Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 61-74 (2015).
Series of Physical-Technical and Mathematical Sciences.

Commutators of multilinear singular integral operators on generalized local Morrey spaces

A.A. Hasanov · Sh.A. Muradova · E.A. Gadjieva

Received: 25.06.2015 / Accepted: 30.10.2015

Abstract. *Let* $\overrightarrow{b} = (b_1, \ldots, b_m)$ *be a finite family of locally integrable functions. In this paper the authors study the boundedness of the commutators of multilinear singular integral operators* $T_m^{\overrightarrow{b}}$ *on product generalized Morrey spaces* $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$. *We find the sufficient conditions on* $(\varphi_1, \ldots, \varphi_m, \varphi)$ *which ensures the boundedness of the operator* $T_m^{\overrightarrow{b}}$ *from* $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ *to* $LM^{\{x_0\}}_{p,\varphi}$. *In all cases, the conditions for the boundedness of* T_m *are given in terms of Zygmund-type integral inequalities on* $(\varphi_1, \ldots, \varphi_m, \varphi)$, *which do not assume any assumption on monotonicity of* $\varphi_1, \ldots, \varphi_m, \varphi$ *in* r.

Keywords. Generalized local Morrey space · multilinear Calderón-Zygmund operator.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderon-Zygmund operators was done by Coifman and Meyer in [12] and was later systematically studied by Grafakos and Torres in [19]-[20].

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_n)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

The research of Sh. Muradova was partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan, Grant EIF-2013-9(15)-46/10/1. The research of E. Gadjieva and Sh. Muradova was partially supported by the grant of Presidium Azerbaijan National Academy of Science 2015.

A.A. Hasanov
E-mail: A.A. Hasanov
amilhasanov@gmail.com
Gandja State University, Gandja, Azerbaijan

Sh.A. Muradova
E-mail: mshams01@yahoo.com
Institute of Mathematics and Mechanics, Baku, Azerbaijan
Baku State University, Baku, Azerbaijan

E.A. Gadjieva
E-mail: egadjieva@gmail.com
Institute of Mathematics and Mechanics, Baku, Azerbaijan

In 2002 Grafakos and Torres [18]-[20] studied the multilinear Calderón-Zygmund operator which can be written for $x \notin \cap_{j=1}^{m} \mathrm{supp} f_j$ as

$$T_m(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) f_1(y_1) \ldots f_m(y_m)\, dy_1 dy_2 \ldots dy_m,$$

where $K(x, y_1, \ldots, y_m)$ is the kernel function defined of the diagonal $x = y_1 = \ldots y_m$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|K(y_0, y_1, \ldots, y_m)| \le c_1 \Big(\sum_{k,l=0}^{m} |y_k - y_l| \Big)^{-nm}, \tag{1.1}$$

and whenever $2|y_j - y_j'| \le \frac{1}{2} \max\limits_{0 \le k \le m} |y_j - y_k|$,

$$|K(y_0, \ldots, y_j, \ldots, y_m) - K(y_0, \ldots, y_j', \ldots, y_m)| \le \frac{c_1 |y_j - y_j'|^{\epsilon}}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l| \Big)^{nm+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \le j \le m$. Grafakos and Torres [18] proved that the operator $K_m : \overrightarrow{f} \to K_m(\overrightarrow{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \ldots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1 (i = 1, \ldots, m)$ and $1/p = 1/p_1 + \ldots + 1/p_m$, and bounded from $L_1(\mathbb{R}^n) \times \ldots \times L_1(\mathbb{R}^n)$ to $L_{\frac{1}{m},\infty}(\mathbb{R}^n)$.

Let $\overrightarrow{b} = (b_1, \ldots, b_m)$ be a finite family of locally integrable functions, then the commutators generated by the m-th Calderón-Zygmund type singular integral and $\overrightarrow{b}$ is defined by:

$$T_m^{\overrightarrow{b}}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \ldots, y_m) \prod_{i=1}^{m} \big(b_i(x) - b_i(y_i) \big) f_i(y_i)\, dy_1 dy_2 \ldots dy_m.$$

In this work, we prove the boundedness of the multilinear singular integral operators T_m from product generalized local Morrey space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$, if $1 < p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$, and from the space $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to the weak space $WLM_{1,\varphi}^{\{x_0\}}$, if $1 \le p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ and at least one exponent p_i equals one. In the case $b_i \in LC_{q_i,\lambda_i}^{\{x_0\}}$, for $0 < \lambda_i < 1/n$, $i = 1, 2, \ldots, m$, we find the sufficient conditions on the pair (φ_1, φ_2) which ensures the boundedness of the commutator operators $T_m^{\overrightarrow{b}}$ from $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$, $1 < p, p_i, q_i < \infty$, for $i = 1, 2, \ldots, m$ such that $1/p = 1/p_1 + \ldots + 1/p_n + 1/q_1 + \ldots + 1/q_n$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 Generalized local Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ play an important role, see [16], [28]. Introduced by C. Morrey [32] in 1938, they are defined by the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \le \lambda < n$, $1 \le p < \infty$.

We also denote by $WM_{p,\lambda} \equiv WM_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

We find it convenient to define the generalized local Morrey spaces in the form as follows.

Definition 2.1 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey space $M_{p,\lambda}$ and weak Morrey space $WM_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \qquad WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

Definition 2.2 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. We denote by $LM_{p,\varphi} \equiv LM_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(0,r))}.$$

Also by $WLM_{p,\varphi} \equiv WLM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}} = \sup_{r>0} \varphi(0,r)^{-1} |B(0,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(0,r))} < \infty.$$

Definition 2.3 *Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in \mathbb{R}^n$ we denote by $LM_{p,\varphi}^{\{x_0\}} \equiv LM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ the generalized local Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm*

$$\|f\|_{LM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi}}.$$

Also by $WLM_{p,\varphi}^{\{x_0\}} \equiv WLM_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WLM_{p,\varphi}^{\{x_0\}}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi}} < \infty.$$

According to this definition, we recover the local Morrey space $LM_{p,\lambda}^{\{x_0\}}$ and weak local Morrey space $WLM_{p,\lambda}^{\{x_0\}}$ under the choice $\varphi(x_0,r) = r^{\frac{\lambda-n}{p}}$:

$$LM_{p,\lambda}^{\{x_0\}} = LM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}, \qquad WLM_{p,\lambda}^{\{x_0\}} = WLM_{p,\varphi}^{\{x_0\}}\Big|_{\varphi(x_0,r)=r^{\frac{\lambda-n}{p}}}.$$

Wiener [35], [36] looked for a way to describe the behavior of a function at the infinity. The conditions he considered are related to appropriate weighted L_q spaces. Beurling [3] extended this idea and defined a pair of dual Banach spaces A_q and $B_{q'}$, where $1/q + 1/q' = 1$. To be precise, A_q is a Banach algebra with respect to the convolution, expressed as a union of certain weighted L_q spaces; the space $B_{q'}$ is expressed as the intersection of the corresponding weighted $L_{q'}$ spaces. Feichtinger [15] observed that the space B_q can be described by

$$\|f\|_{B_q} = \sup_{k\ge 0} 2^{-\frac{kn}{q}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}, \tag{2.1}$$

where χ_0 is the characteristic function of the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$, χ_k is the characteristic function of the annulus $\{x \in \mathbb{R}^n : 2^{k-1} < |x| \leq 2^k\}$, $k = 1, 2, \ldots$. By duality, the space $A_q(\mathbb{R}^n)$, called Beurling algebra now, can be described by

$$\|f\|_{A_q} = \sum_{k=0}^{\infty} 2^{-\frac{kn}{q'}} \|f\chi_k\|_{L_q(\mathbb{R}^n)}. \tag{2.2}$$

Let $\dot{B}_q(\mathbb{R}^n)$ and $\dot{A}_q(\mathbb{R}^n)$ be the homogeneous versions of $B_q(\mathbb{R}^n)$ and $A_q(\mathbb{R}^n)$ by taking $k \in \mathbb{Z}$ in (2.1) and (2.2) instead of $k \geq 0$ there.

If $\lambda < 0$ or $\lambda > n$, then $LM_{p,\lambda}^{\{x_0\}}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. Note that $LM_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $LM_{p,n}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$.

$$\dot{B}_{p,\mu} = LM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}, \qquad W\dot{B}_{p,\mu} = WLM_{p,\varphi}\Big|_{\varphi(0,r)=r^{\mu n}}.$$

Alvarez, Guzman-Partida and Lakey [2] in order to study the relationship between central BMO spaces and Morrey spaces, they introduced λ-central bounded mean oscillation spaces and central Morrey spaces $\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv LM_{p,n+np\mu}(\mathbb{R}^n)$, $\mu \in [-\frac{1}{p}, 0]$. If $\mu < -\frac{1}{p}$ or $\mu > 0$, then $\dot{B}_{p,\mu}(\mathbb{R}^n) = \Theta$. Note that $\dot{B}_{p,-\frac{1}{p}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ and $\dot{B}_{p,0}(\mathbb{R}^n) = \dot{B}_p(\mathbb{R}^n)$. Also define the weak central Morrey spaces $W\dot{B}_{p,\mu}(\mathbb{R}^n) \equiv WLM_{p,n+np\mu}(\mathbb{R}^n)$.

The following statements, containing results obtained in [31], [33] was proved in [21], [23] (see also [1], [4]-[7], [22]).

Theorem 2.1 *Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \varphi_1(x_0, t) \frac{dt}{t} \leq C\, \varphi_2(x_0, r), \tag{2.3}$$

where C does not depend on x_0 and r. Then the operator T is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi_2}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi_2}^{\{x_0\}}$ for $p = 1$.

Corollary 2.1 *Let $1 \leq p < \infty$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \varphi_1(x, t) \frac{dt}{t} \leq C\, \varphi_2(x, r), \tag{2.4}$$

where C does not depend on x and r. Then the operator T is bounded from M_{p,φ_1} to M_{p,φ_2} for $p > 1$ and from M_{1,φ_1} to WM_{1,φ_2} for $p = 1$.

The following statements, containing results Theorem 2.1 was proved in [1], see also [24].

Theorem 2.2 *Let $x_0 \in \mathbb{R}^n$, $1 \leq p < \infty$ and (φ_1, φ) satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x_0, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C\, \varphi(x_0, r), \tag{2.5}$$

where C does not depend on x_0 and r. Let the operator T is bounded from $LM_{p,\varphi_1}^{\{x_0\}}$ to $LM_{p,\varphi}^{\{x_0\}}$ for $p > 1$ and from $LM_{1,\varphi_1}^{\{x_0\}}$ to $WLM_{1,\varphi}^{\{x_0\}}$.

Remark 2.1 It is obvious that if condition (2.3) holds, then condition (2.5) holds too. In general, condition (2.5) does not imply condition (2.3). For example, the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1,\infty)}(r) r^{\frac{n}{p}-\beta}}, \quad \varphi_2(r) = r^{-\frac{n}{p}}\left(1 + r^{\beta}\right), \quad 0 < \beta < \frac{n}{p}$$

satisfy condition (2.5) but do not satisfy condition (2.3) (see [24]).

Inspired by this, we consider the boundedness of the commutators of multilinear singular integral operators $T_m^{\vec{b}}$ on product generalized Morrey spaces $LM_{p_1,\varphi_1}^{\{x_0\}} \times \ldots \times LM_{p_m,\varphi_m}^{\{x_0\}}$.

3 The multilinear singular integral operators in the product spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t}\int_0^t g(r)dr,\ 0 < t < \infty.$$

Theorem 3.1 ([8]) *The inequality*

$$\operatorname*{ess\,sup}_{t>0} w(t)Hg(t) \le c \operatorname*{ess\,sup}_{t>0} v(t)g(t)$$

holds for all non-negative and non-increasing g on $(0,\infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t}\int_0^t \frac{dr}{\operatorname*{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

In this section, we will prove the boundedness of multilinear singular integral operators on product generalized local Morrey space. The following theorem was proved in [27].

Theorem 3.2 *Let $x_0 \in \mathbb{R}^n$, $1 \le p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \cdots + 1/p_m$. Then, for $1 < p_1, \ldots, p_m < \infty$ the inequality*

$$\|T_m(\overrightarrow{f})\|_{L_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} t^{-\frac{n}{p}-1} dt \tag{3.1}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m}(\mathbb{R}^n)$.

Moreover, if at least one exponent p_i equals one, the inequality

$$\|T_m(\overrightarrow{f})\|_{WL_p(B(x_0,r))} \lesssim r^{\frac{n}{p}} \int_{2r}^{\infty} \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0,t))} t^{-\frac{n}{p}-1} dt \tag{3.2}$$

holds for any ball $B(x_0,r)$ and for all $\overrightarrow{f} \in L^{loc}_{p_1}(\mathbb{R}^n) \times \ldots \times L^{loc}_{p_m}(\mathbb{R}^n)$.

Now we give the boundedness of multilinear singular integral operators on product generalized local Morrey space.

Theorem 3.3 *Let $x_0 \in \mathbb{R}^n$, $1 \le p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \prod_{i=1}^m \varphi_i(x_0,s)s^{\frac{n}{p_i}}}{t^{\frac{n}{p_i}+1}} dt \lesssim \varphi(x_0,r). \tag{3.3}$$

Then the operator T_m is bounded from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $LM^{\{x_0\}}_{p,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$, and from product space $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$ to $WLM^{\{x_0\}}_{p,\varphi}$ for $p_i \ge 1$, $i = 1, \ldots, m$.

Proof. Let $1 < p_1, \ldots, p_m < \infty$ and $\overrightarrow{f} \in LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$. By Theorems 3.1 and 3.2 we have

$$\begin{aligned}\|T_m(\overrightarrow{f})\|_{LM^{\{x_0\}}_{p,\varphi}} &\lesssim \sup_{r>0} \varphi(x_0,r)^{-1} \prod_{i=1}^m \int_r^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x_0,t))} dt \\ &\approx \sup_{r>0} \prod_{i=1}^m \varphi(x_0,r)^{-\frac{1}{m}} \int_0^{r^{-\frac{n}{p_i}}} \|f_i\|_{L_{p_i}(B(x_0,t^{-\frac{p_i}{n}}))} dt \\ &= \sup_{r>0} \prod_{i=1}^m \varphi(x_0,r^{-\frac{p}{n}})^{-\frac{1}{m}} \int_0^r \|f_i\|_{L_{p_i}(B(x_0,t^{-\frac{p_i}{n}}))} dt \\ &\lesssim \prod_{i=1}^m \sup_{r>0} \varphi_i(x_0,r^{-\frac{p}{n}})^{-1}\, r\, \|f_i\|_{L_{p_i}(B(x_0,r^{-\frac{p_i}{n}}))} \\ &= \prod_{i=1}^m \|f_i\|_{LM^{\{x_0\}}_{p_i,\varphi_i}}.\end{aligned}$$

When $p_i = 1$, $i = 1, \ldots, m$, the proof is similar and we omit the details here.

From Theorem 3.3 we get the following corollary proven in [27] about boundedness of multilinear singular integral operators on product generalized Morrey space.

Corollary 3.1 *Let $1 \le p_1, \ldots, p_m < \infty$ with $1/p = 1/p_1 + \ldots + 1/p_m$ and $(\varphi_1, \ldots, \varphi_m, \varphi)$ satisfies the condition*

$$\prod_{i=1}^m \int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_i(x,s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p_i}+1}} dt \lesssim \varphi(x,r). \tag{3.4}$$

Then the operator T_m is bounded from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $M_{p,\varphi}$ for $p_i > 1$, $i = 1, \ldots, m$, and from product space $M_{p_1,\varphi_1} \times \ldots \times M_{p_m,\varphi_m}$ to $WM_{p,\varphi}$ for $p_i \ge 1$, $i = 1, \ldots, m$.

4 Commutators of multilinear singular integral operators in the product spaces $LM^{\{x_0\}}_{p_1,\varphi_1} \times \ldots \times LM^{\{x_0\}}_{p_m,\varphi_m}$

Let T be a linear operator, for a function b, we define the commutator $[b,T]$ by

$$[b,T]f(x) = b(x)\,Tf(x) - T(bf)(x)$$

for any suitable function f. If $\widetilde{T}$ be a Calderón-Zygmund singular integral operator, a well known result of Coifman, Rochberg and Weiss [13] states that the commutator $[b,\widetilde{T}]f = b\,\widetilde{T}f - \widetilde{T}(bf)$ is bounded on $L_p(\mathbb{R}^n)$, $1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [10], [11], [14]). In [9], Chanillo proved that the commutator $[b,I_\alpha]f = b\,I_\alpha f - I_\alpha(bf)$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $(1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n})$ if and only if $b \in BMO(\mathbb{R}^n)$.

The definition of local Campanato space as follows.

Definition 4.1 *Let $1 \le q < \infty$ and $0 \le \lambda < \frac{1}{n}$. A function $f \in L_q^{\rm loc}(\mathbb{R}^n)$ is said to belong to the $CBMO^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$ (central Campanato space), if*

$$\|f\|_{CBMO^{\{x_0\}}_{q,\lambda}} = \sup_{r>0} \Big(\frac{1}{|B(x_0,r)|^{1+\lambda q}} \int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|^q dy \Big)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(y) dy.$$

Define

$$CBMO^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n) = \{ f \in L_q^{\rm loc}(\mathbb{R}^n) \,:\, \|f\|_{CBMO^{\{x_0\}}_{q,\lambda}} < \infty \}.$$

In [29], Lu and Yang introduced the central BMO space $CBMO_q(\mathbb{R}^n) = CBMO^{\{0\}}_{q,0}(\mathbb{R}^n)$. Note that, $BMO(\mathbb{R}^n) \subset CBMO^{\{x_0\}}_q(\mathbb{R}^n)$, $1 \le q < \infty$. The space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ can be regarded as a local version of $BMO(\mathbb{R}^n)$, the space of bounded mean oscillation, at the origin. But, they have quite different properties. The classical John-Nirenberg inequality shows that functions in $BMO(\mathbb{R}^n)$ are locally exponentially integrable. This implies that, for any $1 \le q < \infty$, the functions in $BMO(\mathbb{R}^n)$ can be described by means of the condition:

$$\sup_{r>0} \Big(\frac{1}{|B|} \int_B |f(y) - f_B|^q dy \Big)^{1/q} < \infty,$$

where B denotes an arbitrary ball in $\mathbb{R}^n$. However, the space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ depends on q. If $q_1 < q_2$, then $CBMO^{\{x_0\}}_{q_2}(\mathbb{R}^n) \subsetneqq CBMO^{\{x_0\}}_{q_1}(\mathbb{R}^n)$. Therefore, there is no analogy of the famous John-Nirenberg inequality of $BMO(\mathbb{R}^n)$ for the space $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$. One can imagine that the behavior of $CBMO^{\{x_0\}}_q(\mathbb{R}^n)$ may be quite different from that of $BMO(\mathbb{R}^n)$.

Lemma 4.1 *[25, 26, 30] Let* b *be a function in* $CBMO^{\{x_0\}}_{q,\lambda}(\mathbb{R}^n)$*,* $1 \le q < \infty$*,* $0 \le \lambda < \frac{1}{n}$ *and* $r_1, r_2 > 0$*. Then*

$$\left(\frac{1}{|B(x_0, r_1)|^{1+\lambda q}} \int_{B(x_0,r_1)} |b(y) - b_{B(x_0,r_2)}|^q dy \right)^{\frac{1}{q}} \le C \left(1 + \left| \ln \frac{r_1}{r_2} \right| \right) \|b\|_{CBMO^{\{x_0\}}_{q,\lambda}},$$

where $C > 0$ *is independent of* b*,* r_1 *and* r_2*.*

Theorem 4.1 *Let* $x_0 \in \mathbb{R}$*,* $1 < p, p_i, q_i < \infty$*,* $b_i \in LC^{\{x_0\}}_{q_i,\lambda_i}$ *for* $0 < \lambda_i < \frac{1}{n}$*,* $i = 1, 2, \dots, m$ *and* $\frac{1}{p} = \frac{1}{p_1} + \ldots + \frac{1}{p_m} + \frac{1}{p_1} + \ldots + \frac{1}{q_m}$*. Then the inequality*

$$\|T^{\mathbf{b}}_m(\mathbf{f})\|_{L^p(B(x_0,r))} \lesssim \prod_{i=1}^m \|b_i\|_{LC^{\{x_0\}}_{q_i,\lambda_i}} r^{\frac{n}{p}} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big)^m t^{n \sum\limits_{i=1}^m \lambda_i - n \sum\limits_{i=1}^m \frac{1}{p_i} - 1} \prod_{i=1}^m \|f_i\|_{L^{p_i}(B(x_0,t))} dt$$

holds for all ball $B(x_0, r)$ *and all* $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$*,* $i = 1, 2, \dots, m$*.*

Proof. Without loss of generality, it is suffice for us to show that the conclusion holds for $m = 2$.

Let $B = B(x_0, r)$, $f_1 = f_1^0 + f_1^\infty$ and $f_2 = f_2^0 + f_2^\infty$, where f_i^0 and f_i^∞ are as in the proof of Theorem 3.1, for $i = 1.2$. Thus, we have

$$\begin{aligned} &T_2^{(b_1,b_2)}(f_1, f_2)(x) \\ &= T_2^{(b_1,b_2)}(f_1^0, f_2^0)(x) + T_2^{(b_2,b_2)}(f_1^0, f_2^\infty)(x) + T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)(x) + T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)(x). \end{aligned}$$

So,

$$\begin{aligned} \|T_2^{(b_1,b_2)}(f_1, f_2)\|_{L_p(B)} &\le \|T_2^{(b_1,b_2)}(f_1^0, f_2^0)\|_{L_p(B)} + \|T_2^{(b_1,b_2)}(f_1^0, f_2^\infty)\|_{L_p(B)} \\ &+ \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^0)\|_{L_p(B)} + \|T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)\|_{L_p(B)} \\ &=: I + II + III + IV. \end{aligned}$$

Let us estimate I, II, III and IV, respectively.

Since,

$$\begin{aligned} &(b_1(x) - b_1(y))(b_2(x) - b_2(y)) \\ &= (b_1(x) - (b_1)_B)(b_2(x) - (b_2)_B) - (b_1(x) - (b_1)_B)(b_2(y) - (b_2)_B) \\ &- (b_1(y) - (b_1)_B)(b_2(x) - (b_2)_B) + (b_1(y) - (b_1)_B)(b_2(y) - (b_2)_B). \end{aligned} \tag{4.1}$$

Then,

$$\begin{aligned}
&\|T_2^{(b_1,b_2)}(f_1^0,f_2^0)\|_{L_p(B)}\\
&=\|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^0,f_2^0)\|_{L_p(B)}+\|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_p(B)}\\
&+\|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^0,f_2^0)_{L_p(B)}+\|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_p(B)}\\
&=: I_1+I_2+I_3+I_4. \qquad (4.2)
\end{aligned}$$

Let $1<\overline{p},\overline{q}<\infty$, such that $\frac{1}{\overline{p}}=\frac{1}{p_1}+\frac{1}{p_2}$ and $\frac{1}{\overline{q}}=\frac{1}{q_1}+\frac{1}{q_2}$. Then, using the Holder's inequality and Lemma 2.3, we have

$$\begin{aligned}
I_1&\lesssim\|(b_1-(b_1)_B)(b_2-(b_2))_B\|_{L_{\overline{q}}(B)}\|T_2(f_1^0,f_2^0)\|_{L_{\overline{p}}(B)}\\
&\lesssim\|b_1-(b_1)_B\|_{L_{q_2}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_1}(2B)}\\
&\lesssim\|b_1-(b_1)_B\|_{L_{q_1}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}r^{(\frac{1}{p_1}+\frac{1}{p_2})n}\\
&\times\int_{2r}^{\infty}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}\frac{dt}{t^{(\frac{1}{p_1}+\frac{1}{p_2})n+1}}\\
&\lesssim\|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}\\
&\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \qquad (4.3)
\end{aligned}$$

Let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$. Then similarly to the estimate of (4.8), we have

$$\begin{aligned}
I_2&\lesssim\|b_1-(b_1)_B\|_{L_{q_1}(B)}\|T_2(f_1^0,(b_2-(b_2)_B)f_2^0)\|_{L_\tau(B)}\\
&\lesssim\|b_1-(b_1)_B\|_{L_{q_1}(B)}\|f_1^0\|_{L_{p_1}(\mathbb{R}^n)}\|(b_2-(b_2)_{2B})f_2^0\|_{L_s(\mathbb{R}^n)}\\
&\lesssim\|b_1-(b_1)_B\|_{L_{q_1}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(2B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_2}(2B)}, \qquad (4.4)
\end{aligned}$$

where $1<s<\infty$, such that $\frac{1}{s}=\frac{1}{p_2}+\frac{1}{q_2}=\frac{1}{\tau}-\frac{1}{p_1}$.

From Lemma 2.1, it is easy to see that

$$\|b_i-(b_i)_B\|_{L_{q_i}(B)}=Cr^{\frac{n}{q_i}+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{q_1,\lambda_1}},$$

and

$$\begin{aligned}
\|b_i-(b_i)_B\|_{L_{q_i}(B)}&\le\|b_i-(b_i)_{2B}\|_{L_{q_i}(2B)}+\|(b_i)_B-(b_i)_{2B}\|_{L_{q_i}(2B)}\\
&\le Cr^{\frac{n}{q_i}+n\lambda_i}\|b_i\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}, \qquad (4.5)
\end{aligned}$$

for $i=1,2$.

Then,

$$\begin{aligned}
I_2&\lesssim\|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}\\
&\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_3&\lesssim\|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}\\
&\times\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)^2t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}$$

Moreover, let $1 < \tau_1, \tau_2 < \infty$, such that $\frac{1}{\tau_1} = \frac{1}{p_1} + \frac{1}{q_1}$ and $\frac{1}{\tau_2} = \frac{1}{p_2} + \frac{1}{q_2}$. It is easy to see that $\frac{1}{p} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$. Then by Lemma 2.3, Holder's inequality and (4.5), we obtain

$$\begin{aligned} I_4 &\lesssim \|(b_1-(b_1)_B)f_1^0\|_{L_{\tau_1}(\mathbb{R}^n)}\|(b_2-(b_2)_B)f_2^0\|_{L_{\tau_2}(\mathbb{R}^n)} \\ &\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(2B)}\|b_2-(b_2)_B\|_{L_{q_2}(2B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_2}(2B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned} \tag{4.6}$$

Therefore, combining the estimates of I_1, I_2, I_3 and I_4, we have

$$\begin{aligned} I &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Let us estimate II.

It's analogues to (4.2), we have

$$\begin{aligned} &\|T_2^{(b_1,b_2)}(f_1^0,f_2^\infty)\|_{L_p(B)} \\ &= \|(b_1-(b_1)_B)(b_2-(b_2)_B)T_2(f_1^0,f_2^\infty)\|_{L_p(B)} + \|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &+ \|(b_2-(b_2)_B)T_2((b_1-(b_1)_B)f_1^0,f_2^\infty)_{L_p(B)} + \|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &=: I_1+I_2+I_3+I_4. \end{aligned} \tag{4.7}$$

Let $1 < \overline{p}, \overline{q} < \infty$, such that $\frac{1}{\overline{p}} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{1}{\overline{q}} = \frac{1}{q_1} + \frac{1}{q_2}$. Then, using the Holder's inequality and (3.4), we have

$$\begin{aligned} II_1 &\lesssim \|(b_1-(b_1)_B)(b_2-(b_2))_{2B}\|_{L_{\overline{q}}(B)}\|T_2(f_1^0,f_2^\infty)\|_{L_{\overline{p}}(B)} \\ &\lesssim \|b_1-(b_1)_B\|_{L_{q_2}(B)}\|b_2-(b_2)_B\|_{L_{q_2}(B)}\|f_1\|_{L_{p_1}(2B)}\|f_2\|_{L_{p_1}(2B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{(\frac{1}{q_1}+\frac{1}{q_2})n+(\lambda_1+\lambda_2)n} r^{(\frac{1}{p_1}+\frac{1}{p_2})n} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned} \tag{4.8}$$

Moreover, using (1.2) and (3.2), we have

$$\begin{aligned} &|T_2(f_1^0,(b_2-(b_2)_B)f_2^\infty)(x)| \\ &\lesssim \int_{2B}|f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2. \end{aligned}$$

It's obvious that

$$\int_{2B}|f_1(y_1)|dy_1 \lesssim \|f_1\|_{L_{p_1}(2B)}|2B|^{1-1/p_1}, \tag{4.9}$$

and

$$\int_{(2B)^c} \frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2$$
$$\lesssim \int_{(2B)^c} |b_2(y_2)-(b_2)_B||f_2(y_2)|\Big[\int_{|x_0-y_2|}^{\infty}\frac{dt}{t^{2n+1}}\Big]dy_2$$
$$\lesssim \int_{2r}^{\infty} \|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\frac{dt}{t^{2n+1}}$$
$$+\int_{2r}^{\infty} \||(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}|\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}}$$
$$\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty} |B(x_0,t)|^{\frac{1}{q_2}+\lambda_2}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\frac{dt}{t^{2n+1}}$$
$$+\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)|B(x_0,t)|^{\lambda_2}\|f_2\|_{L_{p_2}(B(x_0,t))}|B(x_0,t)|^{1-\frac{1}{p_2}}\frac{dt}{t^{2n+1}}$$
$$\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-n+n\lambda_2-\frac{n}{p_2}-1}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \tag{4.10}$$

Therefore, from (4.9) and (4.10), it follows that

$$|T_2(f_1^0,(b_2-(b_2)_B)f_2^{\infty})(x)|$$
$$\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\|f_1\|_{L_{p_1}(2B)}|2B|^{1-\frac{1}{p_1}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{-n+n\lambda_2-\frac{n}{p_2}-1}\|f_2\|_{L_{p_2}(B(x_0,t))}dt$$
$$\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Thus, let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$, then similarly to the estimate of (4.8), we have

$$II_2 = \|(b_1-(b_1)_B)T_2(f_1^0,(b_2-(b_2)_B)f_2^{\infty})\|_{L_p(B)}$$
$$\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\|T_2(f_1^0,(b_2-(b_2)_B)f_2^{\infty})\|_{L_\tau(B)}$$
$$\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}B^{\lambda_1+\frac{1}{q_1}+\frac{1}{\tau}}$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt$$
$$\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \tag{4.11}$$

Similarly, we have

$$II_3 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}$$
$$\times\int_{2r}^{\infty}\Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Let us estimate II_4.
Since,

$$|T_2((b_1-(b_1)_B)f_1^0,(b_2-(b_2)_B)f_2^{\infty})(x)|$$
$$\lesssim \int_{2B}|b_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_2(y_2)|}{|x_0-y_2|^{2n}}dy_2,$$

and

$$\int_{2B} |b_1(y_1) - (b_1)_B||f_1(y_1)|dy_1 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} |B|^{\lambda_1+1-\frac{1}{p_1}} \|f_1\|_{L_{p_1}(2B)}. \quad (4.12)$$

Then, by (4.10) and (4.13), we have

$$|T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^\infty)(x)|$$
$$\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.$$

Therefore,

$$\begin{aligned} II_4 =& \|T_2((b_1 - (b_1)_B)f_1^0, (b_2 - (b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Combining the estimates of II_1-II_4, we have

$$\begin{aligned} II &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Similarly,

$$\begin{aligned} III &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

For IV, we have

$$\begin{aligned} &\|T_2^{(b_1,b_2)}(f_1^\infty, f_2^\infty)\|_{L_p(B)} \\ &\le \|(b_1 - (b_1)_B)(b_2 - (b_2)_B)T_2(f_1^\infty, f_2^\infty)\|_{L_p(B)} + \|(b_1 - (b_1)_B)T_2(f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &+ \|(b_2 - (b_2)_B)T_2((b_1 - (b_1)_B)f_1^\infty, f_2^\infty)_{L_p(B)} + \|T_2((b_1 - (b_1)_B)f_1^\infty, (b_2 - (b_2)_B)f_2^\infty)\|_{L_p(B)} \\ &=: IV_1 + IV_2 + IV_3 + IV_4. \end{aligned}$$

Let us estimate IV_1, IV_2, IV_3 and IV_4, respectively.

Let $1 < \tau < \infty$, such that $\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{\tau}$. Then, from Holder's inequality and (3.5), we get

$$\begin{aligned} IV_1 &\lesssim \|(b_1 - (b_1)_B)\|_{L_{q_1}(B)}\|(b_2 - (b_2)_B)\|_{L_{q_2}(B)}\|T_2(f_1^\infty, f_2^\infty)\|_{L_\tau(B)} \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} |B|^{(\lambda_1+\lambda_2)+(\frac{1}{q_1}+\frac{1}{q_2})+\frac{1}{\tau}} \\ &\times \int_{2r}^\infty \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))} t^{-n(\frac{1}{p_1}+\frac{1}{p_2})-1}dt \\ &\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}} \\ &\times \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1} \|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \end{aligned}$$

Moreover, by (1.2) and (3.2), we have

$$
\begin{aligned}
&|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty(x)|\\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}\frac{|b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|}{(|x_0-y_1|+|x_0-y_2|)^{2n}}dy_1dy_2\\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}|f_1(y_1)||b_2(y_2)-(b_2)_B||f_2(y_2)|\Big[\int_{|x_0-y_1|+|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big]dy_1dy_2\\
&\lesssim \int_{2r}^\infty\Big[\int_{B(x_0,t)}|f_1(y_1)|dy_1\Big]\Big[\int_{B(x_0,t)}|b_2(y_2)-(b_2)||f_2(y_2)|dy_2\Big]\frac{dt}{t^{2n+1}}.
\end{aligned}
$$

Since,

$$
\int_{B(x_0,t)}|f_1(y_1)|dy_1\lesssim \|f_1\|_{L_{p_1}(B(x_0,t))}t^{n(1-\frac{1}{p_1})},
$$

and

$$
\begin{aligned}
&\int_{B(x_0,t)}|b_2(y_2)-(b_2)B||f_2(y_2)|\\
&\lesssim \|b_2(y_2)-(b_2)_{B(x_0,t)}\|_{L_{q_2}(B(x_0,t))}\|f_2\|_{L_{p_2}}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\\
&+|(b_2)_{B(x_0,t)}-(b_2)_{B(x_0,r)}|\|f_2\|_{L_{p_2}}|B(x_0,t)|^{1-\frac{1}{p_2}}\\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}|B(x_0,t)|^{\frac{1}{q_2}+\lambda_2}\|f_2\|_{L_{p_2}}|B(x_0,t)|^{1-(\frac{1}{p_2}+\frac{1}{q_2})}\\
&+\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\Big(1+\ln\frac{t}{r}\Big)|B(x_0,t)|^{\lambda_2}\|f_2\|_{L_{p_2}}|B(x_0,t)|^{1-\frac{1}{p_2}}\\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\Big(1+\ln\frac{t}{r}\Big)^2t^{n\lambda_2-\frac{n}{p_2}+n}\|f_2\|_{L_{p_2}(B(x_0,t))}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
&|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)(x)|\\
&\lesssim \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{n\lambda_2-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt. \qquad (4.13)
\end{aligned}
$$

Let $1<\tau<\infty$, such that $\frac{1}{p}=\frac{1}{q_1}+\frac{1}{\tau}$. Then, from Holder's inequality and (4.13), we have

$$
\begin{aligned}
IV_2&\lesssim \|b_1-(b_1)_B\|_{L_{q_1}(B)}\|T_2(f_1^\infty,(b_2-(b_2)_B)f_2^\infty)\|_{L_\tau(B)}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}\\
&\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
IV_3&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}r^{\frac{n}{p}}\\
&\times\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}
$$

Similar to the estimate of (4.13), we have

$$
\begin{aligned}
&|T_2(b_1-(b_1)_B)f_1^\infty,(b_2-(b_2)_B)f_2^\infty(x)|\\
&\lesssim \int_{(2B)^c}\int_{(2B)^c}|b_1(y_1)-(b_1)_B||b_2(y_2)-(b_2)_B||f_1(y_1)||f_2(y_2)|\Big[\int_{|x_0-y_1|+|x_0-y_2|}^\infty\frac{dt}{t^{2n+1}}\Big]dy_1dy_2\\
&\lesssim \int_{2r}^\infty\Big[\int_{B(x_0,t)}|f_1(y_1)-(b_1)_B||f_1(y_1)|dy_1\Big]\Big[\int_{B(x_0,t)}|b_2(y_2)-(b_2)_B||f_2(y_2)|dy_2\Big]\frac{dt}{t^{2n+1}}\\
&\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}}\|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}}\int_{2r}^\infty\Big(1+\ln\frac{t}{r}\Big)^2t^{n(\lambda_1+\lambda_2)-n(\frac{1}{p_1}+\frac{1}{p_2})-1}\|f_1\|_{L_{p_1}(B(x_0,t))}\|f_2\|_{L_{p_2}(B(x_0,t))}dt.
\end{aligned}
$$

Thus,

$$IV_4 \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

Then, from the estimate of IV_1-IV_4, we deduced that

$$IV \lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt.$$

So, combining the estimates for I, II, III and IV, we have

$$\|T_2^{(b_1,b_2)}(f_1,f_2)\|_{L_p(B)}$$
$$\lesssim \|b_1\|_{LC^{\{x_0\}}_{q_1,\lambda_1}} \|b_2\|_{LC^{\{x_0\}}_{q_2,\lambda_2}} r^{\frac{n}{p}}$$
$$\times \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)^2 t^{(\lambda_1+\lambda_2)n-(\frac{1}{p_1}+\frac{1}{p_2})n-1} \|f_1\|_{L_{p_1}(B(x_0,t))} \|f_2\|_{L_{p_2}(B(x_0,t))} dt. \qquad (4.14)$$

Therefore, we complete the proof of Theorem 4.1.

References

1. Akbulut, A., Guliyev, V.S. and Mustafayev, R.: *Boundedness of the maximal operator and singular integral operator in generalized Morrey spaces*, Math. Bohem. **137** (1), 27-43 (2012).
2. Alvarez, J., Guzman-Partida, M., Lakey, J.: *Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures*, Collect. Math. **51**, 1-47 (2000).
3. Beurling, A.: *Construction and analysis of some convolution algebras*, Ann. Inst. Fourier (Grenoble) **14**, 132 (1964).
4. Burenkov, V.I., Guliyev, H.V., Guliyev, V.S.: *Necessary and sufficient conditions for boundedness of the fractional maximal operators in the local Morrey-type spaces*, J. Comput. Appl. Math. **208** (1), 280-301 (2007).
5. Burenkov, V.I., Guliyev, V.S.: *Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **30** (3), 211-249 (2009).
6. Burenkov, V., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the fractional maximal operator in local Morrey-type spaces*, Complex Var. Elliptic Equ. **55** (8-10), 739-758 (2010).
7. Burenkov, V., Gogatishvili, A., Guliyev, V.S., Mustafayev, R.: *Boundedness of the Riesz potential in local Morrey-type spaces*, Potential Anal. **35** (1), 67-87 (2011).
8. Carro, M., Pick, L., Soria, J., Stepanov, V.D.: *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4**, 397-428 (2001).
9. Chanillo, S.: *A note on commutators*, Indiana Univ. Math. J. **23**, 7-16 (1982).
10. Chiarenza, F., Frasca, M., Longo, P.: *Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients*, Ricerche Mat. **40**, 149-168 (1991).
11. Chiarenza, F., Frasca, M., Longo, P.: *$W^{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc. **336**, 841-853 (1993).
12. Coifman R.R., Meyer, Y.: *On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212**, 315-331 (1975).
13. Coifman, R., Rochberg, R., Weiss, G.: *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (2), 611-635 (1976).
14. Di Fazio, G., Ragusa, M.A.: *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112**, 241-256 (1993).
15. Feichtinger, H.: *An elementary approach to Wieners third Tauberian theorem on Euclidean n-space*, In:*Harmonic Analysis, Symmetric Spaces and Probability Theory, Cortona/Italy 1984, Symp. Math.* 267-301, Academic Press, London (1987).
16. Giaquinta, M.: Multiple integrals in the calculus of variations and nonlinear elliptic systems. *Princeton Univ. Press, Princeton, NJ* (1983).
17. Grafakos, L.: *On multilinear fractional integrals*, Studia Math. **102**, 49-56 (1992).
18. Grafakos, L., Torres, R.H.: *Maximal operator and weighted norm inequalities for multilinear singular integrals*, Indiana Univ. Math. J. **51** (5), 1261-1276 (2002).
19. Grafakos, L., Torres, R.H.: *Multilinear Calderón-Zygmund theory*, Advances in Mathematics, **165** (1), 124-164 (2002).
20. Grafakos, L., Torres, R.H.: *On multilinear singular integrals of Calderón-Zygmund type*, Publicacions Matem'atiques., **46**, 57-91 (2002).

21. Guliyev, V.S.: *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 329 pp. (1994) (in Russian).
22. Guliyev, V.S.: *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups. Some Applications*, Casioglu, Baku, 332 pp. (1999) (in Russian).
23. Guliyev, V.S.: *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp.
24. Guliyev, V.S., Aliyev, S.S., Karaman, T., Shukurov, P.S.: *Boundedness of sublinear operators and commutators on generalized Morrey Space*, Integral Equations Operator Theory **71** (3), 327-355 (2011).
25. Guliyev, V.S.: *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math. **3** (2), 79-94 (2013).
26. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2), 211-227 (2013).
27. Guliyev, V.S., Ismayilova, A.F.: *Multi-sublinear maximal operator and multilinear singular integral operators on generalized Morrey spaces*, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **40** (2), 65-77 (2014).
28. Kufner, A., John, O., Fuçik, S.: Function Spaces. *Noordhoff International Publishing: Leyden, Publishing House Czechoslovak Academy of Sciences: Prague* (1977).
29. Lu, S.Z., Yang, D.C.: *The central BMO spaces and Littlewood-Paley operators*, J. Approx. Theory Appl. (N.S.) **11**, 72-94 (1995).
30. Lu, S.Z., Wu, Q.: *CBMO estimates for commutators and multilinear singular integrals*, Math. Nachr. **276**, 75-88 (2004).
31. Mizuhara, T.: *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer - Verlag, Tokyo, 183-189 (1991).
32. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126-166 (1938).
33. Nakai, E.: *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95-103 (1994).
34. Sawano, Y.: *Generalized Morrey space for non-doubling measures*, Nonlinear Differential Equations Appl. **15**, 413-425 (2008).
35. Wiener, N.: *Generalized Harmonic Analysis*, Acta Math. **55**, 117-258 (1930).
36. Wiener, N.: *Tauberian theorems*, Ann. Math. **33**, 1-100 (1932).