0^1 y^0(x)\varphi(x,0)dx,$$

where $\hat{\varphi}(x,t)$ *is the solution of (3) with initial data* $\hat{\varphi}^0$*.*

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