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Commutators of potential and singular integral operators in generalized variable exponent Morrey spaces

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Abstract. *In this paper we prove the Sobolev-Morrey type $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega) \to \mathcal{M}^{q(\cdot),\omega(\cdot)}(\Omega)$-theorem for the commutator of potential operators $[b, I^{\alpha(\cdot)}]$, also of variable order. Also prove the $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ boundedness commutator of singular integral operators $[b, T]$.*

Keywords. Maximal function, fraction maximal function, potential type operator, singular integral, Morrey space, BMO space, commutator.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

We consider the variable exponent Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over an open set $\Omega \subseteq \mathbb{R}^n$, introduced in [2]. In [2] there was proved the boundedness in $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ of the maximal operator and a Sobolev type $L^{p(\cdot),\lambda(\cdot)}(\Omega) \to L^{q(\cdot),\lambda(\cdot)}(\Omega)$-theorem for the potential operator $I^{\alpha(\cdot)}$ of variable order, under the assumptions $\inf\limits_{x\in\Omega} \alpha(x) > 0$, $\sup\limits_{x\in\Omega}[\lambda(x) + \alpha(x)p(x)] < n$, under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$. In the case of constant α, for potential operators there was also proved the boundedness theorem $L^{p(\cdot),\lambda(\cdot)} \to BMO$ in the limiting case $p(x) = \frac{n-\lambda(x)}{\alpha}$. The corresponding results for the case where p, λ and α are constant, these results are well known, see for instance [18], [35]. In the recent paper [23] there was proved the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces in the setting of homogeneous spaces.

Last decade there was a real boom in investigation of variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$ and the corresponding Sobolev spaces $W^m_{p(\cdot)}(\Omega)$, we refer to surveys [9], [22], [38] on the progress in this field, including topics of Harmonic Analysis and Operator Theory.

For mapping properties of singular integrals operators and commutator of singular integral operators on Lebesgue spaces with variable exponent we refer to [8], [27], [28], [33] and [38].

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In this paper, within the frameworks of variable Morrey spaces $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ over bounded sets $\Omega \subseteq \mathbb{R}^n$, we continue the study of the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy,$$

the potential type operator

$$I^{\alpha(x)}f(x) = \int_{\Omega} |x-y|^{\alpha(x)-n} f(y)dy, \ 0 < \alpha(x) < n,$$

and the fractional maximal operator

$$M^{\alpha(x)}f(x) = \sup_{r>0} |B(x,r)|^{\frac{\alpha(x)}{n}-1} \int_{\widetilde{B}(x,r)} |f(y)|dy, \ 0 < \alpha(x) < n$$

of variable order $\alpha(x)$ and Calderon-Zygmund type singular operator

$$Tf(x) = \int_{\Omega} K(x,y)f(y)dy$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \ \text{ for all } \ x \neq y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } \ |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } \ |x-y| > 2|x-\xi|.$$

Under the log-conditions, we prove the $L^{p(\cdot)}(\Omega,\omega) \to L^{q(\cdot)}(\Omega,\omega)$-boundedness of commutators of potential operators. and the $L^{p(\cdot),\lambda(\cdot)}(\Omega)$-boundedness of singular integral operators and their commutators.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

Notation:
$\mathbb{R}^n$ is the n-dimensional Euclidean space,
$\Omega \subseteq \mathbb{R}^n$ is an open set, $\ell = \operatorname{diam} \Omega$;
$\chi_E(x)$ is a characteristic function of a set $E \subseteq \mathbb{R}^n$;
$B(x,t) = \{y \in \mathbb{R}^n : |x-y| < t\}$, $\widetilde{B}(x,t) = B(x,t) \cap \Omega$;
by c and C we denote various absolute positive constants, which may have different values even in the same line.

2 Preliminaries on variable exponent Lebesgue spaces

Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. We assume that

$$1 < p_- \le p(x) \le p_+ < \infty, \tag{2.1}$$

where we use the standard notation

$$p_- := \operatorname*{ess\,inf}_{x\in\Omega} p(x) > 1, \quad p_+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x) < \infty. \tag{2.2}$$

By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)}dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

this is a Banach function space. As is known, the following inequalities hold

$$\|f\|_{p(\cdot)}^{p+} \le I_p(f) \le \|f\|_{p(\cdot)}^{p-}, \quad \text{if} \quad \|f\|_{p(\cdot)} \le 1, \tag{2.3}$$

$$\|f\|_{p(\cdot)}^{p-} \le I_p(f) \le \|f\|_{p(\cdot)}^{p+}, \quad \text{if} \quad \|f\|_{p(\cdot)} \ge 1 \tag{2.4}$$

from which there follows that

$$c_1 \le \|f\|_{p(\cdot)} \le c_2 \implies c_3 \le I_p(f) \le c_4 \tag{2.5}$$

and

$$C_1 \le I_p(f) \le C_2 \implies C_3 \le \|f\|_{p(\cdot)} \le C_4 \tag{2.6}$$

with $c_3 = \min\{c_1^{p-}, c_1^{p+}\}$, $c_4 = \max\{c_2^{p-}, c_2^{p+}\}$, $C_3 = \min\{C_1^{p-}, C_1^{p+}\}$, $C_4 = \max\{C_2^{p-}, C_2^{p+}\}$.

By $p'(\cdot) = \frac{p(x)}{p(x)-1}$, $x \in \Omega$, we denote the conjugate exponent. The Hölder inequality is valid in the form

$$\int_\Omega |f(x)||g(x)|dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}. \tag{2.7}$$

For the basics of variable exponent Lebesgue spaces we refer to [41], [33].

Definition 2.1 By $\mathcal{P}^{\log}(\Omega)$ (weak Lipshitz) we denote the class of functions defined on Ω satisfying the log-condition

$$|p(x) - p(y)| \le \frac{C}{-\ln|x-y|}, \quad |x-y| \le \frac{1}{2}, \quad x, y \in \Omega, \tag{2.8}$$

where $C = C(p) > 0$ does not depend on x, y.

We treat $p(x)$ as a function on $\mathbb{R}^n$ by the unique infinite point. To manage with the weighted case under the consideration, we introduce an assumption on $p(x)$ at infinity stronger than the usually considered assumption

$$|p(x) - p(\infty)| \le \frac{C}{\ln(e + |x|)}, \quad x \in \mathbb{R}^n, \tag{2.9}$$

where $p(\infty) := \lim\limits_{x\to\infty} p(x)$.

The space $L^{p(\cdot)}$ coincides with the space

$$\left\{f(x) : \left|\int_\Omega f(y)g(y)dy\right| < \infty \ \ for\ all\ \ g \in L^{p'(\cdot)}(\Omega)\right\} \tag{2.10}$$

up to the equivalence of the norms

$$\|f\|_{L^{p(\cdot)}} \approx \sup_{\|g\|_{L^{p'(\cdot)}} \le 1} \left|\int_\Omega f(y)g(y)dy\right| \tag{2.11}$$

see [33], Theorem 2.3 or [39], Theorem 3.5.

The $L^{p(\cdot)}$-boundedness of the Hardy-Littlewood maximal operator was proved by L. Diening [7] under conditions (2.1)-(2.8).

By φ we always denote a weight, i.e. a locally integrable function with range Ω. The weighted Lebesgue space $L^{p(\cdot),\varphi}(\Omega)$ is defined as the set of all measurable functions for which

$$\|f\|_{L^{p(\cdot),\varphi}(\Omega)} = \inf\left\{\eta > 0 :\ \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \varphi(x)dx \le 1\right\}.$$

Let us define the class $A_{p(\cdot)}(\Omega)$ (see [9], [34]) to consist of those weights φ for which

$$\sup_B |B|^{-1} \|\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \|\varphi^{-\frac{1}{p(\cdot)}}\|_{L^{p'(\cdot)}(\widetilde{B}(x,r))} < \infty.$$

The following theorem for bounded sets Ω, but for variable $\alpha(x)$, was proved in [37] under the condition that the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$, which became an unconditional result after the result of Diening [7] on maximal operators.

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p, \alpha \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1) *and the conditions*

$$\inf_{x\in\Omega} \alpha(x) > 0, \ \sup_{x\in\Omega} \alpha(x)p(x) < n. \tag{2.12}$$

Then the operator $I^{\alpha(\cdot)}$ is bounded from $L^{p(\cdot)}(\Omega)$ to $L^{q(\cdot)}(\Omega)$ with

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n}. \tag{2.13}$$

2.1 Variable exponent Morrey spaces

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.2 Let $\omega(x,r)$ be a non-negative measurable function on $\Omega \times (0,\ell)$ and $1 \le p < \infty$. The generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ is defined by the norm

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, 0<r<\ell} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

According to this definition, we recover the space $\mathcal{M}^{p(\cdot),\lambda(\cdot)}(\Omega)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$:

$$\mathcal{M}^{p(\cdot),\lambda(\cdot)}(\Omega) = \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)\Big|_{\omega(x,r)=r^{\frac{\lambda(x)-n}{p(x)}}}.$$

Everywhere in the sequel we assume that

$$\inf_{x\in\Omega, 0<r<\ell} \omega(x,r) > 0 \tag{2.14}$$

which makes the space $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ nontrivial. Note that when p is constant, in the case of $w(x,r) \equiv const > 0$, we have the space $L^\infty(\Omega)$.

In [13] the following three theorems were proved.

Theorem 2.2 *[13] Let $\Omega \subset \mathbb{R}^n$ be bounded, $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1) *and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ satisfy the condition*

$$\int_r^\ell \omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{2.15}$$

where C does not depend on x and t. Then the operators M and T are bounded from the space $\mathcal{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ the space $\mathcal{M}^{p(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 2.3 *[13] Let $\Omega \subset \mathbb{R}^n$ be bounded, $p, q \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\alpha(x), q(x)$ satisfy the conditions in* (2.12), (2.13) *and the functions $\omega_1(x,r)$ and $\omega_2(x,r)$ fulfill the condition*

$$\int_r^\ell t^{\alpha(x)}\omega_1(x,t)\frac{dt}{t} \le C\,\omega_2(x,r), \tag{2.16}$$

where C does not depend on x and r. Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega_1(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega_2(\cdot)}(\Omega)$.

Theorem 2.4 *[13] Let $\Omega \subset \mathbb{R}^n$ be bounded, $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\alpha(x)$ fulfill the conditions in* (2.12) *and let $\omega(x,t)$ satisfy condition* (2.15) *and the conditions*

$$\omega(x,r) \le \frac{C}{r^{\frac{\alpha(x)}{1-\frac{p(x)}{q(x)}}}}, \tag{2.17}$$

$$\int_r^\ell t^{\alpha(x)-1}\,\omega(x,t)dt \le C\omega(x,r)^{\frac{p(x)}{q(x)}}, \tag{2.18}$$

where $q(x) > p(x)$ and C does not depend on $x \in \Omega$ and $r \in (0,\ell]$. Suppose also that for almost every $x \in \Omega$, the function $w(x,r)$ fulfills the condition

$$\text{there exist an } a = a(x) > 0 \text{ such that } \omega(x,\cdot) : [0,\ell] \to [a,\infty) \text{ is surjective}.$$

Then the operators $M^{\alpha(\cdot)}$ and $I^{\alpha(\cdot)}$ are bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega^{q(\cdot)/p(\cdot)}(\cdot)}(\Omega)$.

Remark 2.1 Note that in the case $p(x) = const$ the Theorems 2.2 and 2.3 was proved in [11], [12] and Theorem 2.4 in [12].

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}|dy,$$

where $f_{\widetilde{B}(x,t)} = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(z)dz$.

3 Commutators of the potential operators

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\inf\limits_{x\in\Omega} \alpha(x) > 0$ and let there exists a positive constant C such that*

$$\omega(x,r) \le Cr^{-\alpha(x)}, \quad r > 0. \tag{3.1}$$

Then the operators $M^{\alpha(\cdot)}$ is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $L^\infty(\Omega)$.

Proof. Let $x \in \Omega$ and $r > 0$. By the Hölder inequality we get successively

$$\begin{aligned}
&r^{\alpha(x)-n} \int_{\widetilde{B}(x,r)} |f(y)|d\mu(y)\\
&= r^{\alpha(x)-n}\omega(x,r)\omega^{-1}(x,r) \int_{\widetilde{B}(x,r)} |f(y)|d\mu(y)\\
&\le Cr^{\alpha(x)-\frac{n}{p'(x)}}\omega(x,r)r^{-\frac{n}{p(x)}}\omega^{-1}(x,r)\|f\|_{L^{p(\cdot)}(B(x,r))}\|\chi_{\widetilde{B}(x,r)}\|_{L^{p'}(\cdot)}\\
&\le Cr^{\alpha(x)}\omega(x,r)\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}
\end{aligned}$$

Then

$$\|M^{\alpha(\cdot)}f\|_{L^\infty(\Omega)} \le C\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)}.$$

Theorem 3.2 *Let $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1) *and let $\omega(x,t)$ satisfy condition* (3.1)*. Then the operator I^α is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $BMO(\Omega)$.*

Proof. Let $\Omega \subset \mathbb{R}^n$ be bounded, $f \in \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$. In [1] was proved

$$M^\sharp(I^\alpha f)(x) \le CM^\alpha f(x), \tag{3.2}$$

where $C > 0$ is independent of $x \in \Omega$.

The proof Theorem 3.2, by the Theorem 3.1 and inequality (3.2).

Now we consider the commutators

$$[b, I^{\alpha(x)}]f(x) = \int_{\Omega} [b(x) - b(y)]f(y)|x-y|^{\alpha(x)-n} dy.$$

The following statement holds:

Lemma 3.1 *[10] Let $b \in BMO(\Omega)$, $1 < s < \infty$. Then*

$$M^{\sharp}([b, I^{\alpha(\cdot)}]f(x)) \le C\|b\|_{BMO}\left[\left(M|I^{\alpha(\cdot)}f(x)|^s\right)^{\frac{1}{s}} + \left(M^{s\alpha(\cdot)}|f(x)|^s\right)^{\frac{1}{s}}\right],$$

where $C > 0$ is independed of f and x.

Proposition A.(see [8], Lemma 3.5) *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ satisfy the conditions* (2.1), (2.9)*. Then for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ there holds*

$$\left|\int_{\mathbb{R}^n} f(y)g(y)dy\right| \le C\left|\int_{\mathbb{R}^n} M^{\sharp}f(y)Mg(y)dy\right|$$

with a constant $C > 0$ not depending on f.

Theorem 3.3 *[19] Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ satisfy the conditions* (2.1), (2.9)*, then M is bounded from $L^{p(\cdot),\varphi}(\mathbb{R}^n)$ to $L^{p(\cdot),\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_{p(\cdot)}(\mathbb{R}^n)$.*

The following lemma is valid.

Lemma 3.2 *Let Ω be bounded and $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\varphi \in A_{p(\cdot)}(\Omega)$. Then*

$$\|f\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\Omega)} \le C\|\varphi^{\frac{1}{p(\cdot)}}M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)}$$

with a constant $C > 0$ not depending on f.

Proof. By (2.11) we have

$$\|f\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\Omega)} \le C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \le 1} \left|\int_{\Omega} f(y)g(y)\varphi^{\frac{1}{p(y)}}(y)dy\right|.$$

According to Proposition A,

$$\|f\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\Omega)} \le C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \le 1} \left|\int_{\Omega} M^{\sharp}f(y)M(g\varphi^{\frac{1}{p(\cdot)}})(y)dy\right|.$$

By the Hölder inequality and Theorem 3.3, we derive

$$\|f\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\Omega)} \le C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \le 1} \|\varphi^{\frac{1}{p(\cdot)}}M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)}\|\varphi^{-\frac{1}{p(\cdot)}}M(g\varphi^{\frac{1}{p(\cdot)}})\|_{L^{p'(\cdot)}(\Omega)}$$

$$\le C \sup_{\|g\|_{L^{p'(\cdot)}(\Omega)} \le 1} \|\varphi^{\frac{1}{p(\cdot)}}M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)}\|g\|_{L^{p'(\cdot)}(\Omega)} \le C\|\varphi^{\frac{1}{p(\cdot)}}M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)}.$$

Lemma 3.3 *Let Ω be bounded and $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\omega(x, r)$ be a non-negative measurable function. Then the following inequality holds*

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C\left\|M^{\sharp}f\right\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

where $C > 0$ is independent of $x \in \Omega$.

Proof. If $0 < \theta < 1$, $\varphi(x) = (M\chi_{\widetilde{B}(x,r)})^{\theta} \in A_{p(\cdot)}(\Omega)$, by the Lemma 3.2 we have

$$\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))} \le \|f\varphi^{\frac{1}{p(\cdot)}}\|_{L^{p(\cdot)}(\Omega)} \le C\|\varphi^{\frac{1}{p(\cdot)}}M^{\sharp}f\|_{L^{p(\cdot)}(\Omega)} \le C\|M^{\sharp}f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))}.$$

Thus

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} = \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)}\|f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))}$$

$$\le C \sup_{x\in\Omega, r>0} \frac{r^{-\frac{n}{p(x)}}}{\omega(x,r)}\|M^{\sharp}f\|_{L^{p(\cdot)}(\widetilde{B}(x,t))} = C\|M^{\sharp}f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

The Lemma has been proved.

Theorem 3.4 *Let Ω be bounded, $p \in \mathcal{P}^{log}(\Omega)$ satisfy assumption* (2.1)*, $\alpha(x)$ fulfill the conditions in* (2.12) *and let $\omega(x,t)$ satisfy condition* (2.16)*. Then the operator $[b, I^{\alpha(\cdot)}]$ is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $\mathcal{M}^{q(\cdot),\omega(\cdot)}(\Omega)$, where $b \in BMO(\Omega)$.*

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ and $b \in BMO(\Omega)$. By the Lemma 3.3, we have

$$\|[b, I^{\alpha(\cdot)}]f\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} \le C_1\|M^{\sharp}([b, I^{\alpha(\cdot)}]f)\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}}.$$

From Lemma 3.1, we have

$$\|M^{\sharp}([b, I^{\alpha(\cdot)}]f)\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} \le C_2\|b\|_{BMO}\left\|\left(M|I^{\alpha(\cdot)}f|^s\right)^{\frac{1}{s}} + \left(M^{\alpha(\cdot)s}|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}}$$
$$\le C_3\|b\|_{BMO}\left[\left\|\left(M|I^{\alpha(\cdot)}f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} + \left\|\left(M^{\alpha(\cdot)s}|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}}\right].$$

By the Theorem 2.2 and Theorem 2.3 (in the case $\omega_1 = \omega_2$), we have

$$\left\|\left(M|I^{\alpha(\cdot)}f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} = \left\|M|I^{\alpha(\cdot)}f|^s\right\|^{\frac{1}{s}}_{\mathcal{M}^{\frac{q(\cdot)}{s},\omega(\cdot)}}$$
$$\le C\left\||I^{\alpha(\cdot)}f|^s\right\|^{\frac{1}{s}}_{\mathcal{M}^{\frac{q(\cdot)}{s},\omega(\cdot)}} = C\left\|I^{\alpha(\cdot)}f\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} \le C\,\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

Similar we can proved

$$\left\|\left(M^{\alpha(\cdot)s}|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} \le C\,\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

Therefore

$$\|[b, I^{\alpha(\cdot)}]f\|_{\mathcal{M}^{q(\cdot),\omega(\cdot)}} \le C_2\|b\|_{BMO}\,\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

The theorem has been proved.

4 Commutators of the singular integral operators

The following statement holds:

Proposition B. [3] *Let T be a Calderon-Zygmund operator. Then for arbitrary s: $0 < s < 1$, there exists a constant $C_s > 0$ such that*

$$[(|Tf|^s)^\sharp]^{\frac{1}{s}}(x) \le C_s Mf(x)$$

for all $f \in C_0^\infty(\mathbb{R}^n)$ and for all $x \in \mathbb{R}^n$.

Lemma 4.1 *[10] Let $1 < s < \infty$, $b \in BMO(\mathbb{R}^n)$, then there exists $C > 0$ such that for all $x \in \mathbb{R}^n$, the following inequality holds*

$$M^\sharp([b,T]f)(x) \le C\|b\|_{BMO}\left(\left(M|Tf|^s\right)^{\frac{1}{s}}(x) + \left(M|f|^s\right)^{\frac{1}{s}}(x)\right).$$

Theorem 4.1 *[21] Let $p \in WL(\mathbb{R}^n)$ under conditions* (2.1), (2.9)*, then the operator $[b,T]$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$, where $b \in BMO(\mathbb{R}^n)$.*

The following Theorem is valid.

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^n$ be bounded, $p \in \mathcal{P}^{log}(\Omega)$ satisfy condition* (2.1) *and $\omega(x,t)$ fulfill condition* (2.15)*. Then the operator $[b,T]$ is bounded from $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ to $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$, where $b \in BMO(\Omega)$.*

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega(\cdot)}(\Omega)$ and $b \in BMO(\Omega)$. By the Lemma 3.3, we have

$$\|[b,T]f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C_1\|M^\sharp([b,T]f)\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

From Lemma 4.1, we get

$$\begin{aligned}
\|M^\sharp([b,T]f)\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} &\le C_2\|b\|_{BMO}\left\|\left(M|Tf|^s\right)^{\frac{1}{s}} + \left(M|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \\
&\le C_3\|b\|_{BMO}\left[\left\|\left(M|Tf|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} + \left\|\left(M|f|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}\right].
\end{aligned}$$

Then by the Theorem 2.2, we have

$$\begin{aligned}
\left\|\left(M|Tf|^s\right)^{\frac{1}{s}}\right\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} &= \left\|M|Tf|^s\right\|^{\frac{1}{s}}_{\mathcal{M}^{\frac{p(\cdot)}{s},\omega(\cdot)}} \le C\left\||Tf|^s\right\|^{\frac{1}{s}}_{\mathcal{M}^{\frac{p(\cdot)}{s},\omega(\cdot)}} \\
&= C\|Tf\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.
\end{aligned}$$

Therefore

$$\|[b,T]f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}} \le C_1\|b\|_{BMO}\|f\|_{\mathcal{M}^{p(\cdot),\omega(\cdot)}}.$$

The theorem has been proved.

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