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$p(x)$-admissible sublinear singular operators in the generalized variable exponent Morrey spaces

Ali Akbulut · **Xayyam A. Badalov** · **Javanshir J. Hasanov** ·
Ayhan Serbetci

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Abstract. *In this paper we prove the boundedness of the $p(x)$-admissible sublinear singular operators on generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\mathbb{R}^n)$ with variable exponent.*

Keywords. Maximal function, singular integral, $p(x)$-admissible sublinear singular operator, generalized variable exponent Morrey space.

Mathematics Subject Classification (2010): Primary 42B20, 42B25, 42B35

1 Introduction

Nowadays there is an evident increase of investigations, last two decades related to both the theory of variable exponent function spaces and operator theory in these spaces. We refer for instance to the surveying papers [2], [4], [19], [22], on the progress in this field, including topics of Harmonic Analysis and Operator Theory, see also references therein. Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$, were introduced and studied in [1] in the Euclidean setting. In [1] the boundedness of the maximal operator was proved in variable

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Ali Akbulut
E-mail: aakbulut@ahievran.edu.tr
Department of Mathematics, Ahi Evran University, Kirsehir, Turkey

Xayyam A. Badalov
E-mail: xeyyam@yahoo.com
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan

Javanshir J. Hasanov
E-mail: hasanovjavanshir@yahoo.com.tr
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
Azerbaijan State Oil and Industry University, Baku, Azerbaijan

Ayhan Serbetci
E-mail: serbetci@ankara.edu.tr
Department of Mathematics, Ankara University, Ankara, Turkey

exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ under the log-condition on $p(\cdot)$ and $\lambda(\cdot)$, and for potential operators a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$- theorem was proved under the same log-condition in the case of bounded sets. Hästö in [18] used his new "local-to-global" approach to extend the result of [1] on the maximal operator to the case of the whole space $\mathbb{R}^n$.

The generalized variable exponent Morrey spaces were introduced and studied in [12] in the case of bounded sets. In [12] the boundedness of the maximal operator, potential operators and singular integral operators in variable exponent Morrey spaces under the certain conditions were proved.

The paper is organized as follows. In Section 2 we provide necessary preliminaries on the topics. In Section 3 we give the main results in Theorems 3.1 and 3.2. We find the condition on the function $\omega(x,r)$ for the boundedness of the $p(x)$-admissible sublinear singular operators T in generalized Morrey space $\mathcal{M}^{p(\cdot),\omega}(\mathbb{R}^n)$ with variable exponent under the log-condition on $p(\cdot)$.

2 Preliminaries

We recall the definition of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$: Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with values in $[1,\infty)$. We mainly suppose that

$$1 < p_- \le p(x) \le p_+ < \infty,$$

where $p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x) > 1,\ \ p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$. We denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the space of all measurable functions $f(x)$ on $\mathbb{R}^n$ such that

$$I_{p(\cdot)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\eta > 0 :\ I_{p(\cdot)}\left(\frac{f}{\eta}\right) \le 1\right\},$$

$L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach function space. For the basics on variable exponent Lebesgue spaces we refer to [19], [26].

The generalized variable exponent Morrey spaces is defined as follows.

Definition 2.1 *Let $\omega(x,r)$ be a non-negative measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \le p < \infty$. The generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\mathbb{R}^n)$ is defined by the norm*

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\omega(x,r)} \|f\|_{L^{p(\cdot)}(B(x,r))}.$$

According to this definition, we recover the space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ under the choice $\omega(x,r) = r^{\frac{\lambda(x)}{p(x)}}$:

$$\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) = \mathcal{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)\Bigg|_{\omega(x,r)=r^{\frac{\lambda(x)}{p(x)}}}.$$

Everywhere in the sequel we assume that

$$\inf_{x\in\mathbb{R}^n, r>0} \omega(x,r) > 0$$

which makes the space $\mathcal{M}^{p(\cdot),\omega}(\mathbb{R}^n)$ nontrivial. Note that when p is constant, in the case of $\omega(x,r) = r^{\frac{n}{p}}$, we have the space $L^\infty(\mathbb{R}^n)$.

Within the frameworks of the spaces $\mathcal{M}^{p(\cdot),\omega}(\mathbb{R}^n)$, we consider the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy$$

and the Calderon-Zygmund singular operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \leq C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \leq C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \leq C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \ \ \sigma > 0, \ \text{ if } |x-y| > 2|x-\xi|.$$

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, that is,

$$\Omega(tx) = \Omega(x), \ \text{ for any } t > 0, \ x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, that is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \leq 1$, that is, there exists a constant $C > 0$ such that,

$$|\Omega(x') - \Omega(y')| \leq C|x'-y'|^\gamma \text{ for any } x', y' \in S^{n-1}.$$

In 1958, Stein [24] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3}\right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y)dy.$$

The continuity of Marcinkiewicz operator μ_Ω has been extensively studied in [20], [25] and [27]. Recall that if T is a sublinear operator, then $|T(f+g)| \leq |Tf| + |Tg|$.

Definition 2.2 *($p(x)$-admissible singular operator). A sublinear operator T will be called $p(x)$-admissible singular operator, if:*

1) *T satisfies the size condition of the form*

$$\chi_{B(x,r)}(z)\left|T\Big(f\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right| \le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|f(y)|}{|y-z|^n}dy \tag{2.1}$$

for $x\in\mathbb{R}^n$ and $r>0$;

2) *T is bounded in $L_{p(\cdot)}(\mathbb{R}^n)$.*

Let $\mathcal{P}(\mathbb{R}^n)$ be the set of bounded measurable functions $p:\mathbb{R}^n\to[1,\infty)$ and $\mathcal{P}^{log}(\mathbb{R}^n)$ be the set of exponents $p\in\mathcal{P}(\mathbb{R}^n)$ satisfying the local log-condition

$$|p(x)-p(y)|\le\frac{A}{-\ln|x-y|},\ \ |x-y|\le\frac{1}{2}\ \ x,y\in\mathbb{R}^n, \tag{2.2}$$

$$|p(x)-p(\infty)|\le\frac{A_\infty}{\ln(2+|x|)},\ \ x\in\mathbb{R}^n, \tag{2.3}$$

where $A=A(p)>0$ does not depend on x,y and $p_\infty=\lim\limits_{x\to\infty}p(x)>1$.

Further, let $\mathbb{P}^{log}_\infty(\mathbb{R}^n)$ be the set of exponents $p\in\mathcal{P}^{log}(\mathbb{R}^n)$ with $1<p_-\le p_+<\infty$ satisfying the conditions (2.2) and (2.3).

We will also make use of the estimate provided by the following lemma (see [5], Corollary 4.5.9).

$$\|\chi_{B(x,r)}(\cdot)\|_{p(\cdot)}\le Cr^{\theta_p(x,r)},\quad x\in\mathbb{R}^n,\ p\in\mathbb{P}^{log}_\infty(\mathbb{R}^n), \tag{2.4}$$

where $\theta_p(x,r)=\begin{cases}\frac{n}{p(x)}, & \text{if } r\le1,\\ \frac{n}{p(\infty)}, & \text{if } r\ge1.\end{cases}$

Theorem A. ([3], [4]) *Let $p\in\mathbb{P}^{log}_\infty(\mathbb{R}^n)$. Then the maximal operator M is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.*

Theorem B. ([6]) *Let $p\in\mathbb{P}^{log}_\infty(\mathbb{R}^n)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.*

We will use the following statement on the boundedness of the weighted Hardy operator

$$H^*_w g(t):=\int_t^\infty g(s)w(s)ds,\quad 0<t<\infty,$$

where w is a weight.

Theorem C. ([9]) *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0}v_2(t)H^*_w g(t)\le C\sup_{t>0}v_1(t)g(t) \tag{2.5}$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B:=\sup_{t>0}v_2(t)\int_t^\infty\frac{w(s)ds}{\sup_{s<\tau<\infty}v_1(\tau)}<\infty.$$

Moreover, the value $C=B$ is the best constant for (2.5).

3 $p(x)$-admissible sublinear singular operators in the spaces $\mathcal{M}^{p(\cdot),\omega(\cdot)}(\mathbb{R}^n)$

Everywhere in the sequel the functions $\omega(x,r),\ \omega_1(x,r)$ and $\omega_2(x,r)$ used in the body of the paper, are non-negative measurable function on $\mathbb{R}^n \times (0,\infty)$.

Theorem 3.1 *Let $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(B(x,r))$ for every $r \in (0,\infty)$. Then for the $p(x)$-admissible singular integral operator T the following inequality is valid*

$$\|Tf\|_{L^{p(\cdot)}(B(x,t))} \leq Ct^{\theta_p(x,t)} \int_t^{\infty} r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr, \tag{3.1}$$

where C does not depend on f and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\mathbb{R}^n \setminus B(x,t)}(y) \quad f_2(y) = f(y)\chi_{B(x,t)}(y)$$

and have

$$\|Tf\|_{L^{p(\cdot)}(B(x,t))} \leq \|Tf_1\|_{L^{p(\cdot)}(B(x,t))} + \|Tf_2\|_{L^{p(\cdot)}(B(x,t))}.$$

Since $f_1 \in L^{p(\cdot)}(\mathbb{R}^n)$, $Tf_1 \in L^{p(\cdot)}(\mathbb{R}^n)$ and from the boundedness of T from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p(\cdot)}(\mathbb{R}^n)$ it follows that:

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leq \|Tf_1\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C\|f_1\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

so that

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leq C\|f\|_{L^{p(\cdot)}(B(x,2t))}.$$

Taking into account the inequality

$$\|f\|_{L^{p(\cdot)}(B(x,t))} \leq Ct^{\theta_p(x,t)} \int_{2t}^{\infty} r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr,$$

we get

$$\|Tf_1\|_{L^{p(\cdot)}(B(x,t))} \leq Ct^{\theta_p(x,t)} \int_{2t}^{\infty} r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr. \tag{3.2}$$

To estimate $\|Tf_2\|_{L^{p(\cdot)}(B(x,t))}$, by the equation (2.1) we have

$$|Tf_2(z)| \leq C \int\limits_{\mathbb{R}^n \setminus B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \leq t$, $|z-y| \geq 2t$ imply $\frac{1}{2}|z-y| \leq |x-y| \leq \frac{3}{2}|z-y|$, and therefore

$$\|Tf_2\|_{L^{p(\cdot)}(B(x,t))} \leq C \int_{\Omega \setminus B(x,2t)} |x-y|^{-n}|f(y)|dy \|\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

Hence by estimate (2.4), we get

$$\|Tf_2\|_{L^{p(\cdot)}(B(x,t))} \leq Ct^{\theta_p(x,t)} \int_{2t}^{\infty} r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr. \tag{3.3}$$

From (3.2) and (3.3) we arrive at (3.1).

Corollary 3.1 *[14] Let $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ and for every $r \in (0,\infty)$, $f \in L^{p(\cdot)}(B(x,r))$. Then for the maximal operator M, singular integral operator T and Marcinkiewicz integral operator μ_Ω the inequality* (3.1) *are valid.*

Theorem 3.2 *Let $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ and $\omega_1(x,t)$ and $\omega_2(x,t)$ fulfill condition*

$$\int_r^{\infty} t^{-\theta_p(x,t)} \operatorname*{ess\,inf}_{t<s<\infty} \omega_1(x,s) \frac{dt}{t} \le c_1 \, r^{-\theta_p(x,r)} \omega_2(x,r). \tag{3.4}$$

Then a $p(x)$-admissible singular integral operator T is bounded from the space $\mathcal{M}^{p(\cdot),\omega_1}(\mathbb{R}^n)$ to the space $\mathcal{M}^{p(\cdot),\omega_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}^{p(\cdot),\omega_1}(\mathbb{R}^n)$ we have

$$\|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2}(\mathbb{R}^n)} = \sup_{t>0} \frac{t^{-\theta_p(x,t)}}{\omega_2(x,t)} \|Tf\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

we estimate $\|Tf\chi_{B(x,t)}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ by means of Theorems C and 3.1, we obtain

$$\begin{aligned} \|Tf\|_{\mathcal{M}^{p(\cdot),\omega_2}(\mathbb{R}^n)} &\le C \sup_{t>0} \frac{1}{\omega_2(x,t)} \int_t^{\infty} r^{-\theta_p(x,r)-1} \|f\|_{L^{p(\cdot)}(B(x,r))} dr \\ &\le C \sup_{t>0} \frac{t^{-\theta_p(x,t)}}{\omega_1(x,t)} \|f\|_{L^{p(\cdot)}(B(x,t))} \\ &= C\|f\|_{\mathcal{M}^{p(\cdot),\omega_1}(\mathbb{R}^n)}. \end{aligned}$$

Corollary 3.2 *[14] Let $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ and $\omega_1(x,t)$ and $\omega_2(x,t)$ fulfill condition*

$$\int_r^{\infty} t^{-\theta_p(x,t)} \operatorname*{ess\,inf}_{t<s<\infty} \omega_1(x,s) \frac{dt}{t} \le c_1 \, r^{-\theta_p(x,r)} \omega_2(x,r).$$

Then the maximal operator M, singular integral operator T and Marcinkiewicz integral operator μ_Ω are bounded from the space $\mathcal{M}^{p(\cdot),\omega_1}(\mathbb{R}^n)$ to the space $\mathcal{M}^{p(\cdot),\omega_2}(\mathbb{R}^n)$.

The following Corollary is new.

Corollary 3.3 *Let $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$ and $0 \le \lambda_- \le \lambda_+ < n$. Then the $p(x)$-admissible sublinear singular operator T is bounded from $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ to $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$.*

Note that the case of the constant exponent p in Theorems 3.1 and 3.2 were proved in [10] (see also [11]).

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