Transactions of NAS of Azerbaijan, Issue Mathematics, **35** (4), 131-142 (2015).
Series of Physical-Technical and Mathematical Sciences.

Interpolation theorems of the generalized Besov- Morrey type spaces with dominant mixed derivatives

Aygun T. Orujova

Received: 18.03.2015 / Accepted: 11.11.2015

Abstract. In this paper was proved Riesz-Thorin type theorems for functions from of the generalized Besov-Morrey type spaces with dominat mixed derivatives.

Keywords. Besov-Morrey type spaces with dominant mixed derivatives, interpolation theorems, embedding theorems, Holder condition.

Mathematics Subject Classification (2010): 26A33, 46E30, 35A315, 35Q35

1 Introduction

It is known that spaces with parameters constructed on the base of the Sobolev isotropic spaces $W_p^{(l)}(G)$, for some special values of indices, were first studied in the papers of Morrey [10,11]. Further these results were developed and generalized in the papers of V.P. Ilyin [6], I. Ross [20], Yu. V. Netrusov [19], A. Mazzucato [9], V.S. Guliyev [2, 3], V.S. Guliyev, L.G. Softova [4,5], Y. Sawano [21], L.Tang and J. Xu [22], A.M.Nadjafov [8, 13, 16, 17] and e.t.c.

Note that the interpolation theorems in the spaces of Bessov-Morrey, Lizorkin-Triebel-Morrey $B^l_{p,\theta,a,æ,\tau}$, $F^l_{p,\theta,a,æ,\tau}$, $S^l_{p,\theta,a,æ,\tau}B(G)$ and $S^l_{p,\theta,a,æ,\tau}F(G)$ earlier was obtained in [12, 14, 15].

The goal of this paper is to investigate differential properties of functions from the intersection of generalized Besov-Morrey type space with dominant mixed derivatives of the form

$$\bigcap_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p^i_\mu,\theta^i_\mu,a,\varkappa,\tau}(G_h)\,(\mu=1,2,...,N). \tag{1.1}$$

Let $e_n=\{1,...,n\}$- be the set of natural numbers $1,...,n$. We denote by e any fixed subset of e_n (including the sets empty set and e_n). The number of this subsets is equal to 2^n . Assume that all these subsets are numbered, i.e. e^i $(i=1,...,2^n)$- are subsets (including empty set and e_n) of e_n. With each set e^i, associate some fixed vector $l^{i,\mu}=\left(l_1^{i,\mu},...,l_n^{i,\mu}\right)$ with the components $l_j^{i,\mu}\geq 0$ and the support $e_{l^i}\supseteq e^i$, i.e. $l_j^{i,\mu}>0\left(j\in e^i\right)$, $l_j^{i,\mu}\geq 0\left(j\in e_{l^i}/e^i\right)(\mu=1,2,...,N)$. The support of a vector $l=(l_1,...,l_n)$ is the set of indices of nonzero components of the vector l. It is denoted by e_l.

A.T. Orujova
Institute of Mathematics and Mechanics of NAS of Azerbaijan
9, B.Vahabzadeh str., AZ1141, Baku, Azerbaijan
E-mail: aygun.orucova@imm.az

2 Preliminaries

Definition 2.1 A space with parameters of the form (1.1) is defined and studied in [18] as a normed space of functions f on G with the finite norm

$$\|f\|_{\bigcap\limits_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p^i_\mu,\theta^i_\mu,a,\varkappa,\tau}(G_h)} = \sum_{i=1}^{2^n} \|f\|_{L^{<l^{i,\mu}>}_{p^i_\mu,\theta^i_\mu,a,\varkappa,\tau}(G_h)}, \tag{2.1}$$

$$\|f\|_{L^{<l^{i,\mu}>}_{p^i_\mu,\theta^i_\mu,a,\varkappa,\tau}(G_h)} \left\{ \int\limits_0^{h_{01}} \cdots \int\limits_0^{h_{0n}} \left[\frac{\left\| \Delta^{m^i}(h,G_h) D^{k^i} f \right\|_{p^i_\mu,a,\varkappa,\tau}}{\prod\limits_{j\in e_{l^i}} h_j^{l^{i,\mu}_j - k^i_j}} \right]^{\theta^i_\mu} \prod_{j\in e_{l^i}} \frac{dh_j}{h_j} \right\}^{\frac{1}{\theta^i_\mu}}, \tag{2.2}$$

$$\|f\|_{p^i_\mu,a,\varkappa,\tau;G} = \|f\|_{L_{p^i_\mu,a,\varkappa,\tau}(G)} = \sup_{x\in G} \left\{ \int\limits_0^{\infty} \cdots \int\limits_0^{\infty} \left[\prod_{j\in e_n} [t_j]_1^{-\frac{\varkappa_j a_j}{p^i_\mu}} \times \right.\right.$$

$$\left.\left. \times \|f\|^{\tau}_{p^i_\mu, G_{t^\varkappa}(x)} \right] \prod_{j\in e_n} \frac{dt_j}{t_j} \right\}^{\frac{1}{\tau}}, \tag{2.3}$$

where $p^i_\mu \in [1,\infty)$,$1 \le \theta^i_\mu \le \infty$, $\tau \in [1,\infty]$, $l^{i,\mu} = \left(l^{i,\mu}_1,...,l^{i,\mu}_n\right)$, $l^{i,\mu}_j > 0 \left(j\in e^i\right)$, $l^{i,\mu}_j \ge 0$ $\left(j \in e_{l^i}/e^i\right)$; $m^i = \left(m^i_1,...,m^i_n\right)$, $m^i_j > 0 \left(j\in e^i\right)$, $m^i_j \ge 0 \left(j\in e_{l^i}/e^i\right)$ are integers, $k^i = \left(k^i_1,...,k^i_n\right)$, k^i_j are non-negative integers. In (2.2) the integrated is over by the index $j \in e_{l^i}$. Assume that $m^i_j \ge l^{i,\mu}_j - k^i_j \ge 0$; $\left(j\in e_{l^i}/e^i\right)$, $m^i_j > l^{i,\mu}_j - k^i_j > 0$, $\left(j\in e^i\right)$, $(i = 1,...,2^n)$, $(\mu = 1,2,\dots,N)$

$$\Delta^{m^i}(h,G_h) f(x) = \Delta_1^{m^i_1}(h_1)...\Delta_n^{m^i_n}(h_n) f(x);\ D^{k^i} f = D_1^{k^i_1}...D_n^{k^i_n} f(x);$$

$G_h = \{x : x + hI \in G\}$; $G_{t^\varkappa}(x) = G \cap I_{t^\varkappa}(x) = G \cap \left\{ y : \left|y_j - x_j\right| < \frac{1}{2} t_j^{\varkappa_j}, j\in e_n \right\}$, $\left[t_j\right]_1 = \min\left\{1, t_j\right\}$, $j \in e_n$, $h_0 = (h_{01},\dots,h_{0n})$ fixed a positive vector; G be an open set of $n-$dimensional Euclidean space R^n.

The space of (1.1) in the case, when

$$l^{i,\mu}_j = \begin{cases} l_j > 0,\ j\in e^i; \\ 0, \quad j\in e_{l^i}/e^i, \end{cases}$$

$p^i_\mu = p_\mu, \theta^i_\mu = \theta_\mu$ $(i = 1,...,2^n)$, coincides with the space $\bigcap\limits_{\mu=1}^{N} S^{l_\mu}_{p_\mu,\theta_\mu,a,\varkappa,\tau} B(G_h)$, introduced and studied in the paper [15] and in the case $\mu = 1, a = (0,...,0), \tau = \infty$ coincides with the space

$$\bigcap_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p^i_\mu,\theta^i_\mu}(G),$$

which was introduced and studied by A.D. Jabrailov's in [7].

When $\tau = \infty$ the space $L_{p,a,\varkappa,\tau}(G)$ coincides with the space $L_{p,a,\varkappa}(G)$ introduced and studied by V.P. Ilyin [6].

Let $\psi_{e^i}(\cdot,y,z) \in C_0^\infty(R^n)$ be such that $S(\psi_{e^i}) = supp\psi_{e^i}$, $S(\psi_{e^i}) \subset I_1 = \left\{x : \left|x_j\right| < \frac{1}{2}, j\in e_n\right\}$, $0 < T_j \le 1, j \in e_n$. Set $V = \bigcup\limits_{0\le t_j\le T_j,\ j\in e_n} \left\{ y : \left(\frac{y}{t}\right) \in S(\psi_{e^i}) \right\}$. It is clear that $V \subset I_T, U$ is an open set contained in the domain G. We'll also assume that $U + V \subset G$. Let

$$G_{T^\varkappa}(U) = \bigcup_{x\in G} G_{T^\varkappa}(x) = (U + I_{T^\varkappa}(x)) \cap G.$$

Note that, if $0 < \varkappa_j \le 1$, $0 < T_j \le 1, j \in e_n$ and $I_T \subset I_{T^{\varkappa}}$, then

$$U + V \subset G_{T^{\varkappa e^i}}(U) = Z.$$

Let $\beta_\mu \ge 1$ $(\mu = 1, 2, \ldots, N)$, $\sum\limits_{\mu=1}^{N} \beta_\mu = 1$, $\frac{1}{p^i} = \sum\limits_{\mu=1}^{N} \frac{\beta_\mu}{p^i_\mu}$, $\frac{1}{p} = \sum\limits_{\mu=1}^{N} \frac{\beta_\mu}{p_\mu}$, $\frac{1}{\theta^i} = \sum\limits_{\mu=1}^{N} \frac{\beta_\mu}{\theta^i_\mu}$, $l^i = \sum\limits_{\mu=1}^{N} \beta_\mu l^{i,\mu}$.

The follows property is valid

Lemma 2.1 *Let* $1 \le p^i_\mu \le p \le r \le \infty, (\mu = 1, 2, \ldots, N), 0 < \varkappa_j \le 1,\ 0 < t_j, \eta_j \le T_j \le 1, 0 < \rho_j < \infty\, (j \in e_n)$ $1 \le \tau \le \infty$, $\nu = (\nu_1, ..., \nu_n)$, $\nu_j \ge 0$ *be integers* $(j \in e_n)$,

$$\varepsilon^i_j = \sum_{\mu=1}^{N} l^{i,\mu}_j \beta_\mu - \nu_j - \left(1 - \varkappa_j a_j\right)\left(\frac{1}{p^i} - \frac{1}{p}\right),$$

and

$$B^i_\eta(x) = \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int\limits_{0^{e^i}}^{\eta^{e^i}} \prod_{j \in e^i} t_j^{-3-\nu_i} \varphi_i(x,t)\, dt^{e^i} \tag{2.6}$$

$$B^i_{\eta,T}(x) = \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int\limits_{\eta^{e^i}}^{T^{e^i}} \prod_{j \in e^i} t_j^{-3-\nu_i} \varphi_i(x,t)\, dt^{e^i}, \tag{2.7}$$

where

$$\varphi_i(x,t) = \int\limits_{R^n} \int\limits_{-\infty}^{\infty} ... \int\limits_{-\infty}^{\infty} \psi_{e^i}\left(\frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho\left(t^{e^i} + T^{e_n \setminus e^i}, x\right)}{t^{e^i} + T^{e_n \setminus e^i}}\right) \times$$

$$\times \zeta^i\left(\frac{u}{t}, \frac{\rho(t,x)}{2t}, \frac{1}{2}\rho'(t,x)\right) \Delta^{m^i}(\delta u)\, f(x + y + u)\, du dy. \tag{2.8}$$

Then the following inequalities hold

$$\sup_{\bar{x} \in U} \left\| B^i_\eta(x) \right\|_{p, U_{\rho^\varkappa}(\bar{x})} \le C_1 \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j \in e^{l^i}} t_j^{-l^{i,\mu}_j} \Delta^{m^i}(t; Z) f \right\|_{p^i_\mu, a, \varkappa, \tau} \right\}^{\beta_\mu} \prod_{j \in e_n} [\rho_j]_1^{\frac{\varkappa_j a_j}{p}} \times$$

$$\times \prod_{j \in e^{l^i}} \eta_j^{\varepsilon^i_j} \ \left(\varepsilon^i_j > 0\right) \tag{2.9}$$

$$\sup_{\bar{x} \in U} \left\| B^i_{\eta,T}(x) \right\|_{p, U_{\rho^\varkappa}(\bar{x})} \le C_1 \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j \in e^{l^i}} t_j^{-l^{i,\mu}_j} \Delta^{m^i}(t; Z) f \right\|_{p^i_\mu, a, \varkappa, \tau} \right\}^{\beta_\mu}$$

$$\times \prod_{j \in e_n} [\rho_j]_1^{-\frac{\varkappa_j a_j}{p}} \begin{cases} \prod\limits_{j \in e^{l^i}} T_j^{\varepsilon^i_j}, \text{if } \varepsilon^i_j > 0; \\ \prod\limits_{j \in e^{l^i}} \ln \frac{T_j}{\eta_j}, \text{if } \varepsilon^i_j = 0; \\ \prod\limits_{j \in e^{l^i}} \eta_j^{\varepsilon^i_j}, \text{if } \varepsilon^i_j < 0, \end{cases} \tag{2.10}$$

here $U_{\rho^\varkappa}(\bar{x}) = \left\{ x : \left| x_j - \bar{x}_j \right| < \frac{\rho_j^{\varkappa_j}}{2}, j \in e_n \right\}$, C_1 *and* C_2 *are constants independent of* Φ, ρ, η *and* T.

Proof. Apply the generalized Minkowski inequality for $B_\eta^i(x)$ determined by equality (2.6). Then we get

$$\left\|B_\eta^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})} \leq \prod_{j\in e_n\setminus e^i} T_j^{-1-\nu_j} \int_{0^{e^i}}^{\eta^{e^i}} \prod_{j\in e^i} t_j^{-3-\nu_i} \left\|\varphi_i(x,t)\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})} dt^{e^i}. \tag{2.11}$$

Applying the Holder inequality with exponents $\alpha_\mu = \frac{p_\mu}{p\beta_\mu}, \mu = 1,2,\ldots,N; \sum_{\mu=1}^{N} \frac{1}{\alpha_\mu} = p\sum_{\mu=1}^{N}\frac{\beta_\mu}{p_\mu} = 1$ and we estimate the norm $\left\|\varphi_i(\cdot,t)\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}$

$$\left\|\varphi_i(\cdot,t)\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})} \leq C_1 \prod_{\mu=1}^{N} \left\{\left\|\varphi_i(\cdot,t)\right\|_{p_\mu,U_{\rho^{\varkappa}}(\bar{x})}\right\}^{\beta_\mu}.$$

From the Holder inequality ($p_\mu \leq r_\mu$) we have

$$\left\|\varphi_i(\cdot,t)\right\|_{p_\mu,U_{\rho^{\varkappa}}(\bar{x})} \leq \left\|\varphi_i(\cdot,t)\right\|_{r_\mu,U_{\rho^{\varkappa}}(\bar{x})} \prod_{j\in e_n} \rho_j^{\varkappa_j\left(\frac{1}{p_\mu}-\frac{1}{r_\mu}\right)}. \tag{2.12}$$

Let X be a characteristic function of the set $S(\psi_{e^i})$. Considering that $1 \leq p_\mu \leq r_\mu \leq \infty, s_\mu \leq r_\mu \left(\frac{1}{s_\mu} = 1 - \frac{1}{p_\mu^i} + \frac{1}{r_\mu}\right)$, represent the subintegrand function in (2.8) in the form

$$\left|\int_{-\infty^{e^i}}^{\infty^{e^i}} \psi_{e^i}\zeta_{e^i}\Delta^{m^i} f du\right| = \left(\left|\int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i}\Delta^{m^i} f du\right|^{p_\mu^i} |\psi_{e^i}|^{s_\mu}\right)^{\frac{1}{r_\mu}}$$

$$\times\left(\left|\int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i}\Delta^{m^i} f du\right|^{p_\mu^i} X\left(\frac{y}{t^{e^i}+T^{e_n\setminus e^i}}\right)\right)^{\frac{1}{p_\mu^i}-\frac{1}{r_\mu}} \left(|\psi_{e^i}|^{s_\mu}\right)^{\frac{1}{p_\mu^i}-\frac{1}{r_\mu}}.$$

We can apply the Holder inequality again and get

$$\left\|\varphi_i(\cdot,t)\right\|_{r_\mu,U_{\rho^{\varkappa}}(\bar{x})}$$

$$\leq \sup_{x\in U}\left(\int_{R^n}\left|\int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.\right.$$

$$\left.\left.\times\Delta^{m^i}(\delta u) f\left(x+y+u^{e^i}\right)du\right|^{p_\mu^i} X\left(\frac{y}{t}\right)dy\right)^{\frac{1}{p_\mu^i}-\frac{1}{r_\mu}}$$

$$\times\sup_{y\in V}\left(\int_{U_{\rho^{\varkappa}}(\bar{x})}\left|\int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.\right.$$

$$\left.\left.\times\Delta^{m^i}(\delta u) f\left(x+y+u^{e^i}\right)du\right|^{p_\mu^i} X\left(\frac{y}{t}\right)dy\right)^{\frac{1}{r_\mu}}$$

$$\times\left(\int_{R^n}\left|\psi_{e^i}\left(\frac{y}{t^{e^i}+T^{e_n\setminus e^i}},\frac{\rho\left(t^{e^i}+T^{e_n\setminus e^i},x\right)}{t^{e^i}+T^{e_n\setminus e^i}}\right)\right|^{s_\mu}dy\right)^{\frac{1}{s_\mu}}. \tag{2.13}$$

Taking into account that $G_{t^{e^i}+T^{e_n\setminus e^i}}(x)\subset G_{(t^{\varkappa})^{e^i}+(T^{\varkappa})^{e_n\setminus e^i}}(x)$ for $0<t\le 1, \varkappa_j\le 1\ (j\in e_n)$ for any $x\in U$ we have

$$\int\limits_{R^n}\left|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.$$

$$\left.\times\Delta^{m^i}(\delta u)f\left(x+y+u^{e^i}\right)du\right|^{p^i_\mu}X\left(\frac{y}{t^{e^i}+T^{e_n\setminus e^i}}\right)dy$$

$$\le\int\limits_{(U+V)_{(t^{\varkappa})^{e^i}+(T^{\varkappa})^{e_n\setminus e^i}}(x)}\left|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.$$

$$\left.\times\Delta^{m^i}(\delta u)f\left(y+u^{e^i}\right)du\right|^{p^i_\mu}dy$$

$$\le\prod_{j\in e_{l^i}}t^{p^i_\mu l^{i,\mu}_j}\left\|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.$$

$$\left.\times\prod_{j\in e_{l^i}}t_j^{-l^{i,\mu}_j}\Delta^{m^i}\left(\delta u^{e^i}\right)fdu^{e^i}\right\|^{p^i_\mu}_{p^i_\mu,G_{(t^{\varkappa})^{e^i}+(T^{\varkappa})^{e_n\setminus e^i}}(x)}$$

$$\le\prod_{j\in e_{l^i}}t^{p^i_\mu+p^i_\mu l^{i,\mu}_j}\left\|\prod_{j\in e_{l^i}}t_j^{-l^{i,\mu}_j}\Delta^{m^i}(t,Z_t)f\right\|^{p^i_\mu}_{p^i_\mu,a,\varkappa}\prod_{j\in e_n\setminus e^i}T_j^{\varkappa_j a_j},\tag{2.14}$$

for $y\in V$

$$\int\limits_{U_{\rho^{\varkappa}}(\bar{x})}\left|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\Delta^{m^i}\left(\delta u^{e^i}\right)f\left(x+y+u^{e^i}\right)du^{e^i}\right|^{p^i_\mu}dx$$

$$\le\int\limits_{Z_{\rho^{\varkappa}}(\bar{x}+y)}\left|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_{e^i}\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\Delta^{m^i}\left(\delta u^{e^i}\right)f\left(x+u^{e^i}\right)du^{e^i}\right|^{p^i_\mu}dx$$

$$\le\prod_{j\in e_{l^i}}t^{p^i_\mu l^{i,\mu}_j}\left\|\int\limits_{-\infty^{e^i}}^{\infty^{e^i}}\zeta_i\left(\frac{u}{t},\frac{\rho(t,x)}{t},\frac{1}{2}\rho'(t,x)\right)\right.$$

$$\left.\times\Delta^{m^i}\left(\delta u^{e^i}\right)f\left(x+u^{e^i}\right)du^{e^i}\right\|^{p^i_\mu}_{p^i_\mu,Z_{\rho^{\varkappa}}(\bar{x}+y)}$$

$$\le\prod_{j\in e_{l^i}}t^{p^i_\mu+p^i_\mu l^{i,\mu}_j}\left\|\prod_{j\in e_{l^i}}t_j^{-l^{i,\mu}_j}\Delta^{m^i}(t,Z_t)f\right\|^{p^i_\mu}_{p^i_\mu,a,\varkappa}\prod_{j\in e_n}[\rho]_1^{\varkappa_j a_j},\tag{2.15}$$

$$\int\limits_{R^n}\left|\psi^{(\nu)}_{e^i}\left(\frac{y}{t^{e^i}+T^{e_n\setminus e^i}},\frac{\rho\left(t^{e^i}+T^{e_n\setminus e^i},x\right)}{t^{e^i}+T^{e_n\setminus e^i}}\right)\right|^{s_\mu}dy=\prod_{j\in e^i}t_j\prod_{j\in e_n\setminus e^i}T_j\left\|\psi^{(\nu)}_{e^i}\right\|^{s_\mu}.\tag{2.16}$$

From inequalities $(2.12) - (2.16)$, we have

$$\|\varphi_i(\cdot,t)\|_{r,U_{\rho^{\varkappa}}(\bar{x})} \leq C\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa}\right\}^{\beta_\mu}\prod_{j\in e_n\setminus e^i} T_j^{1-\left(\frac{1}{p^i}-\frac{1}{r}\right)(1-\varkappa_j a_j)}$$

$$\times\prod_{j\in e_{l^i}} t_j^{2+l_j^{i,\mu}-\left(\frac{1}{p^i}-\frac{1}{r}\right)(1-\varkappa_j a_j)}\prod_{j\in e_n}[\rho_j]_1^{\varkappa_j a_j}\prod_{j\in e_n}\rho_j^{\left(\frac{1}{p^i}-\frac{1}{r}\right)\varkappa_j}. \tag{2.17}$$

For any $\tau, 1\leq\tau\leq\infty$ taking into account the inequality $\|\cdot\|_{p,a,\varkappa}\leq c\|\cdot\|_{p,a,\varkappa,\tau}$ and (2.17) in (2.11) for $r=p$ we get inequality (2.9). Inequality (2.10) is proved by the same way.

Corollary 2.1 *For $1\leq\tau_1\leq\tau_2\leq\infty$ the following inequality is valid:*

$$\|\varphi_i(\cdot,t)\|_{p,b,\varkappa,\tau_2,U}\leq C^1\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa,\tau_1}\right\}^{\beta_\mu}, \tag{2.18}$$

where $b\in[0,1]^n$, C^1 is a constant dependent on t and M_i, but independent of f.

Lemma 2.2 *Let $1\leq p_\mu^i\leq p\leq\infty, (\mu=1,2,\dots,N), 0<\varkappa_j\leq 1,\ 0<T_j\leq 1, (j\in e_n)$, $\nu=(\nu_1,...,\nu_n)$, $\nu_j\geq 0$ be integers $(j=1,...,n)$; $1\leq\tau_1\leq\tau_2\leq\infty, \varepsilon_j^i>0$ and*

$$\varepsilon_j^{i,0}=\sum_{\mu=1}^{N} l_j^{i,\mu}\beta_\mu-\nu_j-\left(1-\varkappa_j a_j\right)\frac{1}{p^i}.$$

Then for the function B_T^i determined by equality (2.6) the following inequality is valid:

$$\left\|B_T^i\right\|_{p,b,\varkappa,\tau_2,U}\leq C^2\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa,\tau_1}\right\}^{\beta_\mu}, \tag{2.19}$$

where $b=(b_1,...,b_n)$, b_j are arbitrary numbers satisfying the inequalities:

$$\begin{array}{ll} 0\leq b_j\leq 1, & \text{if } \varepsilon_j^{i,0}>0\ (j\in e^i)\\ 0\leq b_j<1, & \text{if } \varepsilon_j^{i,0}=0\ (j\in e^i)\\ 0\leq b_j<a_j+\frac{\varepsilon_j^{i,0}p(1-a_j)}{1-\varkappa_j a_j}=a_j+\frac{\varepsilon_j^{i}p(1-a_j)}{1-\varkappa_j a_j}, & \text{if } \varepsilon_j^{i,0}<0\ (j\in e^i)\end{array} \tag{2.20}$$

$j\in e^i, 0\leq b_j\leq a_j, j\in e_n\backslash e^i$ and C^2 is a constant independent of f.

Proof. Estimate $\left\|B_T^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}$, $\bar{x}\in U, 0<\rho_j<\infty\,(j\in e_n)$. First let $0<\rho_j<T_j\ (j\in e_n)$. Then

$$\left\|B_T^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}\leq\left\|B_\rho^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}+\left\|B_{\rho T}^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}. \tag{2.21}$$

From inequality (2.9) $\left(\eta_j=\rho_j, j\in e_n\right)$ for $\tau=\infty$ we have

$$\left\|B_\rho^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})}\leq C_1\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa}\right\}^{\beta_\mu}\prod_{j\in e_n}\rho_j^{\varepsilon_j^i+\frac{\varkappa_j a_j}{p_\mu^i}}, \tag{2.22}$$

where C is independent of f and ρ. Further, by the generalized Minkowski inequality and inequality (2.17) we have:

$$\left\|B^i_{\rho T}\right\|_{p,U_{\rho^\varkappa}(\bar{x})} \leq \int\limits_{\rho^{e^i}}^{T^{e^i}} \prod_{j\in e^i} t_j^{-3-\nu_i} \left\|\varphi_i(x,t)\right\|_{p,U_{\rho^\varkappa}(\bar{x})} dt^{e^i}$$

$$\leq C_2 \prod_{\mu=1}^N \left\{ \left\| \prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}} \Delta^{m^i}(t,Z_t) f \right\|_{p^i_\mu,a,\varkappa} \right\}^{\beta_\mu} \psi(\rho,T,r), \tag{2.23}$$

where $\psi(\rho,T,r) = \prod\limits_{j\in e_n} \rho_j^{\delta_j(r)} \int\limits_{\rho^{e^i}}^{T^{e^i}} \prod\limits_{j\in e^i} t_j^{\varepsilon^i_j(r)-1} dt^{e^i}$,

$$\varepsilon^i_j(r) = \sum_{\mu=1}^N l_j^{i,\mu}\beta_\mu - \nu_j - \left(1-\varkappa_j a_j\right)\left(\frac{1}{p^i}-\frac{1}{p}\right), \delta_j(r) = \frac{\varkappa_j}{p} - \frac{\varkappa_j}{r}\left(1-a_j\right),$$

and C_2 is a constant independent of f and ρ. Estimation (2.23) is valid for any $r, (p \leq r \leq \infty)$. Choose r from the all possible numbers such that the index of power of ρ_j in this estimation is maximal. For this, note that $\delta(r)$ monotonically increases, $\varepsilon^i_j(r)$ monotonically decreases on $[p,\infty]$ and $\varepsilon^i_j(p) = \varepsilon^i_j, \varepsilon^i_j(\infty) = \varepsilon^{i,0}_j$. We consider two cases: $\varepsilon^{i,0}_j \geq 0$ and $\varepsilon^{i,0}_j < 0$, If $\varepsilon^{i,0}_j \geq 0$, then maximum index of ρ_j in this estimation we get for $r=\infty$. Considering that $\delta_j(\infty) = \frac{\varkappa_j}{p}$, we have

$$\psi(\rho,T,r) = \begin{cases} \frac{1}{\varepsilon^{i,0}_j} \prod\limits_{j\in e_n} \rho_j^{\frac{\varkappa_j}{p}} \left(\prod\limits_{j\in e_n} T_j^{\varepsilon^{i,0}_j} - \prod\limits_{j\in e_n} \rho_j^{\varepsilon^{i,0}_j} \right), \text{if } \varepsilon^{i,0}_j > 0; \\ \prod\limits_{j\in e_n} \rho_j^{\frac{\varkappa_j}{p}} \ln \prod\limits_{j\in e_n} \frac{T_j}{\rho_j}, \qquad \text{if } \varepsilon^{i,0}_j = 0. \end{cases}$$

Let $\varepsilon^{i,0}_j < 0$. Since $\varepsilon^i_j(p) = \varepsilon^i_j > 0$, and $\varepsilon^i_j(\infty) = \varepsilon^{i,0}_j < 0$, then at some $r_0, (p \leq r_0 \leq \infty)$, $\varepsilon^i_j(r_0) = 0$. It is easy to see that the best estimation in this case is obtained if in $\psi(\rho,T,r)$ we put $r = r_0$. Then $\psi(\rho,T,r_0) = \prod\limits_{j\in e_n} \rho_j^{\delta_j(r_0)} \ln \prod\limits_{j\in e^{l^i}} \frac{T_j}{\rho_j}$, where

$$\delta_j(r_0) = \frac{\varkappa_j}{p}\left(1 + \frac{\varepsilon^{i,0}_j p\left(1-a_j\right)}{1-\varkappa_j a_j}\right) = \frac{\varkappa_j}{p}\left(a + \frac{\varepsilon^{i,0}_j p\left(1-a_j\right)}{1-\varkappa_j a_j}\right),$$

($1-\varkappa_j a_j$, since $\varepsilon^i_j > 0$ and $\varepsilon^{i,0}_j < 0$). Improving, on the base of it, the estimation (2.23) and taking into account that

$$\varepsilon^i_j + \frac{\varkappa_j a_j}{p} \geq \frac{\varkappa_j}{p} \quad \text{for} \quad \varepsilon^{i,0}_j \geq 0$$

$$\varepsilon^i_j + \frac{\varkappa_j a_j}{p} \geq \frac{\varkappa_j}{p}\left(a + \frac{\varepsilon^{i,0}_j p\left(1-a_j\right)}{1-\varkappa_j a_j}\right) = \delta_j(r_0) \quad \text{for} \quad \varepsilon^{i,0}_j < 0$$

and inequalities (2.21) − (2.23), we get $(\tau = \infty)$

$$\left\|B^i_T\right\|_{p,U_{\rho^\varkappa}(\bar{x})} \leq C_3 \prod_{\mu=1}^N \left\{ \left\| \prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}} \Delta^{m^i}(t,Z_t) f \right\|_{p^i_\mu,a,\varkappa} \right\}^{\beta_\mu}$$

$$\times\psi(\rho,T,r) = \begin{cases} \frac{1}{\varepsilon^{i,0}_j} \prod\limits_{j\in e_n} \rho_j^{\frac{\varkappa_j}{p}} \left(\prod\limits_{j\in e_n} T_j^{\varepsilon^{i,0}_j} - \prod\limits_{j\in e_n} \rho_j^{\varepsilon^{i,0}_j} \right), \text{ if } \varepsilon^{i,0}_j > 0 \\ \prod\limits_{j\in e_n} \rho_j^{\frac{\varkappa_j}{p}} \ln \prod\limits_{j\in e_n} \frac{T_j}{\rho_j}, \qquad \text{if } \varepsilon^{i,0}_j = 0 \end{cases} \times$$

$$\times\begin{cases}\prod\limits_{j\in e_n}\rho_j^{\frac{\varkappa_j}{p}}\prod\limits_{j\in e_n}T_j^{\varepsilon_j^{i,0}}, & \text{if } \varepsilon_j^{i,0}>0;\\ \prod\limits_{j\in e_n}\rho_j^{\frac{\varkappa_j}{p}}\ln\prod\limits_{j\in e_n}\frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0}=0;\\ \prod\limits_{j\in e_n}\rho_j^{\delta_j(r_0)}\ln\prod\limits_{j\in e^{l^i}}\frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0}>0.\end{cases}\tag{2.24}$$

Let $\rho_j \geq T_j \ (j\in e_n)$. Again applying estimation (2.9) $\left(\eta_j \leq T_j, j\in e_n, \tau=\infty\right)$ we have

$$\left\|B_T^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})} \leq C_4\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa}\right\}^{\beta_\mu}$$

$$\times\begin{cases}\prod\limits_{j\in e_n}\rho_j^{\varepsilon_j^i+\frac{\varkappa_j a_j}{p}}, & \text{if } T_j\leq\rho_j\leq 1\\ \prod\limits_{j\in e_n}T_j^{\varepsilon_j^i}, & \text{if } 1<\rho_j<\infty.\end{cases}\tag{2.25}$$

It follows from inequalities (2.24) and (2.25) that for any $\bar{x}\in U$ and any $\rho, 0<\rho<\infty$

$$\left\|B_T^i\right\|_{p,U_{\rho^{\varkappa}}(\bar{x})} \leq C_5\prod_{\mu=1}^{N}\left\{\left\|\prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}}\Delta^{m^i}(t,Z_t)f\right\|_{p_\mu^i,a,\varkappa}\right\}^{\beta_\mu}\prod_{j\in e_n}[\rho_j]_1^{\frac{\varkappa_j b_j}{p}},$$

where $b_j\ (j=1,...,n)$ are the numbers satisfying inequalities (2.20), and C_5 is a constant independent of f,ρ and $\bar{x}$. Considering the relation $1\leq\tau_1\leq\tau_2\leq\infty$, from the last inequality we get inequality (2.19)

3 Main results

In this section, we prove two theorems on some differential properties of functions from the intersection $\bigcap\limits_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau}(G_h)\ (\mu=1,2,\ldots,N)$.

Theorem 3.1 *Let the open set $G\subset R^n$ satisfy the condition of flexible (A_1) horn([12]), $1\leq p_\mu^i\leq p\leq\infty, 1\leq\theta_\mu^i\leq\infty, (i=0,1,2,...,2^n);(\mu=1,2,\ldots,N), \bar{\varkappa}=c\varkappa, \frac{1}{c}=\max\limits_{j\in e_n} l_j\varkappa_j$,*

$\nu=(\nu_1,...,\nu_n), \nu_j\geq 0$ be integers $j\in e_n$; $\nu_j\geq l_j^{i,\mu}$ for $j\in e_{l^i}\backslash e^i$, $1\leq\tau_1\leq\tau_2\leq\infty$;

$$\varepsilon_j^i = l_j^{i,\mu}-\nu_j-\left(1-\varkappa_j a_j\right)\left(\frac{1}{p^i}-\frac{1}{p}\right)>0, \left(j\in e^{l^i}, i=1,...,2^n\right),$$

and let $f\in\bigcap\limits_{\mu=1}^{N}\bigcap\limits_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau}(G_h)$. Then $D^\nu:\bigcap\limits_{\mu=1}^{N}\bigcap\limits_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau_1}(G_h)\hookrightarrow L_{p,b,\varkappa,\tau_2}(G)$, more exactly, for $f\in\bigcap\limits_{\mu=1}^{N}\bigcap\limits_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau}(G_h)$ in domain G there exists a generalized derivative $D^\nu f$, for which the following inequalities are valid:

$$\left\|D^\nu f\right\|_{p,G}\leq C_1\sum_{i=1}^{2^n}\prod_{j\in e_n}T_j^{\varepsilon_j^i}\prod_{\mu=1}^{N}\left\{\|f\|_{L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau_1}(G_t)}\right\}^{\beta_\mu},\tag{3.1}$$

and

$$\left\|D^\nu f\right\|_{p,b,\varkappa,\tau_2;G}\leq C_2\prod_{\mu=1}^{N}\left\{\|f\|_{\bigcap\limits_{i=1}^{2^n}L^{<l^{i,\mu}>}_{p_\mu^i,\theta_\mu^i,a,\varkappa,\tau_1}(G_t)}\right\}^{\beta_\mu},\left(p_\mu^i\leq p<\infty, i=1,2,...,2^n\right).\tag{3.2}$$

In particular if,

$$\varepsilon_{j,0}^{i}=\sum_{\mu=1}^{N}l_{j}^{i,\mu}\beta_{\mu}-\nu_{j}-\left(1-\bar{\varkappa}_{j}a_{j}\right)\frac{1}{p^{i}}>0,\left(j\in e^{i},j=1,...,2^{n}\right),$$

then $D^{\nu}f$ *is continuous on* G *and*

$$\sup_{x\in G}\left|D^{\nu}f\left(x\right)\right|\leq C_{1}\sum_{i=1}^{2^{n}}\prod_{j\in e_{n}}T_{j}^{\varepsilon_{j,0}^{i}}\prod_{\mu=1}^{N}\left\{\|f\|_{L^{<l^{i,\mu}>}_{p_{\mu}^{i},\theta_{\mu}^{i},a,\varkappa,\tau_{1}}(G_{t})}\right\}^{\beta_{\mu}},\tag{3.3}$$

where $T_{j}\in\left(0,\min\left(1,T_{j,0}\right)\right]$ $(j\in e_{n})$, C_{1},C_{2} *are the constants independent of* f, C_{1} *is independent also on* T *and* $b=(b_{1},...,b_{n})$, b_{j} *satisfying*

$$0\leq b_{j}\leq 1,\ \textit{if}\ \varepsilon_{j,0}^{i}>0;$$
$$0\leq b_{j}<1,\ \textit{if}\ \varepsilon_{j,0}^{i}=0;$$
$$0\leq b_{j}<a+\frac{\varepsilon_{j}^{i}p\left(1-a_{j}\right)}{1-\varkappa_{j}a_{j}},\textit{if}\ \varepsilon_{j,0}^{i}<0;$$

$j\in e^{j}, 0\leq b_{j}<a_{j}, j\in e_{n}\backslash e^{i}$ *but with substitution* $\bar{\varkappa}$ *for* $\varkappa$.

Proof. First of all note that since $\bar{\varkappa}=c\varkappa, c>0$, then $f\in\bigcap_{\mu=1}^{N}\bigcap_{i=1}^{2^{n}}L^{<l^{i,\mu}>}_{p_{\mu}^{i},\theta_{\mu}^{i},a,\bar{\varkappa},\tau_{1}}(G)$ and everywhere in inequalities $(3.1)-(3.3)$ we can replace $\varkappa$ by $\bar{\varkappa}$. Passage from $\varkappa$ to $\bar{\varkappa}$, is explained with the following properties of the vector $\bar{\varkappa}$:

1) $\bar{\varkappa}_{j}\leq 1, j\in e_{n}$; 2) among the vectors of the form $d\varkappa : d>0$, satisfying the inequality $d\varkappa\leq 1$, the vector $\bar{\varkappa}$ has the greatest absolute value.

The importance of the first property will be seen from the proof of the theorem, and the significance of the second property follows from the theorem statement (the more $\varkappa_{j}$, the more ε_{j}^{i}). Under the conditions of Theorem 3.1 there exists a generalized derivative $D^{\nu}f$. Indeed, if $\varepsilon_{j}^{i}>0\left(j\in e^{i}, i=1,...,2^{n}\right)$, from $p_{\mu}^{i}\leq p, \bar{\varkappa}_{j}\leq 1, (j\in e_{n}), a\in[0,1]^{n}$ it follows that $l_{j}-\nu_{j}>0, j\in e^{i}\left(j\in e^{i}, i=1,...,2^{n}\right)$. This means that for $f\in\bigcap_{\mu=1}^{N}\bigcap_{i=1}^{2^{n}}L^{<l^{i,\mu}>}_{p_{\mu}^{i},\theta_{\mu}^{i},a,\bar{\varkappa},\tau_{1}}(G)\hookrightarrow\bigcap_{\mu=1}^{N}\bigcap_{i=1}^{2^{n}}L^{<l^{i,\mu}>}_{p_{\mu}^{i},\theta_{\mu}^{i}}(G)$ there exists $D^{\nu}f$ and $D^{\nu}f\in L_{p_{\mu}^{i}}(G)$. Then for almost each point $x\in G$ the following equality holds

$$D^{\nu}f\left(x\right)=\sum_{i=1}^{2^{n}}\left(-1\right)^{\left|\omega^{e^{i}}\right|+|\nu|}\prod_{j\in e_{n}\backslash e_{i}}T_{j}^{-1-\nu_{j}}\int_{0^{e^{i}}}^{T^{e^{i}}}\frac{dt^{e^{i}}}{\prod_{j\in e^{i}}t_{j}^{3+\nu_{j}}}$$
$$\times\int_{R^{n}}\int_{-\infty^{e^{i}}}^{\infty^{e^{i}}}\psi_{e^{i}}^{(\nu)}\left(\frac{y}{t^{e^{i}}+T^{e_{n}\backslash e^{i}}},\frac{\rho\left(t^{e^{i}}+T^{e_{n}\backslash e^{i}},x\right)}{t^{e^{i}}+T^{e_{n}\backslash e^{i}}}\right)$$
$$\times\zeta_{e^{i}}\left(\frac{u}{t},\frac{\rho\left(t,x\right)}{t},\frac{1}{2}\rho'\left(t,x\right)\right)\varDelta^{m^{i}}\left(\delta u\right)f\left(x+y+u^{e^{i}}\right)du^{e^{i}}dy.\tag{3.4}$$

Note that $\psi_{e^{i}}^{(\nu)}\left(\cdot,z\right)\in C_{0}^{\infty}\left(R^{n}\right),\zeta_{i}\in C_{0}^{\infty}\left(R\right)$, their supports are contained in I_{1} and are such that the supports of representations (3.4) are contained in the flexible cone $x+V\subset G$. The parameter of the representation $\delta>0$ is rather small, owing to which the replacement of $\varDelta^{m^{i}}\left(\delta u,G\right)f\ \ (i=1,...,2^{n})$ in the right side (3.4) by $\varDelta^{m^{i}}\left(\delta u,G_{\delta u}\right)$ doesn't influence on the value of the right side. On the base of Minkowski inequality we have

$$\left\|D^{\nu}f\right\|_{p,G}=\sum_{i=1}^{2^{n}}\prod_{j\in e_{n}\backslash e_{i}}T_{j}^{-1-\nu_{j}}\left\|B_{T}^{i}\right\|_{p,G},\tag{3.5}$$

and by means of inequality (2.9) for $U \in G, \psi_i^{(\nu)} = M_i, \rho \longrightarrow \infty$, we get

$$\left\|B_T^i\right\|_{p,G} \leq C_2 \prod_{j\in e_n} T_j^{\varepsilon_j^i} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}} \Delta^{m^i}(t, Z_t) f \right\|_{p_\mu^i, a, \bar{\varkappa}, \tau_1} \right\}^{\beta_\mu}. \quad (3.6)$$

By means of inequalities (3.5) – (3.6) under condition $p_\mu^i \leq \theta_\mu^i$ $(\mu = 1, \ldots, N; i = 1, ..., 2^n)$ we get inequality (3.1). By the similar way, based on inequalities (2.19), estimation (3.2) can be established.

Now let $\varepsilon_j^{i,0} > 0 \left(j \in e^i, i = 1, ..., 2^n\right)$. Let us show that $D^\nu f$ is continuous on G. On the base of identities (3.4), from inequality (3.1) for $p = \infty, \varepsilon_j^i = \varepsilon_{j,0}^i > 0 \left(j \in e^i\right), 1 \leq \theta_\mu^i \leq \infty \,(i = 1, 2, ..., 2^n)$, we obtain that

$$\left\|D^\nu f - \bar{f}_T^{(\nu)}\right\|_{\infty,G}$$

$$\leq \sum_{\substack{i=1\\ e^i\neq\varnothing}}^{2^n} \prod_{j\in e_n} T_j^{\varepsilon_{j,0}^i} \prod_{\mu=1}^{N} \left\{ \left\{ \int^{t_{01}} \cdots \int_0^{t_{0n}} \left\| \prod_{j\in e_{l^i}} t_j^{-l_j^{i,\mu}} \Delta^{m^i}(t, Z_t) f \right\|_{p_\mu^i, a, \varkappa, \tau_1}^{\theta_\mu^i} \prod_{j\in e_{l^i}} \frac{dt_j}{t_j} \right\}^{\frac{1}{\theta_\mu^i}} \right\}^{\beta_\mu}.$$

Hence the left hand side of the inequality tends to zero as $T_j \to 0\,(j \in e_n)$. Since $\bar{f}_T^{(\nu)}$ is continuous on G, the convergence in $L_\infty(G)$ coincides at the present case with the uniform one, and consequently the limit function $D^\nu f$ is continuous on G. Theorem 3.1 is proved.

Let $\gamma = (\gamma_1, ..., \gamma_n) \in R^n$.

Theorem 3.2 *Let domain G, parameters $p_\mu^i, p, \tau, (\mu = 1, 2, \ldots, N)$ and vectors $\varkappa, a, \varepsilon_j^i, \varepsilon_{j,0}^i, \nu$ satisfy the conditions of theorem 3.1. Then for $\varepsilon_j^i > 0$, $\left(j \in e^i, i = 1, ..., 2^n\right)$ the derivative $D^\nu f$ satisfies on G the Holder condition with the exponent β_j, more exactly*

$$\left\|\Delta(\gamma, G) D^\nu f\right\|_{p,G} \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i, \theta_\mu^i, a, \varkappa, \tau}(G_t)} \right\}^{\beta_\mu} \prod_{j\in e_n} \left|\gamma_j\right|^{\beta_j}, \quad (3.7)$$

here $\beta = (\beta_1, ..., \beta_n), \beta_j$ is any number satisfying the inequalities:

$$\begin{array}{l} 0 \leq \beta_j \leq 1, \ \text{if } \varepsilon_j^i > 1, \\ 0 \leq \beta_j < 1, \ \text{if } \varepsilon_j^i = 1, \\ 0 \leq \beta_j < \varepsilon_j^i, \text{if } \varepsilon_j^i < 1. \end{array} \quad (3.8)$$

where $(j \in e_n, i = 1, ..., 2^n), C$ is a constant independent of f and $\gamma = (\gamma_1, ..., \gamma_n)$.

In particular, if $\varepsilon_j^{i,0} > 0$, then

$$\sup_{x\in G} \left|\Delta(\gamma, G) D^\nu f(x)\right| \leq C \prod_{\mu=1}^{N} \left\{ \|f\|_{\bigcap_{i=1}^{2^n} L^{<l^{i,\mu}>}_{p_\mu^i, \theta_\mu^i, a, \varkappa, \tau}(G_t)} \right\}^{\beta_\mu} \prod_{j\in e_n} \left|\gamma_j\right|^{\beta_j^0}, \quad (3.9)$$

where β_j^0 satisfies the same conditions as σ_j, but with replacement of ε_j^i by $\varepsilon_j^{i,0}$.

Proof. According to lemma 8.6 in [1], there exists the domain $G_\sigma \subset G \left(\sigma = (\sigma_1, ..., \sigma_n), \sigma_j = \xi_j r(x), \xi_j > 0, r(x) = \rho(x, \partial G), x \in G\right)$. Assume that $\left|\gamma_j\right| < \sigma_j$ $(j \in e_n)$, then for any $x \in G_\sigma$ the interval connecting the points $x, x + \gamma$ is contained in G and for all the points of this interval, identities (3.4) are valid with the same kernels. After some transformations we have

$$\left|\Delta(\gamma, G) D^\nu f(x)\right| \leq C_1 \sum_{i=1}^{2^n} \prod_{j\in e_n\setminus e^i} T_j^{-1-\nu_j} \int_0^{|\gamma|^{e^i}} \frac{dt^{e^i}}{\prod_{j\in e^i} t_j^{3+\nu_j}}$$

$$\times \int_{R^n} \int_{-\infty^{e^i}}^{\infty^{e^i}} \left| \zeta_{e^i} \left(\frac{u}{t}, \frac{\rho(t,x)}{t}, \frac{1}{2}\rho'(t,x) \right) \right| \left| \psi_{e^i}^{(\nu)} \left(\frac{y}{t^{e^i} + T^{e_n \backslash e^i}}, \frac{\rho\left(t^{e^i} + T^{e_n \backslash e^i}, x\right)}{t^{e^i} + T^{e_n \backslash e^i}} \right) \right|$$

$$\times \left| \varDelta(\gamma, G) \varDelta^{m^i}(\delta u) f\left(x + y + u^{e^i}\right) \right| du^{e^i} dy$$

$$\leq C_2 \sum_{i=1}^{2^n} \prod_{j \in e_n \backslash e^i} T_j^{-1-\nu_j} \int_{|\gamma|^{e^i}}^{T^{e^i}} \frac{dt^{e^i}}{\prod_{j \in e^i} t_j^{3+\nu_j}}$$

$$\times \int_{R^n} \int_{-\infty}^{\infty} \left| \psi_{e^i}^{(\nu+1)} \left(\frac{y}{t^{e^i} + T^{e_n \backslash e^i}}, \frac{\rho\left(t^{e^i} + T^{e_n \backslash e^i}, x\right)}{t^{e^i} + T^{e_n \backslash e^i}} \right) \right| \left| \zeta_{e^i} \left(\frac{u}{t}, \frac{\rho(t,x)}{t}, \frac{1}{2}\rho'(t,x) \right) \right|$$

$$\times \int_0^1 \ldots \int_0^1 \left| \varDelta^{m^i}(\delta u) \right| f(x + y + u + \upsilon_1 \gamma_1 + \ldots + \upsilon_n \gamma_n) \, d\upsilon du^{e^i} dy$$

$$= C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \backslash e^i} T_j^{-1-\nu_j} \left(B_\gamma^i(x, \gamma) + B_{\gamma T}^i(x, \gamma) \right), \tag{3.10}$$

here $|\gamma|^{e^i} = \left(|\gamma_1|^{e^i}, \ldots, |\gamma_n|^{e^i}\right), \left|\gamma_j\right|^{e^i} = |\gamma|$ for $j \in e^i$; $\left|\gamma_j\right|^{e^i} = 0$ for $j \in e_n \backslash e^i$, $(i = 1, \ldots, 2^n)$, $0 < T_j \leq \min\left(1, T_{j,0}\right), j \in e_n$. We also assume that $\left|\gamma_j\right| < T_j$ $(j \in e_n)$ and consequently, $|\gamma| < \min\left(\sigma_j, T_j\right), j \in e_n$. If $x \in G \backslash G_\sigma$, then $\varDelta(\gamma, G) D^\nu f(x) = 0$. Therefore,

$$\left\| \varDelta(\gamma, G) D^\nu f(x) \right\|_{p,G}$$

$$\leq C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \backslash e^i} T_j^{-1-\nu_j} \left(\left\| B_\gamma^i(\cdot, \gamma) \right\|_{p,G_\sigma} + \left\| B_{\gamma T}^i(\cdot, \gamma) \right\|_{p,G_\sigma} \right). \tag{3.11}$$

By means of inequality (2.9) for $\eta = |\gamma|, j \in e_n, U = G, \rho \to \infty$ we get

$$\left\| B_\gamma^i(\cdot, \gamma) \right\|_{p,G_\sigma} \leq C_2 \prod_{j \in e_n} \left|\gamma_j\right|^{\varepsilon_j^i} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j \in e_{l^i}} t_j^{-l_j^{i,\mu}} \varDelta^{m^i}(t, Z_t) f \right\|_{p_\mu^i, a, \varkappa, \tau} \right\}^{\beta_\mu}, \tag{3.12}$$

and by means of inequality (2.10) for $U = G, \eta_j = \left|\gamma_j\right|, j \in e_n, \rho \to \infty$ we get

$$\left\| B_{\gamma T}^i(\cdot, \gamma) \right\|_{p,G_\sigma} \leq C_3 \prod_{j \in e_n} \left|\gamma_j\right|^{\beta_j} \prod_{\mu=1}^{N} \left\{ \left\| \prod_{j \in e_{l^i}} t_j^{-l_j^{i,\mu}} \varDelta^{m^i}(t, Z_t) f \right\|_{p_\mu^i, a, \varkappa, \tau} \right\}^{\beta_\mu}, \tag{3.13}$$

where β_j $(j \in e_n)$ is a number satisfying inequalities (3.8).

Then by means of inequalities (3.12) – (3.13) from inequality (3.11) under the condition $p_\mu^i \leq \theta_\mu^i$ $(\mu = 1, \ldots, N; i = 1, \ldots, 2^n)$, we get inequality (3.7).

Now suppose that $\left|\gamma_j\right| \geq \min\left(\sigma_j, T_j\right), (j \in e_n)$. Then

$$\left\| \varDelta(\gamma, G) D^\nu f(x) \right\|_{p,G} \leq C(\sigma, T) \prod_{j \in e_n} \left|\gamma_j\right|^{\beta_j} \left\| D^\nu f(x) \right\|_{p,G}.$$

Estimating $\|D^\nu f(x)\|_{p,G}$ by means of inequality (3.1), we get the required estimation. Theorem 3.2 is proved.

References

1. Besov, O.V., Il'in, V.P., Nikolskii, S.M.: Integral representation of functions and imbedding theorems. *M. Nauka*. 480 p. (1975) (Russian).
2. Guliyev, V.S.: *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*. Eurasian Math. J., **3** (3), 33–61 (2012).
3. Guliyev, V.S.: *Generalized local Morrey spaces and fractional integral operators with rough kernel*. J. Math. Sci.. **193** (2), 211-227 (2013).
4. Guliyev, V.S., Softova, L.G.: *Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients*. Potential Anal.. **38** (3), 843-862 (2013).
5. Guliyev, V.S., Softova, L.G.: *Generalized Morrey regularity for parabolic equation with discontinuity data*. Proc. Edinb. Math. Soc.,. **58** (1), 199-218 (2015).
6. Ilyin, V.P.: *On some properties of functions from the spaces*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), **23**, 33–40 (1971) (Russian).
7. Jabrailov, A.D.: *Familly of spaces of functions whose mixed derivatives satisfy the Holder multiple integral condition*. Tr. Mat. Inst. Steklova, **117**, 139–158 (1972) (Russian).
8. Kadimova, L.Sh., Najafov, A.M.: *Theorems on imbedding of functions from the Sobolev-Morrey generalized space*. Proc. A. Razmadze Math. Inst. 97–109 (2010).
9. Mazzucato, A.I.: *Besov-Morrey spaces. Function space theory and applications to non-linear PDE*. Transactions of the AMS, **355** (4), 1297–1364 (2002).
10. Morrey, C.B.: *On the solutions of quasi-linear elliptic partial differential equations*. Trans. Amer. Math. Soc., **43**, 126–166 (1938).
11. Morrey, C.B.: *Second order elliptic equations in several variables and Holder continuity*. Math. Zet., **72** (2), 146–164 (1959).
12. Najafov, A.M.: *Interpolation theorems for Besov-Morre and Lizerkin-Triebel- Morrey spaces*. "Voprosy funksionalnogo analiza i mathematicheskoy fiziki" devoted to 80 years of M.E. Rassulzadeh BSU. Baku, 363–366 (1999) (Russian).
13. Najafov, A.M.: *The embedding theorems for the space of Besov-Morrey type with dominant mixed derivatives*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.. **XII(XX)**, 97–104 (2000).
14. Najafov, A.M.: *Interpolation theorem of Besov Morrey type spaces and some its applications*. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **XXIV** (4), 125–134 (2004).
15. Najafov, A.M.: *Some properties of functions from the intersection of Besov-Morrey type spaces with dominant mixed derivatives*. Proc. A. Razmadze Math. Inst. **139**, 71–82 (2005).
16. Najafov, A.M., Orujova, A.T.: *On Riesz-Thorin type theorems in Besov-Morrey spaces and its applications*. Amer. J. Math., **1** (2), 139–154 (2012).
17. Najafov, A.M.: *The embedding theorems of space Math*. Aeterna, **3** (4), 299–308 (2013).
18. Najafov, A.M., Orujova, A.T.: *On properties of the generalized Besov-Morrey spaces with dominant mixed derivatives*. Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. **41** (1), 3-15 (2015).
19. Netrusov, Yu.V.: *On some imbedding theorems of Besov-Morrey type spaces*. Zap. nauchn. Sem. LOMI AN SSSR, **139**, 139–147 (1984) (Russian).
20. Ross, J.: *A Morrey-Nicolskii inequality*. Proc. Amer. Math. Soc. **78**, 97–102 (1980).
21. Sawano, Y.: *Idendification of the image of Morrey spaces by the fractional integral operators.* Proc. A. Razmadze Math. Inst., (149), 87-93 (2009).
22. Li, Tang, Xu, Jingshi: *Some properties of Morrey type Besov-Triebel spaces*. Math. Nachr., **278**, (7-8), 904–917 (2005).