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On uniqueness of solution to n-th order ordinary linear differential equation

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Abstract. *The paper is devoted to investigation of solutions of boundary value problems for the n-th order ordinary linear differential equations when a boundary condition is not necessary for the uniqueness of the solution.*

Keywords. Uniqueeness, ordinary differential equations, fundamental solutions, boundary value problems.

Mathematics Subject Classification (2010): Primary 34A12, 34B05, 35A08.

1 Introduction

As is known, for an ordinary linear differential equation, the amount of boundary conditions coincides with the order of differential equation under consideration [3,9].

H.Lewy [7] was the first who constructed differential equations not having even a local solution.

Note that, the necessary conditions for the solution of the equation is obtained by means of the fundamental solutions of the conjugated equation.

If this necessary conditions on one of the boundary conditions of differential equation under consideration don't satisfy, then the equation hasn't any solution even at one boundary condition.

If the necessary condition is given instead of the boundary condition then this condition doesn't put any limatated on the solution of the equation.

The similar way the solution of boundary value problems for the partial differential equations is investigated [1,4,6].

In this paper, we give equations that don't need additional restrictions for uniqueness of a bounded solution. Note that for the second order ordinary linear differential equations the

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analogous question is studied in [2].The similar problem for the first order partial differential equations was considered in [5].

2 Main result

Consider the following ordinary linear differential equation of n-th order

$$Ly \equiv (-1)^n \left[p(x)y(x)\right]^{(n)} - p(x)y(x) = f(x), \;\; x \in (a,b), \tag{2.1}$$

where

$$p(x) = (b-x)^n (x-a)^n p_0(x), \tag{2.2}$$

f is real-valued continuous function, $y, p_0 \in C^{(n)}[a,b]$.

Let H be a space of real -valued functions that was determined on $[a,b]$ with the following scalar product:

$$(u,v) = \int\limits_a^b u(x)v(x)dx, \;\; u(x) \in H, \;\; v(x) \in H.$$

Consider the following scalar product

$$(Ly,z) = \int\limits_a^b (-1)^n \left[p(x)y(x)\right]^{(n)} z(x)dx - \int\limits_a^b p(x)y(x)z(x)dx. \tag{2.3}$$

Integrate by parts the first summand in the right side of equality (2.3) n times we obtain

$$\int\limits_a^b \left[p(x)y(x)\right]^{(n)} z(x)dx = [p(x)y(x)]^{(n-1)} z(x)\Big|_{x=a}^{x=b} - \int_a^b [p(x)y(x)]^{(n-1)} z'(x)dx$$

$$= \left\{ [p(x)y(x)]^{(n-1)} z(x) - [p(x)y(x)]^{(n-2)} z'(x) \right\} \Big|_{x=a}^{x=b}$$

$$+ \int_a^b [p(x)y(x)]^{(n-2)} z'' z(x)dx$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$= \left\{ \sum_{k=0}^{n-1} (-1)^k [p(x)y(x)]^{(n-1-k)} z^{(k)}(x) \right\} \Big|_{x=a}^{x=b} + (-1)^n \int_a^b p(x)y(x)z^{(n)}(x)dx. \tag{2.4}$$

Allowing for (2.2) in (2.4) we have

$$\int_a^b [p(x)y(x)]^{(n)} z(x)dx = (-1)^n \int_a^b p(x)y(x)z^{(n)}(x)dx \tag{2.5}$$

Taking into account (2.5) in the equality (2.3), we have the following Lagrange formula

$$(Ly,z) = \int_a^b p(x)y(x)[z^{(n)}(x) - z(x)]dx, \tag{2.6}$$

where from we get the conjugated differential expression

$$L^*z \equiv p(x)\left[z^{(n)}(x) - z(x)\right].$$

For the fundamental solution of the conjugated equation

$$p(x)\left[z^{(n)}(x) - z(x)\right] = g(x),$$

where $g(x)$ is an arbitrary continuous function, by means of the variation of parameters method we get [8]:

$$Z(x,\xi) = \frac{W(x-\xi)}{p(\xi)}, \tag{2.7}$$

where

$$W(x-\xi) = \sum_{k=1}^{n} \frac{\Delta^{(n,k)}}{\Delta}\theta(x-\xi)e^{\varepsilon^k(x-\xi)} \tag{2.8}$$

is fundamental solution of the equation

$$W^{(n)}(x) - W(x) = g(x).$$

Here

$$\theta(x-\xi) = \begin{cases} \frac{1}{2}, \text{if } x > \xi \\ 0, \text{if } x = \xi \\ -\frac{1}{2}, \text{if } x < \xi \end{cases}$$

is a Heaviside function, ε^k $\left(\varepsilon = e^{\frac{2\pi i}{n}}\right)$, $k = \overline{1,n}$ are roots of the equation

$$\varepsilon^n - 1 = 0, \tag{2.9}$$

and

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \varepsilon & \varepsilon^2 & \varepsilon^3 & \dots & \varepsilon^n \\ \dots & \dots & \dots & \dots & \dots \\ \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \varepsilon^{3(n-1)} & \dots & \varepsilon^{n(n-1)} \end{vmatrix}, \tag{2.10}$$

is a Wronskian determinant of the solutions ε^k, $k = \overline{1,n}$ of equation (2.9), $\Delta^{(n,k)}$ is the algebraic adjunct of the element situated in n-th row and k-th column of the determinant (2.10).

Now, we come back to Lagrange formula (2.6), substitute the fundamental solution (2.8) for $z(x)$ and get:

$$(Ly, Z) \equiv \int_a^b [p(x)y(x)]^{(n)} Z(x,\xi)dx - \int_a^b p(x)y(x)Z(x,\xi)dx$$

$$= \int_a^b y(x)\delta(x-\xi)dx. \tag{2.11}$$

Using the property of Dirac's delta function, from (2.11) we get:

$$y(\xi) = \int_a^b \frac{f(x)}{p(\xi)} W(x-\xi)dx, \quad \xi \in (a,b). \tag{2.12}$$

Thus, we established the following statement:

Theorem 2.1. *Let* $p \in C^n[a,b]$ *and satisfies relation (2.2)*, and f *be continuous real-valued functions. Then equation (2.1) has a unique solution representable in the form (2.12), where* $W(x-\xi)$ *has the form (2.8).*

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