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Estimation of Green function of a spectral problem with quasi-regular boundary condition

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Abstract. *In the paper, we consider a general boundary condition problem for a fourth order, λ complex parameter dependent ordinary differential equation. The asymptotics of eigennumbers of the problem under consideration was constructed within the quasi-regular boundary condition. For the Green function of the appropriate spectral problem the estimation was found at large values of $|\lambda|$.*

Keywords. fundamental solutions, asymptotics, boundary value problem, boundary conditions, sector, characteristic determinant.

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1 Introduction

In the paper we consider the following problem:

$$\mathrm{p}_1 p_2 y^{IV} - (p_1 + p_2)\lambda^2 y'' + \lambda^4 y - b(x) y'' - a(x)\lambda^2 y = f(x,\lambda),\ 0 < x < 1, \tag{1.1}$$

$$\begin{aligned} L_1(\mathrm{y}) &\equiv \sum_{k=1}^{4} \alpha_{1k} y^{(k-1)}(x,\lambda)\,|_{x=0} + \sum_{k=1}^{4} \beta_{1k} y^{(k-1)}(x,\lambda)\,|_{x=1} = 0 \\ L_2(\mathrm{y}) &\equiv \sum_{k=1}^{4} \alpha_{2k} y^{(k-1)}(x,\lambda)\,|_{x=0} + \sum_{k=1}^{4} \beta_{2k} y^{(k-1)}(x,\lambda)\,|_{x=1} = 0 \\ L_3(\mathrm{y}) &\equiv \sum_{k=1}^{4} \alpha_{3k} y^{(k-1)}(x,\lambda)\,|_{x=0} + \sum_{k=1}^{4} \beta_{3k} y^{(k-1)}(x,\lambda)\,|_{x=1} = 0 \\ L_4(\mathrm{y}) &\equiv \sum_{k=1}^{4} \alpha_{4k} y^{(k-1)}(x,\lambda)\,|_{x=0} + \sum_{k=1}^{4} \beta_{4k} y^{(k-1)}(x,\lambda)\,|_{x=1} = 0 \quad , \end{aligned} \tag{1.2}$$

where $a(x),\ b(x)$ are complex valued functions, $p_1, p_2,\ \alpha_{ij},\ \beta_{ij}\ (i,j = \overline{1,4})$ are complex numbers and the conditions $Rep_1 > 0,\ Rep_2 > 0$ are satisfied.

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For constructing the asymptotics of fundamental solutions of equation (1) we divide the λ - complex plane into eight sectors in the following way [1].

$$\mathrm{S}_1 = \{\lambda\, |\lambda_2 > k_1\lambda_1;\ \ \lambda_2 < k_4\lambda_1\},$$

$$\mathrm{S}_2 = \{\lambda\, |\lambda_2 > k_4\lambda_1;\ \ \lambda_2 < k_3\lambda_1\},$$

$$\mathrm{S}_3 = \{\lambda\, |\lambda_2 > k_3\lambda_1;\ \ \lambda_2 < k_2\lambda_1\},$$

$$\mathrm{S}_4 = \{\lambda\, |\lambda_2 > k_1\lambda_1;\ \ \lambda_2 < k_2\lambda_1\},$$

$$\mathrm{S}_5 = \{\lambda\, |\lambda_2 > k_1\lambda_1;\ \ \lambda_2 < k_4\lambda_1\},$$

$$\mathrm{S}_6 = \{\lambda\, |\lambda_2 > k_4\lambda_1;\ \ \lambda_2 < k_3\lambda_1\},$$

$$\mathrm{S}_7 = \{\lambda\, |\lambda_2 > k_3\lambda_1;\ \ \lambda_2 < k_2\lambda_1\},$$

$$\mathrm{S}_8 = \{\lambda\, |\lambda_2 > k_1\lambda_1;\ \ \lambda_2 < k_2\lambda_1\},$$

where

$$k_1 = \frac{\cos\psi_1}{\sin\psi_1};\quad k_2 = \frac{|\omega_3|\cos\psi_3 - |\omega_1|\cos\psi_1}{|\omega_3|\sin\psi_3 - |\omega_1|\sin\psi_1};$$

$$k_3 = \frac{\cos\psi_3}{\sin\psi_3};\quad k_4 = \frac{|\omega_1|\cos\psi_1 + |\omega_3|\cos\psi_3}{|\omega_1|\sin\psi_1 + |\omega_3|\sin\psi_3};$$

$$\omega_1 = |\omega_1|\, e^{\psi_1 i};\ \omega_2 = -\omega_1;\ \omega_3 = |\omega_3|\, e^{\psi_3 i};\ \omega_4 = -\omega_3;$$

$$|\omega_1| = |p_1|^{-\frac{1}{2}};\ \ |\omega_3| = |p_3|^{-\frac{1}{2}};$$
$$\psi_k = -\frac{1}{2}arctg\frac{Imp_k}{Rep_k},\quad k = 1, 2.$$

Here we assume that the following inequalities are valid: $0 < \psi_3 < \psi_1 < \frac{\pi}{4}$, $|\omega_3|\, \sin\psi_3 - |\omega_1|\, \sin\psi_1 > 0$.

For finding the asymptotics of fundamental solutions of equation (1) we give the following theorem ([4]).

Theorem 1.1 *When* $\mathrm{a(x), b(x)} \in \mathrm{C}^1\,[0,1]$, $Rep_1 > 0$, $Rep_2 > 0$, *are satisfied, at each sector* S_p $\left(p = \overline{1,8}\right)$ *the asymptotics of fundamental solutions of equation (1) is as follows:*

$$\frac{\mathrm{d}^{\mathrm{k}} y_m(x,\lambda)}{dx^k} = (\lambda\omega_m)^k \left[1 + \frac{1}{\lambda}y^1_{mk}(x) + \frac{1}{\lambda^2}y^2_{mk}(x) + \frac{E_{mk}(x,\lambda)}{\lambda^3}\right]\exp\left[\lambda\omega_m x\right],$$
$$m = \overline{1,4};\quad k = \overline{0,3},$$

where

$$y^1_{mk}(x) = \frac{1}{4q\omega_m^3 + 2p\omega_m}\left[\int_0^x a\,(\xi)\, d\xi + \omega_m^2 \int_0^x b\,(\xi)\, d\xi\right],$$

$$y^2_{mk}(x) = \frac{k}{\omega_k}\frac{dy^1_{mk}(x)}{dx} + \frac{1}{4q\omega_m^3 + 2p\omega_m}\left[\int_0^x a\,(\eta) + \omega_m^2 b\,[\eta]\right] y^1_{mk}(\eta) d\eta$$
$$- \frac{6q\omega_m^2 + p}{4q\omega_m^3 + 2p\omega_m}\frac{dy^1_{mk}(x)}{dx},$$
$$q = p_1 p_2,\ \ p = -p_1 - p_2.$$

The functions $E_{mk}(x,\lambda)$ $(m = \overline{1,4};\ k = \overline{0,3})$ *are analytic and at large values of* $|\lambda|$ *are bounded.*

By means of the Green formula, the solution of the spectral problem is found as follows [2].

$$\mathrm{y(x,\lambda)} = \int_0^1 \mathrm{G(x,\xi,\lambda)} f(\xi,\lambda) d\xi,$$

where $\mathrm{G(x,\xi,\lambda)}$ is the Green function of problem (1), (2) and is determined in the following way:

$$G(x,\xi,\lambda) = \frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}, \quad \lambda \in S_p \ \ (p = \overline{1,8}),$$

where

$$\Delta(\lambda) = \begin{vmatrix} L_1(y_1(x,\lambda)) & L_1(y_2(x,\lambda)) & L_1(y_3(x,\lambda)) & L_1(y_4(x,\lambda)) \\ L_2(y_1(x,\lambda)) & L_2(y_2(x,\lambda)) & L_3(y_3(x,\lambda)) & L_4(y_4(x,\lambda)) \\ L_3(y_1(x,\lambda)) & L_3(y_2(x,\lambda)) & L_3(y_3(x,\lambda)) & L_3(y_4(x,\lambda)) \\ L_4(y_1(x,\lambda)) & L_4(y_2(x,\lambda)) & L_4(y_3(x,\lambda)) & L_4(y_4(x,\lambda)) \end{vmatrix},$$

$$\Delta(x,\xi,\lambda) = \begin{vmatrix} g(x,\xi,\lambda) & y_1(x,\lambda) & y_2(x,\lambda) & y_4(x,\lambda) & y_5(x,\lambda) \\ L_1(g)_x & L_1(y_1(x,\lambda)) & L_1(y_2(x,\lambda)) & L_1(y_3(x,\lambda)) & L_1(y_4(x,\lambda)) \\ L_2(g)_x & L_2(y_1(x,\lambda)) & L_2(y_2(x,\lambda)) & L_3(y_3(x,\lambda)) & L_4(y_4(x,\lambda)) \\ L_3(g)_x & L_3(y_1(x,\lambda)) & L_3(y_2(x,\lambda)) & L_3(y_3(x,\lambda)) & L_3(y_4(x,\lambda)) \\ L_4(g)_x & L_4(y_1(x,\lambda)) & L_4(y_2(x,\lambda)) & L_4(y_3(x,\lambda)) & L_4(y_4(x,\lambda)) \end{vmatrix},$$

$$g(x,\xi,\lambda) = \pm \sum_{k=1}^{4} \frac{V_{4k}(\xi,\lambda)}{V(\xi,\lambda)} y_k(x,\lambda).$$

The function $g(x,\xi,\lambda)$ is taken "+" for $0 \le \xi \le x \le 1$, and is taken "-" for $0 \le x \le \xi \le 1$.

The function $V(\xi,\lambda)$ is the Wronskian determinant composed of the function $y_1(\xi,\lambda)$, $y_2(\xi,\lambda)$, $y_3(\xi,\lambda)$ and $y_4(\xi,\lambda)$. The function $V_{4k}(\xi,\lambda)$ is the cofactor of the $(4,k)$ element of the determinant $V(\xi,\lambda)$.

For choosing the principal part of the characteristic determinant $\Delta(\lambda)$ in the half-plane $Re\lambda \ge 0$ we give the following two semi-strips

$$\Pi_1(\lambda) = \{\lambda = \lambda_1 + i\lambda_2 \,|{-\delta} < \lambda_2 - \lambda_1 k_1 < \delta, \ \ \lambda_1 > R, \ \delta > 0, \ R > 0\}$$
$$\Pi_2(\lambda) = \{\lambda = \lambda_1 + i\lambda_2 \,|{-\delta} < \lambda_2 - \lambda_1 k_3 < \delta, \ \ \lambda_1 > R, \ \delta > 0, \ R > 0\}.$$

In the semi-strips $\Pi_1(\lambda)$ and $\Pi_2(\lambda)$ we denote the principal part of the determinant $\Delta(\lambda)$ by $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$, respectively.

$$\Delta_1(\lambda) = D_{14}(\lambda)e^{\lambda(\omega_1+\omega_4)} + D_{24}(\lambda)e^{\lambda(\omega_2+\omega_4)} + D_4(\lambda)e^{\lambda\omega_4}$$
$$\Delta_2(\lambda) = D_{13}(\lambda)e^{\lambda(\omega_1+\omega_3)\lambda} + D_{14}(\lambda)e^{(\omega_1+\omega_4)} + D_1(\lambda)e^{\omega_1\lambda} \quad ,$$

where

$$D_{14}(\lambda) = d_{14}^{10}\lambda^{10} + d_{14}^{9}\lambda^9 + d_{14}^{8}\lambda^8 + O(\lambda^7)$$

$$D_{24}(\lambda) = d_{24}^{10}\lambda^{10} + d_{24}^{9}\lambda^9 + d_{24}^{8}\lambda^8 + O(\lambda^7)$$

$$D_{13}(\lambda) = d_{13}^{10}\lambda^{10} + d_{13}^{9}\lambda^9 + d_{13}^{8}\lambda^8 + O(\lambda^7)$$

$$D_1(\lambda) = d_1^9\lambda^9 + d_1^8\lambda^8 + d_1^7\lambda^7 + O(\lambda^6)$$

$$D_4(\lambda) = d_4^9\lambda^9 + d_4^8\lambda^8 + d_4^7\lambda^7 + O(\lambda^6).$$

For finding the asymptotics of eigennumbers of the spectral problem we accept the following denotation

$$L\left(\gamma_n^1, \gamma_m^2, \gamma_p^3, \gamma_q^4\right) = \begin{vmatrix} \gamma_{1n}^1 & \gamma_{1m}^2 & \gamma_{1p}^3 & \gamma_{1q}^4 \\ \gamma_{2n}^1 & \gamma_{2m}^2 & \gamma_{2p}^3 & \gamma_{2q}^4 \\ \gamma_{3n}^1 & \gamma_{3m}^2 & \gamma_{3p}^3 & \gamma_{3q}^4 \\ \gamma_{4n}^1 & \gamma_{4m}^2 & \gamma_{4p}^3 & \gamma_{4q}^4 \end{vmatrix},$$

$$d_1^9 = 2\left(\omega_1^6\omega_3^3 - \omega_1^4\omega_3^5\right)\left(L\left(\alpha_2,\alpha_3,\alpha_4,\beta_4\right) + L\left(\alpha_4,\beta_2,\beta_3,\beta_4\right)\right),$$

$$d_4^9 = 2\left(\omega_1^5\omega_3^4 - \omega_1^3\omega_3^6\right)\left(L\left(\alpha_2,\alpha_3,\alpha_4,\beta_4\right) + L\left(\alpha_4,\beta_2,\beta_3,\beta_4\right)\right),$$

$$\begin{aligned}
d_1^8 = {} & 2\left[\omega_1^3\omega_3^5-\omega_1^5\omega_3^3\right]\left[L\left(\alpha_1,\alpha_3,\alpha_4,\beta_4\right)-L\left(\alpha_2,\alpha_3,\alpha_4,\beta_3\right)-L\left(\alpha_4,\beta_1,\beta_3,\beta_4\right)\right.\\
& \left.+L\left(\alpha_3,\beta_2,\beta_3,\beta_4\right)\right]+2\left(\omega_1^6\omega_3^3-\omega_1^4\omega_3^5\right)y_{13}^1(1)L\left(\alpha_2,\alpha_3,\alpha_4,\beta_4\right)\\
& +\left[2\omega_1^6\omega_3^3y_{13}^1(1)-2\omega_1^4\omega_3^5y_{11}^1(1)+\left(\omega_1^6\omega_3^3-\omega_1^4\omega_3^5\right)\left(y_{42}^1(1)+y_{32}^1(1)\right)\right.\\
& +\left(\omega_1^6\omega_3^3+\omega_1^5\omega_3^4\right)y_{31}^1(1)+\left(\omega_1^6\omega_3^3-\omega_1^5\omega_3^4\right)y_{41}^1(1)\\
& \left.+\left(\omega_1^5\omega_3^4-\omega_1^4\omega_3^5\right)y_{43}^1(1)+\left(-\omega_1^4\omega_3^5-\omega_1^5\omega_3^4\right)y_{33}^1(1)\right]L(\alpha_4,\beta_2,\beta_3,\beta_4)
\end{aligned}$$

$$\begin{aligned}
d_4^8 = {} & 2\left(\omega_1^5\omega_3^3-\omega_1^3\omega_3^5\right)\left(L\left(\alpha_1,\alpha_3,\alpha_4,\beta_4\right)-L\left(\alpha_2,\alpha_3,\alpha_4,\beta_3\right)+L(\alpha_3,\beta_2,\beta_3,\beta_4)-L(\alpha_4,\beta_1,\beta_3,\beta_4)\right)\\
& +2\omega_1^3\omega_3^6y_{43}^1(1)+\left(-\omega_1^3\omega_3^6-\omega_1^4\omega_3^5\right)\left(y_{11}^1(1)+y_{21}^1(1)\right)+\left(-\omega_1^3\omega_3^6+\omega_1^5\omega_3^4\right)\left(y_{22}^1(1)\right.\\
& \left.+y_{12}^1(1)\right)+\left(\omega_1^4\omega_3^5+\omega_1^5\omega_3^4\right)y_{23}^1(1)+\left(-\omega_1^4\omega_3^5+\omega_1^5\omega_4^4\right)y_{13}^1(1)\big]L(\alpha_4,\beta_2,\beta_3,\beta_4),
\end{aligned}$$

$$\begin{aligned}
d_{13}^8 = {} & \left(\omega_1^2\omega_3^6-\omega_1^3\omega_3^5-\omega_1^5\omega_3^3+\omega_1^6\omega_3^2\right)\left(L\left(\alpha_3,\alpha_4,\beta_1,\beta_4\right)+L\left(\alpha_1,\alpha_4,\beta_3,\beta_4\right)\right)\\
& +\left(\omega_1^3\omega_3^5-2\omega_1^4\omega_3^4+\omega_1^5\omega_3^3\right)\left(L\left(\alpha_2,\alpha_3,\beta_3,\beta_4\right)+L\left(\alpha_3,\alpha_4,\beta_2,\beta_3\right)\right)\\
& +\left(-\omega_1^2\omega_3^6+2\omega_1^4\omega_3^4-\omega_1^6\omega_3^2\right)L\left(\alpha_2,\alpha_4,\beta_2,\beta_4\right)\\
& +\left[\left(\omega_1^5\omega_3^4-\omega_1^3\omega_3^6\right)\left(y_{12}^1(1)+y_{33}^1(1)\right)+\left(\omega_1^4\omega_3^5-\omega_1^6\omega_3^3\right)\left(y_{13}^1(1)+y_{32}^1(1)\right)\right]L\left(\alpha_2,\alpha_4,\beta_3,\beta_4\right)
\end{aligned}$$

$$\begin{aligned}
d_{14}^8 = {} & \left(\omega_1^5\omega_3^3+2\omega_1^4\omega_3^4+\omega_1^3\omega_3^5\right)\left(L\left(\alpha_2,\alpha_3,\beta_3,\beta_4\right)+L\left(\alpha_3,\alpha_4,\beta_2,\beta_3\right)\right)\\
& -\left(\omega_1^6\omega_3^2+\omega_1^3\omega_3^5+\omega_1^5\omega_3^3+\omega_1^2\omega_3^6\right)\left(L\left(\alpha_3,\alpha_4,\beta_1,\beta_4\right)+L\left(\alpha_1,\alpha_4,\beta_3,\beta_4\right)\right)\\
& +\left(\omega_1^6\omega_3^2-2\omega_1^4\omega_3^4+\omega_1^2\omega_3^6\right)L\left(\alpha_2,\alpha_4,\beta_2,\beta_4\right)\\
& +\left[\left(\omega_1^5\omega_3^4+\omega_1^6\omega_3^3\right)\left(y_{13}^1(1)+y_{41}^1(1)\right)+\left(-\omega_1^3\omega_3^6-\omega_1^4\omega_3^5\right)\left(y_{11}^1(1)+y_{43}^1(1)\right)\right]L\left(\alpha_3,\alpha_4,\beta_2,\beta_4\right)\\
& +\left[\left(\omega_1^4\omega_3^5-\omega_1^6\omega_3^3\right)\left(y_{13}^1(1)+y_{42}^1(1)\right)+\left(\omega_1^3\omega_3^6-\omega_1^5\omega_3^4\right)\left(y_{12}^1(1)+y_{43}^1(1)\right)\right]L\left(\alpha_2,\alpha_4,\beta_3,\beta_4\right),
\end{aligned}$$

$$\begin{aligned}
d_{24}^8 = {} & \left(\omega_1^5\omega_3^3-2\omega_1^4\omega_3^4+\omega_1^3\omega_3^5\right)\left(L\left(\alpha_2,\alpha_3,\beta_3,\beta_4\right)+L\left(\alpha_3,\alpha_4,\beta_2,\beta_3\right)\right)\\
& +\left(\omega_1^6\omega_3^2-\omega_1^3\omega_3^5-\omega_1^5\omega_3^3+\omega_1^2\omega_3^6\right)\left(L\left(\alpha_1,\alpha_4,\beta_3,\beta_4\right)+L\left(\alpha_3,\alpha_4,\beta_1,\beta_4\right)\right)\\
& +\left(-\omega_1^6\omega_3^2+2\omega_1^4\omega_3^4-\omega_1^2\omega_3^6\right)L\left(\alpha_2,\alpha_4,\beta_2,\beta_4\right)\\
& +\left(-\omega_1^6\omega_3^3+\omega_1^5\omega_3^4\right)y_{23}^1(1)+\left(\omega_1^4\omega_3^5-\omega_1^3\omega_3^6\right)y_{21}^1(1)\big]L\left(\alpha_3,\alpha_4,\beta_2,\beta_4\right)\\
& +\left(\omega_1^6\omega_3^3-\omega_1^4\omega_3^5\right)y_{42}^1(1)+\left(\omega_1^3\omega_3^6-\omega_1^5\omega_3^4\right)y_{43}^1(1)\big]L\left(\alpha_2,\alpha_4,\beta_3,\beta_4\right),
\end{aligned}$$

$$\begin{aligned}
d_1^7 = {} & d_1^7+\big\{-\omega_1^4\omega_3^5\left[2y_{13}^2(1)+2y_{21}^2(0)+2y_{32}^2(0)+y_{43}^2(0)+y_{33}^2(0)+y_{42}^2(0)\right]\\
& +\omega_1^5\omega_3^4\left[-y_{31}^2(0)-y_{43}^2(0)+y_{33}^2(0)+y_{41}^2(0)\right]\\
& +\omega_1^6\omega_3^3\left[2y_{13}^2(1)+2y_{23}^2(1)+y_{31}^2(1)+y_{42}^2(1)+y_{32}^2(1)+y_{41}^2(1)\right]\big\}+L\left(\alpha_2,\alpha_3,\alpha_4,\beta_4\right)\\
& +\big\{-\omega_1^4\omega_3^5\left[2y_{11}^2(1)+2y_{23}^2(0)+y_{32}^2(1)+y_{43}^2(1)+y_{33}^2(1)+y_{42}^2(1)\right.\\
& \left.+y_{11}^1(1)y_{32}^1(1)+y_{11}^1(1)y_{42}^1(1)+y_{32}^1(1)y_{43}^1(1)+y_{11}^1(1)y_{33}^1(1)+y_{11}^1(1)y_{42}^1(1)+y_{33}^1(1)y_{42}^1(1)\right]\\
& +\omega_1^5\omega_3^4\left[y_{13}^2(1)+y_{43}^2(1)-y_{33}^2(1)-y_{41}^2(1)+y_{12}^1(1)y_{31}^1(1)+y_{12}^1(1)y_{43}^1(1)\right.\\
& \left.+y_{31}^1(1)y_{43}^1(1)-y_{12}^1(1)y_{33}^1(1)-y_{12}^1(1)y_{41}^1(1)-y_{33}^1(1)y_{41}^1(1)\right]\\
& +\omega_1^6\omega_3^3\left[2y_{13}^2(1)+2y_{43}^2(0)+y_{31}^2(1)+y_{42}^2(1)+y_{32}^2(1)+y_{41}^2(1)\right.\\
& +y_{13}^1(1)y_{31}^1(1)+y_{13}^1(1)y_{42}^1(1)+y_{31}^1(1)y_{42}^1(1)+y_{13}^1(1)y_{32}^1(1)+y_{13}^1(1)y_{41}^1(1)\\
& \left.+y_{32}^1(1)y_{41}^1(1)\right]\big\}L(\alpha_4,\beta_2,\beta_3,\beta_4)
\end{aligned}$$

$$\begin{aligned}
d_4^7 = {} & d_4^7+\big\{-\omega_1^3\omega_3^6\left[y_{11}^2(0)+y_{22}^2(0)+2y_{33}^2(0)+2y_{43}^2(1)+y_{12}^2(0)+y_{21}^2(0)\right]\\
& +\omega_1^4\omega_3^5\left[y_{13}^2(0)+y_{21}^2(0)-y_{11}^2(0)-y_{23}^2(0)\right]\\
& +\omega_1^5\omega_3^4\left[y_{12}^2(0)+y_{23}^2(0)+2\,y_{31}^2(0)+2\,y_{43}^2(1)+y_{13}^2(0)+y_{22}^2(0)\right]\big\}L\left(\alpha_2,\alpha_3,\alpha_4,\beta_4\right)\\
& +\big\{-\omega_1^3\omega_3^6\left[y_{11}^2(1)+y_{22}^2(1)+2y_{33}^2(0)+y_{43}^2(1)+y_{12}^2(1)+y_{21}^2(1)+y_{43}^2(1)\right.\\
& \left.+y_{11}^1(1)y_{22}^1(1)+y_{11}^1(1)y_{43}^1(1)+y_{22}^1(1)y_{43}^1(1)+y_{12}^1(1)y_{21}^1(1)+y_{12}^1(1)y_{43}^1(1)+y_{21}^1(1)y_{43}^1(1)\right]\\
& +\omega_1^4\omega_3^5\left[y_{11}^2(1)+y_{23}^2(1)-y_{13}^2(1)-y_{21}^2(1)+y_{11}^1(1)y_{23}^1(1)+y_{11}^1(1)y_{42}^1(1)-y_{13}^1(1)y_{21}^1(1)\right.\\
& \left.-y_{13}^1(1)y_{42}^1(1)-y_{21}^1(1)y_{42}^1(1)\right]+\omega_1^5\omega_3^4\left[y_{12}^2(1)+y_{23}^2(1)+y_{13}^2(1)+y_{22}^2(1)+2y_{33}^2(0)+2y_{41}^2(1)\right.\\
& +y_{12}^1(1)y_{23}^1(1)+y_{12}^1(1)y_{41}^1(1)+y_{23}^1(1)y_{41}^1(1)+y_{13}^1(1)y_{22}^1(1)+y_{13}^1(1)y_{41}^1(1)\\
& \left.+y_{22}^1(1)y_{41}^1(1)\right]\big\}L(\alpha_4,\beta_2,\beta_3,\beta_4).
\end{aligned}$$

Now, in the case when the boundary conditions are first order quasi-regular, we find the asymptotics of the roots of the equation $\Delta(\lambda)=0$. For that in the half-strip $\Pi_1(\lambda)$ we give a theorem for the asymptotics of the roots of the equation $\Delta_1(\lambda)=0$.

Theorem 1.2 *Assume that the coefficients of equation (1) and boundary conditions (2) satisfy the following conditions:*

$$\mathrm{Re} p_i > 0 \ (i = 1, 2), \quad a(x),\ b(x) \in C^1[0, 1],$$

$$L(\alpha_3, \alpha_4, \beta_3, \beta_4) = 0,\ \ L(\alpha_3, \alpha_4, \beta_2, \beta_4) = 0,\ \ L(\alpha_2, \alpha_4, \beta_3, \beta_4) = 0,$$

$$L(\alpha_2, \alpha_3, \alpha_4, \beta_4) \neq 0\,,\ \ L(\alpha_4, \beta_2, \beta_3, \beta_4) \neq 0.$$

Satisfying $L(\alpha_2, \alpha_4, \beta_2, \beta_4)$, $L(\alpha_2, \alpha_3, \beta_3, \beta_4) + L(\alpha_3, \alpha_4, \beta_2, \beta_3\,)$ *if one of the expressions* $L(\alpha_3, \alpha_4, \beta_1, \beta_4) + L(\alpha_1, \alpha_4, \beta_3, \beta_4)$ *differs from zero, then far from the vicinity* δ *of the eigennumbers of the spectral problem, the Green function has the following estimation:*

$$|G\,(x, \xi, \lambda)| \leq \frac{M}{|\lambda|^2} \quad , \quad |\lambda| \to \infty\ , \quad \lambda \in S_p \quad \left(p = \overline{1,\,8}\right)$$

So, for the asymptotics of eigennumbers of problem (1), (2) the following formula is valid:

$$\lambda_{k\nu} = -\tfrac{1}{2\omega_k} \left\{ \ln \left| \tfrac{\pi\nu A_k}{\omega_k} \right| + i \left[2\pi\nu + \tfrac{\pi}{2}(2 - sgn\nu) + \arg A_k \right] \right\} + O\left(\tfrac{\ln|\nu|}{\nu} \right),\ \ k = 1, 2,$$

$$k = 1, 2,\ \ (-1)^k \nu \to +\infty,$$

$$A_1 = -\left(\tfrac{d_4^2}{2d_4^9 d_{14}^8} - \tfrac{d_{24}^8}{(d_4^9)^2} - \tfrac{d_4^7}{2d_{14}^8} \right)^{-1}, \quad A_2 = \tfrac{d_4^9}{d_{14}^8}.$$

Proof. Now within the conditions of the theorem we find the asymptotics of the eigennumbers of problem (1), (2). For finding the asymptotics of eigennumbers, at first consider the set $\Pi_1(\lambda)$ located in the first half-plane of the complex plane λ. In the semi-strip $\Pi_1(\lambda)$ we write the expression $\Delta_1(\lambda)$ being the principal part of $\Delta(\lambda)$:

$$\Delta_1(\lambda) = D_{14}(\lambda) e^{\lambda(\omega_1 + \omega_4)} + D_{24}(\lambda) e^{\lambda(\omega_2 + \omega_4)} + D_4(\lambda) e^{\lambda\omega_4}.$$

Within the theorem conditions we write the expressions $D_{14}(\lambda)$, $D_{24}(\lambda)$ and $D_4(\lambda)$ as follows

$$D_{14}(\lambda) = d_{14}^8 \lambda^8 + O(\lambda^7),$$

$$D_{24}(\lambda) = d_{24}^8 \lambda^8 + O(\lambda^7),$$

$$D_4(\lambda) = d_4^9 \lambda^9 + d_4^8 \lambda^8 + d_4^7 \lambda^7 + O(\lambda^6).$$

Taking into account these expressions in the equation $\Delta_1(\lambda) = 0$

$$\left[d_{14}^8 + O\left(\frac{1}{\lambda}\right) \right] e^{\lambda\omega_1} + \left[d_{24}^8 + O\left(\frac{1}{\lambda}\right) \right] e^{\lambda\omega_2} + \left[d_4^9 \lambda + d_4^8 + d_4^7 \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right] = 0. \qquad (1.3)$$

$$D^{\frac{1}{2}} = d_4^9 \lambda + d_4^8 + \frac{d_4^9 d_4^7 - 2 d_{14}^8 d_{24}^8}{\left(d_4^9\right)^2} \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right).$$

We write the asymptotic expressions of the roots of equation (3):

$$e^{\omega_1 \lambda} = \frac{1}{2d_{14}^8 + O\left(\frac{1}{\lambda}\right)} \left[-\left(d_4^9 \lambda + d_4^8 + d_4^7 \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \right) \pm D^{\frac{1}{2}} \right] \qquad (1.4)$$

$$e^{\omega_1 \lambda} = \left(\frac{d_4^7}{2d_4^9 d_{14}^8} - \frac{d_{24}^8}{\left(d_4^9\right)^2} - \frac{d_4^7}{2d_{14}^8} \right) \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad \lambda \in \Pi_1\left(\lambda\right), \quad |\lambda| \to \infty. \qquad (1.5)$$

We can easily see that $O(e^{\omega_2\lambda}) = O(1), \quad \lambda \in \Pi_1(\lambda) \quad |\lambda| \to \infty$. We can write formulas (4) and (5) in the following form

$$\mathrm{A_k}\lambda e^{\omega_k\lambda} + 1 + O\left(\frac{1}{\lambda}\right) = 0, \quad k = 1, 2$$
$$A_1 = \left(\frac{-d_4^7}{2d_4^9 d_{14}^8} + \frac{d_{24}^8}{(d_4^9)^2} + \frac{d_4^7}{2d_{14}^8}\right)^{-1},$$

$$A_2 = \frac{d_4^9}{d_4^8}\ .$$

Using any of the methods from [2], [3], [4], we can get the asymptotic formulas of the roots of the equation determined by formula (4), (5)

$$\mathrm{A_k}\lambda e^{\omega_k\lambda} + 1 = 0, \quad k = 1, 2$$

$$\lambda_{k\nu} = -\frac{1}{2\omega_k}\left\{\ln_0\left[-A_k\alpha_\nu\frac{-\pi i\nu}{\omega_k}\right] + 2\pi i\nu\right\}, \tag{1.6}$$
$$\alpha_\nu = 1 + \frac{\ln|\alpha_\nu|}{2\pi i\nu} + O\left[\frac{\ln|\nu|}{\nu}\right],$$

hence we get

$$\ln_0 \alpha_\nu = O\left[\frac{\ln|\nu|}{\nu}\right]. \tag{1.7}$$

Taking into account formula (7) in (6), we get an asymptotic formula for the roots of equations (4) and (5):

$$\lambda_{\mathrm{k}\nu} = -\frac{1}{2\omega_k}\left\{\ln\left(\left|A_k\frac{\pi}{\omega_k}\nu\right|\right) + i\left[2\pi\nu + \frac{\pi}{2}(2 - sgn\nu) + \arg A_k\right]\right\} + O\left[\frac{\ln|\nu|}{\nu}\right], \quad (-1)^k\nu \to +\infty.$$

In the same way, we can show that the last formula is valid in the complex plane λ.

Now, within the theorem's conditions estimate the Green function outside of the vicinity of δ of the eigennumbers.

For making estimations at first we consider the sector S_1.

It is easy to see that for $\lambda \in S_1$ the following inequality is valid

$$Re\lambda\,\omega_3 \le Re\lambda\,\omega_2 \le 0 \le Re\lambda\,\omega_1 \le Re\lambda\,\omega_4, |\lambda| \to \infty, \lambda \in S_1.$$

Multiplying the second, third, fourth and fifth columns of the determinant $\Delta(x,\xi,\lambda)$ by the functions $-\frac{1}{2}z_1(\xi,\lambda)$, $\frac{1}{2}z_2(\xi,\lambda)$, $\frac{1}{2}z_3(\xi,\lambda)$ and $-\frac{1}{2}z_4(\xi,\lambda)$, adding it to the first column, we find the elements of the first column as follows:

$$g^0(x,\xi,\lambda) = \begin{cases} -z_1(\xi,\lambda)\,y_1(x,\lambda) - z_4(\xi,\lambda)\,y_4(x,\lambda) \quad , \quad 0 \le \xi \le x \le 1\,, \\ z_2(\xi,\lambda)\,y_2(x,\lambda) + z_3(\xi,\lambda)\,y_3(x,\lambda) \quad , \quad 0 \le \xi \le x \le 1\,, \end{cases}$$

$$b_q^0(\xi,\lambda) = -z_1(\xi,\lambda)\sum_{k=1}^{4}\beta_{q\,k}\,y_1^{(k-1)}(0,\lambda) + z_2(\xi,\lambda)\sum_{k=1}^{4}\alpha_{q\,k}\,y_2^{(k-1)}(0,\lambda)$$

$$+z_3(\xi,\lambda)\sum_{k=1}^{4}\alpha_{q\,k}\,y_3^{(k-1)}(0,\lambda) - z_4(\xi,\lambda)\sum_{k=1}^{4}\beta_{q\,k}\,y_4^{(k-1)}(1,\lambda) \quad , \quad q = \overline{1,4} \quad , \quad \lambda \in S_1.$$

Within the theorem conditions, the following estimations are valid

$$\left|\Delta(x,\xi,\lambda)\,e^{\lambda(\omega_1+\omega_4)}\right| \le M_1\,|\lambda|^6 \quad , \quad |\lambda| \to \infty \quad , \quad \lambda \in S_1 \quad , \quad M_1 = const\ , \tag{1.8}$$

$$\left|\Delta(x,\xi,\lambda)\,e^{\lambda(\omega_1+\omega_4)}\right| > N_\delta\,|\lambda|^8 \quad , \quad |\lambda| \to \infty \quad , \quad \lambda \in S_1 \quad , \quad N_\delta = const. \tag{1.9}$$

Using inequalities (8) and (9), we get

$$\left|\frac{\Delta(x,\xi,\lambda)}{\Delta(\lambda)}\right| \le M\,|\lambda|^{-2} \quad , \quad |\lambda| \to \infty \quad , \quad \lambda \in S_1 \quad , \quad M = const\ , \tag{1.10}$$

where λ is valid when the eigennumbers are outside of the vicinity of δ.

In the same way we can show that in the sectors $S_p \quad \left(p = \overline{2,\ 7}\right)$, inequality (10) is valid. So, the following estimation for the Green function of the eigennumbers outside of the vicinity of δ is valid:

$$|G(x, \xi, \lambda)| \leq \frac{M}{|\lambda|^2} \quad , \quad |\lambda| \to \infty \ , \quad \lambda \in S_p \quad \left(p = \overline{1,\ 8}\right).$$

The theorem is proved.

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