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Generalized maximal functions measuring smoothness

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Abstract. This paper is devoted to the study of certain generalized maximal functions measuring smoothness. We study the connection between the two Φ-maximal functions that measuring smoothness of locally integrable function.

Keywords. Maximal function, Smoothness of function, Mean oscillation, Φ-oscillation, generalized maximal functions

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Introduction

It is known that maximal functions measuring smoothness play an important role in the study of properties of integral operators and other objects of Harmonic Analysis. The main topic of this paper is the study of certain generalized maximal function measuring smoothness.

The paper is organized as follows. Section 1 has auxiliary character and presents the basic definitions, some notation and well-known facts. In section 2 the relations between maximal function and metric characteristic are investigated and some useful inequalities were obtained. In section 3 was obtained inequalities for metric characteristics. In section 4 inequalities between generalized maximal functions was proved. In section 5 estimations between generalized maximal function and maximal function was obtained. The main results are given in Theorem 3.1, Theorem 4.2, Propositions 5.3, 5.5 and Corollary 5.2.

1 Some definition and auxiliary facts

Let $\mathbb{R}^n$ be n-dimensional Euclidean space of the points $x = (x_1, x_2, ..., x_n)$, and $B(a,r) := \{ x \in \mathbb{R}^n : \quad |x-a| \le r\}$ be a closed ball in $\mathbb{R}^n$ of radius $r > 0$ with the center at point $a \in \mathbb{R}^n$. Denote by $L_{loc}(\mathbb{R}^n)$ a class of all local summable functions defined on $\mathbb{R}^n$.

Let $f \in L_{loc}(\mathbb{R}^n)$ and

$$f_{B(x,r)} := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(t)\, dt,$$

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where $|B(x,r)|$ denotes the volume of ball $B(a,r)$. Let $s_f(x) = \lim\limits_{r\to+0} f_{B(x,r)}$ if this limit exist and is finite, and $s_f(x) = f(x)$ at the remaining points. It is known that if $f \in L_{loc}(\mathbb{R}^n)$, then almost everywhere in $\mathbb{R}^n$ it holds the equality $s_f(x) = f(x)$.

Let the function $\varphi(x, r)$ be defined on the set $\mathbb{R}^n \times (0, +\infty)$, accept only positive values, and monotonically increase with respect to the argument r on the interval $(0, +\infty)$. We denote the class of all functions $\varphi(x, r)$ with the above mentioned properties by Ψ.

Let $\Phi \in L^1(\mathbb{R}^n)$, $\Phi(x) \geq 0$ $(x \in \mathbb{R}^n)$, $\varphi \in \Psi$, $f \in L_{loc}(\mathbb{R}^n)$. Introduce the following Φ-maximal functions (generalized maximal functions)

$$f_{\varphi}^{\#,\Phi}(x) = \sup_{r>0} \frac{1}{\varphi(x, r)} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - f_{B(x,r)} \right| dt,$$

$$N_{\varphi}^{\Phi} f(x) = \sup_{r>0} \frac{1}{\varphi(x, r)} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - s_f(x) \right| dt.$$

We also introduce the following metric Φ-characteristics

$$m_f^{\Phi}(x; \delta) = \sup_{0<r\leq\delta} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - f_{B(x,r)} \right| dt,$$

$$n_f^{\Phi}(x; \delta) = \sup_{0<r\leq\delta} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - s_f(x) \right| dt, \quad x \in \mathbb{R}^n, \ \delta > 0.$$

Note that the functions $f_{\varphi}^{\#,\Phi}(x)$ and $m_f^{\Phi}(x; \delta)$ were first introduced in [16].

Consider the known special cases of the introduced maximal functions.

1) If $\Phi(x) \equiv \Phi_0(x) = \frac{1}{|B(0,1)|} X_{B(0,1)}(x)$, where $X_E(x)$ is a characteristic function of the set $E \subset \mathbb{R}^n$, and $\varphi(x, r) \equiv 1$, then $f_{\varphi}^{\#,\Phi}(x) = f^{\#}(x)$, where $f^{\#}(x)$ is the maximal function introduced in the paper [4];

2) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x, r) \equiv r^{\alpha}$ $(\alpha > 0)$, then $f_{\varphi}^{\#,\Phi}(x) = f_{\alpha}^{\#}(x)$. The maximal function $f_{\alpha}^{\#}(x)$ was introduced in the paper [5]. In paper [6] the function $f_{\alpha}^{\#}(x)$ was investigated.

3) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x, r) \equiv r^{\alpha}$ $(\alpha > 0)$, then$N_{\varphi}^{\Phi} f(x) = N_{\alpha} f(x)$. The maximal function $N_{\alpha} f(x)$ was introduced in [4] and studied in [5]. The paper [6] was devoted to the investigation of the functions $f_{\alpha}^{\#}(x)$ and $N_{\alpha} f(x)$.

4) If $\Phi(x) \equiv \Phi_0(x)$, $\varphi(x, r) \equiv \varphi(r)$, then the maximal functions $f_{\varphi}^{\#,\Phi}(x) =: f_{\varphi}^{\#}(x)$ and $N_{\varphi}^{\Phi} f(x) =: N_{\varphi} f(x)$ may be found in the papers [7], [8], [9], [10], [11].

Now let's consider special cases of metric Φ-characteristics.

1. If $\Phi(x) \equiv \Phi_0(x)$, then $m_f^{\Phi}(x; \delta) \equiv m_f(x; \delta)$, where

$$m_f(x; \delta) = \sup_{0<r\leq\delta} \frac{1}{|B(x, r)|} \int_{B(x,r)} \left| f(t) - f_{B(x,r)} \right| dt.$$

Note that the function $m_f(x; \delta)$ was first introduced in the paper [12] (see also, [13], [14]).

2. If $\Phi(x) \equiv \Phi_0(x)$, then $n_f^{\Phi}(x; \delta) = n_f(x; \delta)$, where

$$n_f(x; \delta) = \sup_{0<r\leq\delta} \frac{1}{|B(x, r)|} \int_{B(x,r)} \left| f(t) - s_f(x) \right| dt.$$

For almost all $x \in \mathbb{R}^n$ it is valid the equality $n_f(x; \delta) = \omega_f(x; \delta)$, $\delta > 0$, where

$$\omega_f(x; \delta) = \sup_{0<r\leq\delta} \frac{1}{|B(x, r)|} \int_{B(x,r)} |f(t) - f(x)| \, dt, \ x \in \mathbb{R}^n, \ \delta > 0.$$

The function $\omega_f(x; \delta)$ may be found in literature (see e.g. [2], [17]).

3. Let $\Phi(x) \equiv P(x)$, where $P(x)$ is the Poisson-Kernel i.e.

$P(x) = c_n \cdot \left(1+|x|^2\right)^{-\frac{n+1}{2}}$, where $c_n = \Gamma\left(\frac{n+1}{2}\right) \cdot \pi^{-\frac{n+1}{2}}$. Global variant of the characteristic $m_f^{\Phi}(x;\,\delta)$ (more precisely, the equivalent characteristic to it which is called a modulus of harmonic oscillation) for periodic functions of one variable may be found in the papers of O.Blasco and others [1].

It is known that Hardy-Littlewood's maximal function is determined by the equality

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,\,r)|} \int_{B(x,\,r)} |f(t)|\,dt,\ x \in \mathbb{R}^n.$$

If $\Phi \in L^1(\mathbb{R}^n)$, $\Phi(x) \geq 0$ $(x \in \mathbb{R}^n)$, $f \in L_{loc}(\mathbb{R}^n)$, then the following maximal function is also considered [16]

$$M_{\Phi} f(x) = \sup_{r>0} \int_{\mathbb{R}^n} \Phi_r(x-t)\,|f(t)|\,dt,\ x \in \mathbb{R}^n.$$

It is easy to see that if $\Phi(x) \equiv \Phi_0(x)$, $x \in \mathbb{R}^n$, then $M_{\Phi} f(x) = Mf(x)$.

From the definition of a maximal function $f^{\#}(x)$ it follows that,

$$\forall\ x \in \mathbb{R}^n: \quad f^{\#}(x) \leq \sup_{r>0} \frac{2}{|B(x,\,r)|} \int_{B(x,\,r)} |f(t)|\,dt = 2Mf(x).$$

Thus,

$$f^{\#}(x) \leq 2Mf(x),\ \ x \in \mathbb{R}^n. \tag{1.1}$$

On the other hand,

$$\forall x \in \mathbb{R}^n,\ \forall r > 0: \qquad \frac{1}{|B(x,\,r)|} \int_{B(x,\,r)} |f(t)|\,dt \leq Mf(x),$$

and therefore for almost all $x \in \mathbb{R}^n$

$$|f(x)| = \lim_{r \to +0} \frac{1}{|B(x,\,r)|} \int_{B(x,\,r)} |f(t)|\,dt \leq Mf(x).$$

Thus, for almost all $x \in \mathbb{R}^n$ it is valid the inequality

$$|f(x)| \leq Mf(x). \tag{1.2}$$

If

$$Nf(x) := \sup_{r>0} \frac{1}{|B(x,\,r)|} \int_{B(x,\,r)} \left|f(t) - s_f(x)\right| dt,\ x \in \mathbb{R}^n,$$

then for almost all $x \in \mathbb{R}^n$ we have

$$Nf(x) \leq \sup_{r>0} \frac{1}{|B(x,\,r)|} \int_{B(x,\,r)} |f(t)|\,dt + \left|s_f(x)\right| = Mf(x) + |f(x)|,$$

and hence by means of (1.2) we get that for almost all $x \in \mathbb{R}^n$

$$Nf(x) \leq 2Mf(x). \tag{1.3}$$

It is known that (see, e.g. [17]) if $1 < p \leq \infty$ then

$$\exists\, C_p > 0\ \ \forall f \in L^p\left(\mathbb{R}^n\right):\ \|Mf\|_{L^p} \leq C_p \cdot \|f\|_{L^p}.$$

Hence, from (1.1) and (1.3) we get

$$\exists\, A_p > 0\ \ \forall\, f \in L^p\left(\mathbb{R}^n\right):\ \left\|f^{\#}\right\|_{L^p} \leq A_p \cdot \|f\|_{L^p};$$

$$\exists\, B_p > 0\ \ \forall\, f \in L^p\left(\mathbb{R}^n\right):\ \|Nf\|_{L^p} \leq B_p \cdot \|f\|_{L^p}.$$

The last relation mean that the operators $f \to f^{\#}$ and $f \to Nf$ are the operators of the type $(p,\,p)$ for $1 < p \leq \infty$.

It is also known that [17] that if $f \in L^1(\mathbb{R}^n)$, then there exist a number $A > 0$ such that for any $\lambda > 0$

$$m\left\{\, x \in \mathbb{R}^n : \quad Mf(x) > \lambda \right\} \le \frac{A}{\lambda} \cdot \|f\|_{L^1(\mathbb{R}^n)} ,$$

where mE denotes the Lebesque measure of the set $E \subset \mathbb{R}^n$. Hence, by means of (1.1) and (1.3) we get

$$m\left\{\, x \in \mathbb{R}^n : \quad f^{\#}(x) > \lambda \right\} \le m\left\{\, x \in \mathbb{R}^n : Mf(x) > \frac{\lambda}{2} \right\} \le \frac{2A}{\lambda} \cdot \|f\|_{L^1(\mathbb{R}^n)} ,$$

$$m\left\{\, x \in \mathbb{R}^n : \quad Nf(x) > \lambda \right\} \le m\left\{\, x \in \mathbb{R}^n : Mf(x) > \frac{\lambda}{2} \right\} \le \frac{2A}{\lambda} \cdot \|f\|_{L^1(\mathbb{R}^n)} .$$

Thus, if $f \in L^1(\mathbb{R}^n)$, then there exist the numbers $A_1 > 0$ and $B_1 > 0$ such that for any $\lambda > 0$

$$m\left\{\, x \in \mathbb{R}^n : \quad f^{\#}(x) > \lambda \right\} \le \frac{A_1}{\lambda} \cdot \|f\|_{L^1(\mathbb{R}^n)} ,$$

$$m\left\{\, x \in \mathbb{R}^n : \quad Nf(x) > \lambda \right\} \le \frac{B_1}{\lambda} \cdot \|f\|_{L^1(\mathbb{R}^n)} .$$

The last relations mean that the operators $f \to f^{\#}$ and $f \to Nf$ are the operators of weak type (1,1).

In the case $\varphi(x, r) \equiv 1$, we denote the functions $f_\varphi^{\#,\Phi}(x)$ and $N_\varphi^{\Phi} f(x)$ by $f^{\#,\Phi}(x)$ and $N^{\Phi} f(x)$, respectively. Then for the function $f^{\#,\Phi}(x)$ we have

$$f^{\#,\Phi}(x) = \sup_{r>0} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - f_{B(x,r)} \right| dt$$

$$\le \sup_{r>0} \left\{ \int_{\mathbb{R}^n} \Phi_r(x-t) |f(t)|\, dt + \left| f_{B(x,r)} \right| \cdot \int_{\mathbb{R}^n} \Phi_r(x-t)\, dt \right\}$$

$$\le \sup_{r>0} \int_{\mathbb{R}^n} \Phi_r(x-t) |f(t)|\, dt + C \cdot \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(t)|\, dt$$

$$= M_\Phi f(x) + C \cdot Mf(x), \;\; x \in \mathbb{R}^n,$$

where $C = \int_{\mathbb{R}^n} \Phi_r(x-t)\, dt = \int_{\mathbb{R}^n} \Phi(t)\, dt$. Thus,

$$f^{\#,\Phi}(x) \le M_\Phi f(x) + C \cdot Mf(x), \;\; x \in \mathbb{R}^n. \tag{1.4}$$

Similarly, for $N^{\Phi} f(x)$ we have

$$N^{\Phi} f(x) = \sup_{r>0} \int_{\mathbb{R}^n} \Phi_r(x-t) \left| f(t) - s_f(x) \right| dt$$

$$\le \sup_{r>0} \int_{\mathbb{R}^n} \Phi_r(x-t) |f(t)|\, dt + C \cdot \left| s_f(x) \right| = M_\Phi f(x) + C \cdot \left| s_f(x) \right|$$

$$= M_\Phi f(x) + C \cdot |f(x)| ,$$

for almost all $x \in \mathbb{R}^n$. Hence by means of (1.2) it follows that for almost all $x \in \mathbb{R}^n$

$$N^{\Phi} f(x) \le M_\Phi f(x) + C \cdot Mf(x) . \tag{1.5}$$

From these inequalities (1.4), (1.5), from the Hardy-Littlewood maximal theorem and from theorem 2 of chapter 3 [17] we get following facts.

If $\psi(x) = \sup\limits_{|y| \ge |x|} |\Phi(y)|$, $\psi \in L^1(\mathbb{R}^n)$, then for $1 < p \le \infty$

$$\exists\, A_p > 0 \forall\, f \in L^p\left(\mathbb{R}^n\right) : \left\| f^{\#,\Phi} \right\|_{L^p} \le A_p \cdot \|f\|_{L^p} ;$$

$$\exists\, B_p > 0 \forall\, f \in L^p\left(\mathbb{R}^n\right) : \left\| N^{\Phi} f \right\|_{L^p} \le B_p \cdot \|f\|_{L^p}$$

and for $p = 1$ we have

$$m\Big\{\ x \in \mathbb{R}^n :\quad f^{\#,\,\Phi}(x) > \lambda\Big\} \le \frac{A_1}{\lambda}\cdot \|f\|_{L^1}\,,\ f \in L^1\left(\mathbb{R}^n\right),\ \lambda > 0,$$

$$m\Big\{\ x \in \mathbb{R}^n :\quad N^{\Phi}(x) > \lambda\Big\} \le \frac{B_1}{\lambda}\cdot \|f\|_{L^1}\,,\ f \in L^1\left(\mathbb{R}^n\right),\ \lambda > 0,$$

where the positive constants A_1 andB_1 are independent of f and λ.

Thus, at the indicated conditions on the function Φ, the operators $f \to f^{\#,\,\Phi}$ and $f \to N^{\Phi} f$ are the operators of type $(p,\,p)$ for $1 < p \le \infty$, and are also weak type operators (1,1).

2 Relations between maximal functions and metric characteristics

Everywhere in this point we'll assume that $\Phi \in L^1(\mathbb{R}^n)$, $\Phi(x) \ge 0$ ($x \in \mathbb{R}^n$), $\varphi \in \Psi$.

Proposition 2.1.[16] If $f \in L_{loc}(\mathbb{R}^n)$, then the following equality is true

$$f_{\varphi}^{\#,\,\Phi}(x) = \sup_{r>0} \frac{m_f^{\Phi}(x,\,r)}{\varphi(x,\,r)},\ x \in \mathbb{R}^n.$$

Proposition 2.2. If $f \in L_{loc}(\mathbb{R}^n)$, then the following equality is true

$$N_{\varphi}^{\Phi}(x) = \sup_{r>0} \frac{n_f^{\Phi}(x,\,r)}{\varphi(x,\,r)},\ x \in \mathbb{R}^n. \tag{2.1}$$

Proof. From definition of the function $N_{\varphi}^{\Phi}(x)$ it follows that

$$N_{\varphi}^{\Phi} f(x) = \sup_{r>0} \frac{1}{\varphi(x,\,r)} \int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt$$

$$\le \sup_{r>0} \frac{n_f^{\Phi}(x,\,r)}{\varphi(x,\,r)},\ x \in \mathbb{R}^n. \tag{2.2}$$

On the other hand, for any $r > 0$ and $x \in \mathbb{R}^n$ we have

$$N_{\varphi}^{\Phi} f(x) \ge \frac{1}{\varphi(x,\,r)} \int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt.$$

Hence it follows that

$$\int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt \le \varphi(x,\,r)\cdot N_{\varphi}^{\Phi} f(x)\,,$$

therefore

$$n_f^{\Phi}(x;\,r) \le \varphi(x,\,r)\cdot N_{\varphi}^{\Phi} f(x)\,, r > 0,\ x \in \mathbb{R}^n.$$

So,

$$N_{\varphi}^{\Phi} f(x) \ge \frac{n_f^{\Phi}(x;\,r)}{\varphi(x,\,r)}, r > 0,\ x \in \mathbb{R}^n.$$

From the last inequality we get

$$N_{\varphi}^{\Phi} f(x) \ge \sup_{r>0} \frac{n_f^{\Phi}(x;\,r)}{\varphi(x,\,r)},\ x \in \mathbb{R}^n. \tag{2.3}$$

Equality (2.1) follows from inequalities (2.2) and (2.3).

3 Inequalities between metric characteristics $m_f^{\Phi}(x;\,r)$ and $n_f^{\Phi}(x,\,r)$

Lemma 3.1 *[16] Let $f \in L_{loc}(\mathbb{R}^n)$, and*

$$\operatorname{essinf}\{\Phi(x): \quad x \in B(0,1)\} = c_0 > 0. \tag{3.1}$$

Then for any constant C the following inequality is true

$$\int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt \le c_1 \cdot \int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - C\right| dt, r > 0,\ x \in \mathbb{R}^n, \tag{3.2}$$

where the positive constant c_1 depends only on the c_0, dimension n and on the quantity $\|\Phi\|_{L^1(\mathbb{R}^n)}$.

Proposition 3.1. Let $f \in L_{loc}(\mathbb{R}^n),\ x \in \mathbb{R}^n$ and condition (3.1) be fulfilled. Then the following inequality is true

$$m_f^{\Phi}(x;\,r) \le c_1 \cdot n_f^{\Phi}(x;\,r),\ r > 0, \tag{3.3}$$

where $c_1 > 0$ is a constant from inequality (3.2).

Proof. Having taken $C = s_f(x)$ in the (3.2), we have

$$\int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - f_{B(x,r)}\right| dt \le c_1 \cdot \int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt, r > 0.$$

In going to supremum in this inequality, we get inequality (3.3).

Proposition 3.2. Let $f \in L_{loc}(\mathbb{R}^n),\ x \in \mathbb{R}^n$ and condition (3.1) be fulfilled. Then the following inequalities are true

$$m_f(x;\,r) \le \frac{1}{c_0 \cdot |B(0,1)|} \cdot m_f^{\Phi}(x;\,r),\ r > 0, \tag{3.4}$$

$$n_f(x;\,r) \le \frac{1}{c_0 \cdot |B(0,1)|} \cdot n_f^{\Phi}(x;\,r),\ r > 0. \tag{3.5}$$

Proof. Inequality (3.4) was proved in [16]. We now prove the inequality (3.5). By means of condition (3.1) we get

$$\int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt \ge \frac{1}{\mathbb{R}^n}\int_{\mathbb{R}^n} \Phi_r\left(\frac{x-t}{r}\right)\left|f(t) - s_f(x)\right| dt$$

$$\ge c_0 \cdot \frac{|B(0,1)|}{|B(x,r)|}\int_{B(x,r)} \left|f(t) - s_f(x)\right| dt,\ x \in \mathbb{R}^n,\ r > 0.$$

Hence,

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} \left|f(t) - s_f(x)\right| dt \le \frac{1}{c_0 \cdot |B(0,1)|}\int_{\mathbb{R}^n} \Phi_r(x-t)\left|f(t) - s_f(x)\right| dt.$$

From this we obtain (3.5).

Theorem 3.1 *Let $f \in L_{loc}(\mathbb{R}^n),\ x \in \mathbb{R}^n$, the function Φ satisfy the condition* (3.1) *and let*

$$\int_0^1 \frac{m_f^{\Phi}(x;\,t)}{t} dt < +\infty. \tag{3.6}$$

Then it holds the inequality

$$n_f^{\Phi}(x;\,t) \le c \cdot \left(m_f^{\Phi}(x;\,t) + \int_0^r \frac{m_f^{\Phi}(x;\,t)}{t} dt\right),\ r > 0, \tag{3.7}$$

where the constant $c > 0$ is independent of x, r and f.

Proof. By means of lemma 1.1 from [12] for $0<\eta<\xi$ the following inequality is true

$$\left|f_{B(x,\,\xi)}-f_{B(x,\,\eta)}\right|\le\frac{2^n}{\ln 2}\left(m_f\left(x;\,\xi\right)+\int_\eta^\xi\frac{m_f\left(x;\,t\right)}{t}dt\right). \tag{3.8}$$

From inequality (3.8), taking into account (3.4), we get

$$\left|f_{B(x,\,\xi)}-f_{B(x,\,\eta)}\right|\le c_1\cdot\left(m_f^{\Phi}\left(x;\,\xi\right)+\int_\eta^\xi\frac{m_f^{\Phi}\left(x;\,t\right)}{t}dt\right), \tag{3.9}$$

where $c_1=\frac{2^n}{\ln 2}\cdot\frac{1}{c_0\cdot|B(0,\,1)|}$. By means of conditions (3.6) inequality (3.9) shows that there exists the limit $s_f\left(x\right)=\lim\limits_{r\to+0}f_{B(x,\,r)}$. Taking this into account and passing to limit as $\eta\to 0$ in inequality (3.9), we get

$$\left|f_{B(x,\,\xi)}-s_f\left(x\right)\right|\le c_1\cdot\left(m_f^{\Phi}\left(x;\,\xi\right)+\int_0^\xi\frac{m_f^{\Phi}\left(x;\,t\right)}{t}dt\right).$$

Therefore we have

$$\int_{\mathbb{R}^n}\Phi_r\left(x-t\right)\left|f\left(t\right)-s_f\left(x\right)\right|dt\le\int_{\mathbb{R}^n}\Phi_r\left(x-t\right)\left|f\left(t\right)-f_{B(x,\,r)}\right|dt$$
$$+\left|f_{B(x,\,r)}-s_f\left(x\right)\right|\cdot\int_{\mathbb{R}^n}\Phi_r\left(x-t\right)dt\le m_f^{\Phi}\left(x;\,t\right)$$
$$+c_1\cdot\|\Phi\|_{L^1(\mathbb{R}^n)}\cdot\left(m_f^{\Phi}\left(x;\,r\right)+\int_0^r\frac{m_f^{\Phi}\left(x;\,t\right)}{t}dt\right)$$
$$\le\left(1+c_1\cdot\|\Phi\|_{L^1(\mathbb{R}^n)}\right)\cdot\left(m_f^{\Phi}\left(x;\,r\right)+\int_0^r\frac{m_f^{\Phi}\left(x;\,t\right)}{t}dt\right).$$

Hence, the required (3.7) follows.

For $\Phi=\Phi_0$, the unimprovability of estimation (3.9) was shown in [14].

Remark. Note that in place of the function $\Phi\left(x\right),\ \ x\in\mathbb{R}^n$, satisfying condition (3.1) we can take, for instance, the following functions:

1) $\Phi_0\left(x\right)=\frac{1}{|B(0,\,1)|}\chi_{B(0,\,1)}\left(x\right);$

2) $\Phi^{(\alpha)}\left(x\right)=\frac{1}{1+|x|^{n+\alpha}},\ \left(\alpha>0\right);$

3) $P\left(x\right)=c_n\cdot\left(1+|x|^2\right)^{-\frac{n+1}{2}}$, where $c_n=\Gamma\left(\frac{n+1}{2}\right)\pi^{-\frac{n+1}{2}}$.

Verify, that if $\Phi\left(x\right)\equiv\Phi_0\left(x\right)$, then then

$$n_f^{\Phi}\left(x;\,r\right)\equiv n_f\left(x;\,r\right). \tag{3.10}$$

Indeed, if $\Phi\left(x\right)=\frac{1}{|B(0,\,1)|}\chi_{B(0,\,1)}\left(x\right)$, then

$$\Phi_r\left(x-t\right)=r^{-n}\Phi\left(\frac{x-t}{r}\right)=\frac{1}{\mathbb{R}^n\cdot|B\left(0,\,1\right)|}\chi_{B(0,\,1)}\left(\frac{x-t}{r}\right)$$
$$=\frac{1}{|B\left(x,\,r\right)|}\cdot X_{B(x,\,r)}\left(t\right)=\begin{cases}\frac{1}{|B(x,\,r)|} & \text{if }\ t\in B\left(x,\,r\right),\\ 0 & \text{if }\ t\notin B\left(x,\,r\right).\end{cases}$$

Therefore for this function $\Phi\left(x\right)$ we have

$$\int_{\mathbb{R}^n}\Phi_r\left(x-t\right)\left|f\left(t\right)-s_f\left(x\right)\right|dt=\frac{1}{|B\left(x,\,r\right)|}\int_{B(x,\,r)}\left|f\left(t\right)-s_f\left(x\right)\right|dt.$$

Hence equality (3.10) follows.

Similar reasonings show that if $\Phi\left(x\right)\equiv\Phi_0\left(x\right)$, then

$$m_f^{\Phi}\left(x;\,r\right)\equiv m_f\left(x;\,r\right).$$

4 Inequalities between maximal functions

First, we recall some notation and relations.

If $\Phi_0(x) = \frac{1}{|B(0,1)|}\chi_{B(0,1)}(x),\ x \in \mathbb{R}^n$, then

$$f_\varphi^{\#,\Phi_0}(x) = f_\varphi^{\#}(x),\, N_\varphi^{\Phi_0} f(x) = N_\varphi f(x),\ x \in \mathbb{R}^n,$$

$$m_f^{\Phi_0}(x;\delta) = m_f(x;\delta),\, n_f^{\Phi_0}(x;\delta) = n_f(x;\delta),\ x \in \mathbb{R}^n,\ \delta > 0.$$

If $\varphi(x,t) \equiv 1,\ x \in \mathbb{R}^n, t \in (0,+\infty)$, then for any functions $\Phi(x) \geq 0$ ($x \in \mathbb{R}^n$), $\Phi \in L^1(\mathbb{R}^n)$ and $f \in L_{loc}(\mathbb{R}^n)$ we have introduced the notation

$$f_\varphi^{\#,\Phi}(x) = f^{\#,\Phi}(x),\, N_\varphi^{\Phi} f(x) = N^{\Phi} f(x),\ x \in \mathbb{R}^n.$$

Theorem 4.1. Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, and the function Φ satisfy condition (3.1). Then the following inequality is true

$$f_\varphi^{\#,\Phi}(x) \leq c \cdot N_\varphi^{\Phi} f(x),\ x \in \mathbb{R}^n, \tag{4.1}$$

where the constant $c > 0$ is independent of x, f and φ.

Proof. From Propositions 2.1, 3.1 and 2.2 we have

$$f_\varphi^{\#,\Phi}(x) = \sup_{r>0}\frac{m_f^{\Phi}(x;r)}{\varphi(x,r)} \leq c_1 \cdot \sup_{r>0}\frac{n_f^{\Phi}(x;r)}{\varphi(x,r)} = c_1 \cdot N_\varphi^{\Phi} f(x),$$

where $c_1 > 0$ is a constant from inequality (3.3).

Theorem 4.2. *Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, and the function Φ satisfy condition* (3.1) *and let*

$$\int_0^\delta \frac{\varphi(x,t)}{t}dt = O(\varphi(x,\delta))\left(x \in \mathbb{R}^n, \quad \delta > 0\right). \tag{4.2}$$

Then there exists a number $c > 0$ such that the following inequality is true

$$N_\varphi^{\Phi} f(x) \leq c \cdot f_\varphi^{\#,\Phi}(x),\ x \in \mathbb{R}^n, \tag{4.3}$$

where the constant $c > 0$ is independent of f and x.

Proof. If $x \in \mathbb{R}^n$ and $f_\varphi^{\#,\Phi}(x) = +\infty$, fulfillment of inequality (4.3) is obvious.

Let $x \in \mathbb{R}^n$ and $f_\varphi^{\#,\Phi}(x) < +\infty$. Then, applying Proposition 2.2 and Theorem 3.1, we have

$$\begin{aligned}
N_\varphi^{\Phi} f(x) &= \sup_{r>0}\frac{n_f^{\Phi}(x;r)}{\varphi(x,r)} \leq c \cdot \sup_{r>0}\frac{1}{\varphi(x,r)}\left(m_f^{\Phi}(x;r) + \int_0^r \frac{m_f^{\Phi}(x;r)}{t}dt\right) \\
&\leq c \cdot \left(\sup_{r>0}\frac{m_f^{\Phi}(x;r)}{\varphi(x,r)} + \sup_{r>0}\frac{1}{\varphi(x,r)}\int_0^r \frac{m_f^{\Phi}(x;t)}{\varphi(x,t)} \cdot \frac{\varphi(x,t)}{t}dt\right) \\
&\leq c \cdot \left(\sup_{r>0}\frac{m_f^{\Phi}(x;r)}{\varphi(x,r)} + \sup_{t>0}\frac{m_f^{\Phi}(x;t)}{\varphi(x,t)} \cdot \sup_{r>0}\frac{1}{\varphi(x,r)}\int_0^r \frac{\varphi(x,t)}{t}dt\right) \\
&= c \cdot f_\varphi^{\#,\Phi}(x) \cdot \left(1 + \sup_{r>0}\frac{1}{\varphi(x;r)}\int_0^r \frac{\varphi(x,t)}{t}dt\right),
\end{aligned}$$

where the constant $c > 0$ is independent of f, x and φ. Hence from condition (4.2) we get the required inequality.

Corollary 4.1. Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, the function Φ satisfy condition (3.1) and condition (4.2) be fulfilled. Then there exist the numbers $c_1 > 0$, $c_2 > 0$such that

$$c_1 \cdot f_\varphi^{\#,\Phi}(x) \leq N_\varphi^{\Phi} f(x) \leq c_2 \cdot f_\varphi^{\#,\Phi}(x),\ x \in \mathbb{R}^n, \tag{4.4}$$

where the constants c_1 and c_2 are independent of$f$ and x.

5 Estimations of Φ-maximal functions by maximal functions

Proposition 5.1. If $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$ and the function Φ satisfies condition (3.1) then the following inequalities are true

$$f_{\varphi}^{\#}(x) \le c \cdot f_{\varphi}^{\#,\Phi}(x), \ x \in \mathbb{R}^n,$$

$$N_{\varphi} f(x) \le c \cdot N_{\varphi}^{\Phi} f(x), \ x \in \mathbb{R}^n,$$

where $c = \frac{1}{c_0 \cdot |B(0,1)|}$, and c_0 is a constant from inequality (3.1).

Proof. By means of Proportions 2.1, 2.2 and inequalities (3.4), (3.5) we get

$$f_{\varphi}^{\#}(x) = \sup_{r>0} \frac{m_f(x;\, r)}{\varphi(x,\, r)} \le \frac{1}{c_0 \cdot |B(0,\, 1)|} \cdot \sup_{r>0} \frac{m_f^{\Phi}(x;\, r)}{\varphi(x,\, r)} = \frac{1}{c_0 \cdot |B(0,\, 1)|} \cdot f_{\varphi}^{\#,\Phi}(x), \ x \in \mathbb{R}^n,$$

$$N_{\varphi} f(x) = \sup_{r>0} \frac{n_f(x;\, r)}{\varphi(x,\, r)} \le \frac{1}{c_0 \cdot |B(0,\, 1)|} \cdot \sup_{r>0} \frac{n_f^{\Phi}(x;\, r)}{\varphi(x,\, r)} = \frac{1}{c_0 \cdot |B(0,\, 1)|} \cdot N_{\varphi}^{\Phi} f(x), \ x \in \mathbb{R}^n.$$

Proposition 5.2. [15]. Let $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $\alpha > 0$, $x \in \mathbb{R}^n$, $f \in L_{loc}(\mathbb{R}^n)$. Then the following inequality is true

$$m_f^{\Phi_\alpha}(x;\, r) \le c \cdot r^{\alpha} \int_r^{\infty} \frac{m_f(x;\, t)}{t^{\alpha+1}} dt, \ r > 0, \tag{5.1}$$

where the constant $c > 0$ is independent of f, x and r.

Proposition 5.3. Let $f \in L_{loc}(\mathbb{R}^n)$, $x \in \mathbb{R}^n$, $\alpha > 0$. Then the following inequality is true

$$n_f^{\Phi_\alpha}(x;\, r) \le c \left(\int_0^r \frac{n_f(x;\, t)}{t} dt + r^{\alpha} \int_r^{\infty} \frac{n_f(x;\, t)}{t^{\alpha+1}} dt \right), \ r > 0, \tag{5.2}$$

where the constant $c > 0$ is independent of f, x and r.

Proof. It is easy to verify that the function $\Phi_\alpha(x)$ satisfies condition (3.1). By means of inequalities (3.7), (5.1) and (3.3) we get

$$n_f^{\Phi_\alpha}(x;\, r) \le c \left(m_f^{\Phi_\alpha}(x;\, r) + \int_0^r \frac{m_f^{\Phi_\alpha}(x;\, t)}{t} dt \right)$$

$$\le c_1 \left(r^{\alpha} \int_r^{\infty} \frac{m_f(x;\, t)}{t^{\alpha+1}} dt + \int_0^r \frac{1}{t} \left(t^{\alpha} \int_t^{\infty} \frac{m_f(x;\, u)}{u^{\alpha+1}} du \right) dt \right)$$

$$= c_1 \left(r^{\alpha} \int_r^{\infty} \frac{m_f(x;\, t)}{t^{\alpha+1}} dt + \int_0^r \frac{m_f(x;\, u)}{u^{\alpha+1}} \left(\int_0^u t^{\alpha-1} dt \right) du + \int_r^{\infty} \frac{m_f(x;\, u)}{u^{\alpha+1}} \left(\int_0^r t^{\alpha-1} dt \right) du \right)$$

$$= c_1 \left(r^{\alpha} \int_r^{\infty} \frac{m_f(x;\, t)}{t^{\alpha+1}} dt + \frac{1}{\alpha} \int_0^r \frac{m_f(x;\, u)}{u} du + \frac{1}{\alpha} r^{\alpha} \int_r^{\infty} \frac{m_f(x;\, u)}{u^{\alpha+1}} du \right)$$

$$\le c_2 \cdot \left(\int_0^r \frac{n_f(x;\, t)}{t} dt + r^{\alpha} \int_r^{\infty} \frac{n_f(x;\, t)}{t^{\alpha+1}} dt \right),$$

where the constant $c_2 > 0$ is independent of f, x and r.

Proposition 5.4. [16] Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in \mathbb{R}^n$, $\alpha > 0$ and

$$r^\alpha \int_r^\infty \frac{\varphi(x,\, t)}{t^{\alpha+1}} dt = O(\varphi(x,\, r)),\ r > 0,\ x \in \mathbb{R}^n. \tag{5.3}$$

Then the following inequality is true

$$f_\varphi^{\#,\,\Phi_\alpha}(x) \le c \cdot f_\varphi^{\#}(x),\ x \in \mathbb{R}^n,$$

where the positive constant c is independent of f and x.

Proposition 5.5. Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in \mathbb{R}^n$, $\alpha > 0$, conditions (4.2) and (5.3) be fulfilled. Then it holds the inequality

$$N_\varphi^{\Phi_\alpha} f(x) \le c \cdot N_\varphi f(x),\ x \in \mathbb{R}^n,$$

where the positive constant c is independent of f and x.

Proof. From relation (2.1), (5.2), (5.3) and (4.2) we have

$$\begin{aligned} N_\varphi^{\Phi_\alpha} f(x) = \sup_{r>0} \frac{n_f^{\Phi_\alpha}(x;\, r)}{\varphi(x,\, r)} &\le c \cdot \sup_{r>0} \frac{1}{\varphi(x,\, r)} \left(\int_0^r \frac{n_f(x;\, t)}{t} dt + r^\alpha \int_r^\infty \frac{n_f(x;\, t)}{t^{\alpha+1}} dt \right) \\ &\le c \cdot \sup_{r>0} \frac{1}{\varphi(x;\, r)} \int_0^r \frac{n_f(x;\, t)}{\varphi(x,\, t)} \cdot \frac{\varphi(x,\, t)}{t} dt \\ &+ c \cdot \sup_{r>0} \frac{1}{\varphi(x,\, r)} \cdot r^\alpha \int_r^\infty \frac{n_f(x;\, t)}{\varphi(x,\, t)} \cdot \frac{\varphi(x,\, t)}{t^{\alpha+1}} dt \\ \le c \cdot N_\varphi f(x) \left(\sup_{r>0} \frac{1}{\varphi(x,\, r)} \cdot \int_0^r \frac{\varphi(x,\, t)}{t} dt \right. &+ \left. \sup_{r>0} \frac{1}{\varphi(x,\, r)} \cdot r^\alpha \int_r^\infty \frac{\varphi(x,\, t)}{t^{\alpha+1}} dt \right) \\ &\le c_1 \cdot N_\varphi f(x),\ x \in \mathbb{R}^n, \end{aligned}$$

where the constant $c_1 > 0$ is independent of f and x.

Proposition 5.6. [16] Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in \mathbb{R}^n$, $\alpha > 0$, and condition (5.3) be fulfilled. Then there exist the numbers $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \cdot f_\varphi^{\#}(x) \le f_\varphi^{\#,\,\Phi_\alpha}(x) \le c_2 \cdot f_\varphi^{\#}(x),\ x \in \mathbb{R}^n,$$

where the constants c_1 and c_2 are independent of f and x.

Corollary 5.1. Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in \mathbb{R}^n$, $\alpha > 0$, conditions (4.2) and (5.3) be fulfilled. Then there exist the numbers $c_1 > 0$, $c_2 > 0$ such that

$$c_1 \cdot N_\varphi f(x) \le N_\varphi^{\Phi_\alpha} f(x) \le c_2 \cdot N_\varphi f(x),$$

where the constants c_1 and c_2 are independent of f and x.

From Corollary 4.1, Proposition 5.6 and Corollary 5.1 we get the following statements.

Corollary 5.2. Let $f \in L_{loc}(\mathbb{R}^n)$, $\varphi \in \Psi$, $\Phi_\alpha(x) = \frac{1}{1+|x|^{n+\alpha}}$, $x \in \mathbb{R}^n$, $\alpha > 0$, conditions (4.2) and (5.3) be fulfilled. Then the following relation is true

$$f_\varphi^{\#,\,\Phi_\alpha}(x) \approx N_\varphi^{\Phi_\alpha} f(x) \approx f_\varphi^{\#}(x) \approx N_\varphi f(x)\,^1,\ x \in \mathbb{R}^n,$$

[1] For non-negative functions F and G we will use the notation $F(u) \approx G(u)$, $u \in U$, if there are positive constants c_1 and c_2 such that $\forall u \in U: \quad c_1 \cdot F(u) \le G(u) \le c_2 \cdot F(u)$.

where the constants in the ratio "$\approx$" are independent of f and x.

Now consider the case of the function $\Phi(x) \equiv P(x)$, where $P(x)$ is a Poisson kernel, i.e. $P(x) = c_n \cdot \left(1 + |x|^2\right)^{-\frac{n+1}{2}}$, where $c_n = \Gamma\left(\frac{n+1}{2}\right) \cdot \pi^{-\frac{n+1}{2}}$. It is easy to see that there exist the numbers $c_1 > 0$, $c_2 > 0$ such that for all $x \in \mathbb{R}^n$ it holds the relation

$$c_1 \cdot \frac{1}{1 + |x|^{n+1}} \le P(x) \le c_2 \cdot \frac{1}{1 + |x|^{n+1}}.$$

In other words, $P(x) \approx \Phi_1(x),\ x \in \mathbb{R}^n$, where $\Phi_1(x) = \frac{1}{1+|x|^{n+1}}$. Hence it follows that if $f \in L_{loc}(\mathbb{R}^n)$ and $\varphi \in \Psi$, then the following relations are true

$$f_\varphi^{\#,\, P}(x) \approx f_\varphi^{\#,\, \Phi_1}(x),\, N_\varphi^P f(x) \approx N_\varphi^{\Phi_1} f(x),\ x \in \mathbb{R}^n,$$

$$m_f^P(x;\, r) \approx m_f^{\Phi_1}(x;\, r),\, n_f^P(x;\, r) \approx n_f^{\Phi_1}(x;\, r),\ x \in \mathbb{R}^n,\ r > 0.$$

By means of these reasonings, from Corollary 5.2 we get.

Corollary 5.3. Let $f \in L_{loc}(\mathbb{R}^n)$, $P = P(x)$ be a Poisson kernel, $\varphi \in \Psi$, condition (4.2) be fulfilled, and

$$r \int_r^\infty \frac{\varphi(x,\, t)}{t^2} dt = O(\varphi(x,\, r)),\ r > 0,\ x \in \mathbb{R}^n.$$

Then the following relation is true

$$f_\varphi^{\#,\, P}(x) \approx N_\varphi^P f(x) \approx f_\varphi^{\#}(x) \approx N_\varphi f(x),\ x \in \mathbb{R}^n.$$

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