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Maximal commutator and commutator of maximal function on modified Morrey spaces

Canay Aykol · **Hatice Armutcu** · **Mehriban N. Omarova**

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Abstract. *We study the boundedness of the maximal commutator operator M_b and commutator of maximal operator $[M, b]$ in the modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.*

Keywords. Maximal commutator · commutator of maximal function · modified Morrey space · BMO space.

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1 Introduction

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration for the development of the general class of singular and potential operators (see, for instance [7], [16]).

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$. The Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x,t)| = v_n t^n$ is the Lebesgue measure of the ball $B(x,t)$ and v_n is the volume of the unit ball in $\mathbb{R}^n$.

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Canay Aykol
Ankara University, Department of Mathematics, Ankara, Turkey
E-mail: Canay.Aykol@science.ankara.edu.tr

Hatice Armutcu
Yildiz Technical University, Istanbul, Turkey
E-mail: haticexarmutcu@gmail.com

Mehriban N. Omarova
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: mehriban_omarova@yahoo.com

Let $f \in L_1^{loc}(\mathbb{R}^n)$. Then f is said to be in $BMO(\mathbb{R}^n)$ if the seminorm given by

$$\|f\|_* := \sup_B \frac{1}{|B|} \int_B |f(y) - f_B| dy$$

is finite.

Definition 1 *Given a measurable function b the maximal commutator is defined by*

$$M_b(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)| dy,$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance [5], [12], [15]). The maximal operator M_b has been studied intensively and there exist plenty of results about it. Garcia-Cuerva et al. [5] proved the following statement.

Theorem 1 *Let $1 < p < \infty$. The operator M_b is bounded on $L_p(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.*

Definition 2 *Given a measurable function b the commutator of the Hardy-Littlewood maximal operator M and b is defined by*

$$[M, b]f(x) := M(bf)(x) - b(x)Mf(x)$$

for all $x \in \mathbb{R}^n$.

The operator $[M, b]$ was studied by Milman et al. in [13] and [2]. This operator arises, for example, when one tries to give a meaning to the product of a function in H^1 and a function in BMO (which may not be a locally integrable function, see, for instance, [3]). Using real interpolation techniques, in [13], Milman and Schonbek proved the L_p-boundedness of the operator $[M, b]$. Bastero, Milman and Ruiz [2] proved the next theorem.

Theorem 2 *Let $1 < p < \infty$. Then the following assertions are equivalent:*
(i) *$[M, b]$ is bounded on $L_p(\mathbb{R}^n)$.*
(ii) *$b \in BMO(\mathbb{R}^n)$ and $b^- \in L_\infty(\mathbb{R}^n)$.*[1]
The operators M_b and $[M, b]$ enjoy weak-type $L(1 + \log^+ L)$ estimate.

Operators M_b and $[M, b]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[M, b]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[M, b]$.

Lemma 1 *Let b be any non-negative locally integrable function. Then*

$$|[M, b]f(x)| \le M_b(f)(x), \;\; x \in \mathbb{R}^n \tag{1}$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$.

If b is any locally integrable function on $\mathbb{R}^n$, then

$$|[M, b]f(x)| \le M_b(f)(x) + 2b^-(x)Mf(x), \;\; x \in \mathbb{R}^n \tag{2}$$

holds for all $f \in L_1^{loc}(\mathbb{R}^n)$.

[1] Denote by $b^+(x) = \max\{b(x), 0\}$ and $b^-(x) = -\min\{b(x), 0\}$, consequently $b = b^+ - b^-$ and $|b| = b^+ + b^-$.

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C.B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [14]).

Definition 3 *Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1, t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ Morrey space, and by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}$$

respectively.

The modified Morrey space $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ firstly was defined and investigated by [10] (see also [8],[9]).

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset\!\!\!\succ L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}} \tag{3}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Lemma 2 *[9] Let $1 \le p < \infty$, $0 \le \lambda \le n$. Then*

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) = L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \max\left\{ \|f\|_{L_{p,\lambda}}, \|f\|_{L_p} \right\}.$$

2 $\widetilde{L}_{p,\lambda}$-boundedness of the maximal commutator operator M_b

In this section we study the $\widetilde{L}_{p,\lambda}$-boundedness of the maximal commutator operator M_b.

Theorem 3 *[6] Let $1 < p < \infty$, $0 \le \lambda \le n$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) *The operator M_b is bounded on $L_{p,\lambda}(\mathbb{R}^n)$.*

Applying Theorem 3, we obtain the following result.

Theorem 4 *Let $1 < p < \infty$, $0 \le \lambda \le n$. The following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$.
(ii) *The operator M_b is bounded on $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. It is obvious that (see Lemma 2)

$$\|M_b f\|_{\widetilde{L}_{p,\lambda}} = \max\left\{\|M_b f\|_{L_{p,\lambda}}, \|M_b f\|_{L_p}\right\}$$

for $1 < p < \infty$.

By the boundedness of M_b on $L_p(\mathbb{R}^n)$, $1 < p < \infty$ (see Theorem 1) and from Theorem 3 we get

$$\|M_b f\|_{\widetilde{L}_{p,\lambda}} \le \max\{C_p, C_{p,\lambda}\} \|f\|_{\widetilde{L}_{p,\lambda}}.$$

$(ii) \Rightarrow (i)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\begin{aligned}\|\chi_B\|_{\widetilde{L}_{p,\lambda}} &\approx \sup_{y\in\mathbb{R}^n, t>0} \left([t]_1^{\lambda-n} \int_{B(y,t)} \chi_B(z)dz\right)^{\frac{1}{p}} = \sup_{y\in\mathbb{R}^n, t>0} \left(|B(y,t)\cap B| [t]_1^{\lambda-n}\right)^{\frac{1}{p}} \\ &= \sup_{B(y,t)\subseteq B} \left(|B(y,t)| [t]_1^{\lambda-n}\right)^{\frac{1}{p}} = r^{\frac{n}{p}} \, [r]_1^{\frac{\lambda-n}{p}}. \end{aligned} \tag{4}$$

On the other hand, since

$$M_b(\chi_B)(x) \gtrsim \frac{1}{|B|}\int_B |b(z) - b_B|dz \quad \text{for all} \quad x \in B,$$

then

$$\begin{aligned}\|M_b(\chi_B)\|_{\widetilde{L}_{p,\lambda}} &\approx \sup_{B(y,t)} \left([t]_1^{\lambda-n} \int_{B(y,t)} |M_b(\chi_B)(z)|^p dz\right)^{\frac{1}{p}} \\ &\gtrsim r^{\frac{n}{p}} \, [r]_1^{\frac{\lambda-n}{p}} \frac{1}{|B|}\int_B |b(z) - b_B|dz. \end{aligned} \tag{5}$$

Since by assumption

$$\|M_b(\chi_B)\|_{\widetilde{L}_{p,\lambda}} \approx \|\chi_B\|_{\widetilde{L}_{p,\lambda}},$$

by (4) and (5), we get that

$$\frac{1}{|B|}\int_B |b(z) - b_B|dz \lesssim c.$$

3 $\widetilde{L}_{p,\lambda}$-boundedness of the commutator of maximal operator $[M, b]$

In this section we study the $\widetilde{L}_{p,\lambda}$-boundedness of the commutator of maximal operator $[M, b]$.

Theorem 5 *[6] Let $1 < p < \infty$, $0 \le \lambda \le n$. Suppose that b be a real valued, locally integrable function in $\mathbb{R}^n$.*

The following assertions are equivalent:

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty$.

(ii) The operator $[M, b]$ is bounded on $L_{p,\lambda}(\mathbb{R}^n)$.

Applying Theorem 3, we obtain the following result.

Theorem 6 *Let $1 < p < \infty$, $0 \leq \lambda \leq n$. Suppose that b be a real valued, locally integrable function in $\mathbb{R}^n$.*

The following assertions are equivalent:

(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L_\infty$.

(ii) The operator $[M, b]$ is bounded on $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. It is obvious that (see Lemma 2)

$$\|[M,b]f\|_{\widetilde{L}_{p,\lambda}} = \max\left\{\|[M,b]f\|_{L_{p,\lambda}}, \|[M,b]f\|_{L_p}\right\},$$

for $1 < p < \infty$.

By the boundedness of $[M, b]$ on $L_p(\mathbb{R}^n)$, $1 < p < \infty$ (see Theorem 2) and from Theorem 5 we get

$$\|[M,b]f\|_{\widetilde{L}_{p,\lambda}} \leq \max\{C_p, C_{p,\lambda}\} \|f\|_{\widetilde{L}_{p,\lambda}}.$$

$(ii) \Rightarrow (i)$. Assume that $[M, b]$ is bounded on $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$. Let $B = B(x, r)$ be a fixed ball.

Denote by $M_B f$ the local maximal function of f:

$$M_B f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)|\, dy \quad (x \in \mathbb{R}^n).$$

Since

$$M(b\chi_B)\chi_B = M_B(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

then

$$\begin{aligned} |M_B(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\ &\leq |M(b\chi_B) - bM(\chi_B)| = |[M,b]\chi_B|. \end{aligned}$$

Hence

$$\|M_B(b) - b\chi_B\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n)} \leq \|[M,b]\chi_B\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n)}.$$

Thus

$$\begin{aligned} \frac{1}{|B|}\int_B |b - M_B(b)| &\leq \left(\frac{1}{|B|}\int_B |b - M_B(b)|^p\right)^{\frac{1}{p}} \\ &\leq |B|^{-\frac{1}{p}} [r]_1^{\frac{n-\lambda}{p}} \|b\chi_B - M_B(b)\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n)} \\ &\leq |B|^{-\frac{1}{p}} [r]_1^{\frac{n-\lambda}{p}} \|[M,b]\chi_B\|_{\widetilde{L}_{p,\lambda}(\mathbb{R}^n)} \\ &\leq c|B|^{-\frac{1}{p}} [r]_1^{\frac{n-\lambda}{p}} \|\chi_B\|_{\widetilde{L}_{p,\lambda}} \\ &\approx c|B|^{-\frac{1}{p}} [r]_1^{\frac{n-\lambda}{p}} |B|^{\frac{1}{p}} [r]_1^{\frac{\lambda-n}{p}} = c. \end{aligned}$$

Denote by

$$E := \{x \in B : b(x) \leq b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, dt = \int_F |b(t) - b_B|\, dt,$$

in view of the inequality $b(x) \le b_B \le M_B(b)$, $x \in E$, we get that

$$\begin{aligned}\frac{1}{|B|}\int_B |b - b_B| &= \frac{2}{|B|}\int_E |b - b_B| \\ &\le \frac{2}{|B|}\int_E |b - M_B(b)| \\ &\le \frac{2}{|B|}\int_B |b - M_B(b)| \lesssim c.\end{aligned}$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L^\infty(\mathbb{R}^n)$, note that $M_B(b) \ge |b|$. Hence

$$0 \le b^- = |b| - b^+ \le M_B(b) - b^+ + b^- = M_B(b) - b.$$

Thus

$$(b^-)_B \le c,$$

and by the Lebesgue Differentiation Theorem we get that

$$b^-(x) \le c \quad \text{for a.e. } x \in \mathbb{R}^n.$$

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