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Convergence of biorthogonal expansion of a vector-function from the class $W_{2,m}^{1}(G)$ in eigen and associated vector-functions of fourth order differential operator with matrix coefficients.

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Abstract. *In the paper an ordinary differential operator of fourth order with summable coefficients is considered. Absolute and uniform convergence of biorthogonal expansion of a vector-function from the class $W_{2,m}^{1}(G)$, $G=(0,1)$ in eigen and associated vector-functions of this operator is studied, and the rate of uniform convergence of this expansion in $\overline{G}$ is estimated.*

Keywords. absolute convergence, uniform convergence, eigen and associated functions biorthogonal expansion

Mathematics Subject Classification (2010): 34L10, 42A20

1 Introduction and formulation of results

Consider on the interval $G=(0,1)$ a formal differential operator

$$L\psi=\psi^{(4)}+U_2(x)\,\psi^{(2)}+U_3(x)\,\psi^{(1)}+U_4(x)\,\psi$$

with complex-valued matrix coefficients $U_l(x)\equiv(u_{lij}(x))_{i,j=1}^{m}$, $l=\overline{2,4}$; $u_{lij}(x)\in L_1(G)$.

Denote by $D(G)$ a class of m-component vector-functions absolutely continuous together with their derivatives to third order, inclusively on a closed interval $\overline{G}=[0,1]$ $\left(D(G)\equiv W_{1,m}^{4}(G)\right)$.

Following [1] under the eigen vector-function of the operator L responding to the complex eigen value λ, we will understand any complex-valued vector-function $\overset{0}{\psi}(x)\in D(G)$ not identically equal to zero and satisfying almost everywhere in G the equation $L\overset{0}{\psi}+\lambda\overset{0}{\psi}=0$. In the similar way, under an associated vector-function of order r $(r\geq 1)$ of the operator L, responding to the same eigen value λ and eigen-vector-function $\overset{0}{\psi}(x)$ we will understand

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any complex-valued vector-function $\overset{r}{\psi}(x) \in D(G)$ satisfying almost everywhere on G the equation $L\overset{r}{\psi} + \lambda\overset{r}{\psi} = \overset{r-1}{\psi}$.

We will consider each eigen vector-function an associated vector-function of zero order. The highest order of the root (associated) vector-functions responding to the given eigen vector-function will be said to be a rang of this eigen vector function.

Consider an arbitrary system $\{\psi_k(x)\}_{k=1}^{\infty}$ consisting of eigen and associated vector-functions of the operator L, responding to the system of eigen values $\{\lambda_k\}_{k=1}^{\infty}$ and require that together with each root vector-function of order $r \geq 1$ this system include appropriate root vector-functions of order less than r and the rang of eigen vector-functions be uniformly bounded. This means that $\psi_k(x) \in D(G)$ and satisfies almost everywhere in G the equation $L\psi_k + \lambda_k\psi_k = \theta_k\psi_{k-1}$, where θ_k is equal either to 0 (in this case $\psi_k(x)$ is an eigen vector-function) or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $\psi_k(x)$ an associated vector-function).

Denote $\mu_k = \sqrt[3]{-\lambda_k}$, where $\left(\rho e^{i\varphi}\right)^{1/4} = \rho^{\frac{1}{4}} e^{i\varphi/4}$, $-\dfrac{\pi}{2} < \varphi \leq \dfrac{3\pi}{2}$. Obviously, $Re\mu_k \geq 0$.

Let $U_i(x) \in W_1^{4-l}(G)$, $l = \overline{2,4}$, i.e. $u_{lij}(x) \in W_1^{4-l}(G)$. Therewith $W_1^0(G) \equiv L_1(G)$. Denote by $L_p^m(G)$, $1 \leq p \leq \infty$ a space of m-component vector-functions $f(x) = (f_1(x), f_2(x), ..., f_m(x))^T$ with the norm

$$\|f\|_{p,m} = \left\{\int\limits_G |f(x)|^p\right\}^{\frac{1}{p}} = \left\{\int\limits_G \left(\sum_{i=1}^m |f_i(x)|^2\right)^{\frac{p}{2}} dx\right\}^{\frac{1}{p}},$$

moreover for $p = \infty$

$$\|f\|_{\infty,m} = \nu rai\sup_{x\in\overline{G}} |f(x)|.$$

We require the system $\{\psi_k(x)\}_{k=1}^{\infty}$ to satisfy the conditions A:

1) the system $\{\psi_k(x)\}_{k=1}^{\infty}$ is complete and minimal in $L_2^m(G)$;

2) the Carleman and "sum of units" conditions are fulfilled:

$$|Im\mu_k| \leq const, \quad k = 1, 2, ..., \tag{1.1}$$

$$\sum_{\tau\leq\rho_k\leq\tau+1} 1 \leq const, \quad \forall\tau \geq 0, \ \ \rho_k = Re\mu_k; \tag{1.2}$$

3) the system $\{\varphi_k(x)\}_{k=1}^{\infty}$ is biorthogonally conjugated to $\{\psi_k(x)\}_{k=1}^{\infty}$ is the system of root vector-functions of the formally conjugated operator

$$L^*y = y^{(4)} + (U_2^*y)^{(2)} - (U_3^*y)^{(1)} + U_4^*y,$$

i.e. $L^*\varphi_k + \overline{\lambda}_k\varphi_k = \theta_{k+1}\varphi_{k+1}$;

4) the following antiapriori estimations are fulfilled:

$$\theta_k \|\psi_{k-1}\|_{2,m} \leq const\,(1 + |\mu_k|)^3 \|\psi_k\|_{2,m}; \tag{1.3}$$

$$\theta_{k+1} \|\varphi_{k+1}\|_{2,m} \leq const\,(1 + |\mu_k|)^3 \|\varphi_k\|_{2,m}; \tag{1.4}$$

5) there exists a constant C_0 such that

$$\|\psi_k\|_{2,m} \|\varphi_k\|_{2,m} \le C_0, \quad k = 1, 2, ... \tag{1.5}$$

6) for any $\tau \ge 0$ the following estimations are fulfilled:

$$\sum_{0 \le \rho_k \le \tau} \|\psi_k\|^2_{\infty,m} \|\psi_k\|^{-2}_{2,m} \le const\,(1+\tau)\,, \tag{1.6}$$

$$\sum_{0 \le \rho_k \le \tau} \|\varphi_k\|^2_{\infty,m} \|\varphi_k\|^{-2}_{2,m} \le const\,(1+\tau)\,. \tag{1.7}$$

For an arbitrary vector-function $f(x) \in W^1_{2,m}(G)$ we introduce the partial sum

$$\sigma_\nu(x, f) = \sum_{\rho_k \le \nu} f_k \psi_k(x)\,, \nu > 0$$

where

$$f_k = (f, \varphi_k) = \int_0^1 < f(x)\,, \varphi_k(x) > dx = \int_0^1 \sum_{j=1}^m f_j(x)\,\overline{\varphi_{kj}(x)}dx,$$

$$\varphi_k(x) = (\varphi_{k1}(x)\,, \varphi_{k2}(x)\,, ..., \varphi_{km}(x))^T\,.$$

In the paper we prove the following theorem an absolute and uniform convergence of biorthogonal expansion.

Theorem 1.1 *Let the conditions A be fulfilled, the vector-function $f(x) \in W^1_{2,m}(G)$ and the biorthogonal system $\{\varphi_k(x)\}_{k=1}^\infty$ satisfy the condition*

$$\left| < f(x)\,, \varphi^{(3)}_k(x) > \Big|_0^1 \right| \le C_1(f)\,|\mu_k|^\alpha \|\varphi_k\|_{\infty,m}\,, \quad 0 \le \alpha < 3, \tag{1.8}$$

for $\rho_k \ge 1$.

Then biorthogonal expansion of the vector-function $f(x)$ converges absolutely and uniformly on the segment $\overline{G} = [0, 1]$ and the following estimation is valid:

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} \le const \left\{ C_1(f) \left(1 + \frac{1}{3-\alpha}\right) \nu^{-(3-\alpha)} + \nu^{-\frac{1}{2}} \|f'\|_{2,m} + \right.$$

$$\left. + \nu^{-\frac{3}{2}} \|Q_2 f\|_{2,m} + \nu^{-\frac{5}{2}} \|Q_3 f\|_{2,m} + \nu^{-3} \|Q_4 f\|_{1,m} \right\}, \tag{1.9}$$

where

$$Q_2 = -U_2; \;\; Q_3 = U_3 - 2\,(U_2)'\,; \;\; Q_4 = (U_3)' - (U_2)'' - U_4;$$

$const$ is idependent of the vector-function $f(x)$.

Corollary 1.1 *Let the conditions A be fulfilled. Then biorthogonal expansion of the vector-function $f(x) \in W^1_{2,m}(G)$, $f(0) = f(1) = 0$, converges absolutely and uniformly and the following estimations are valid:*

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} \le const \nu^{-\frac{1}{2}} \|f\|_{W^1_{2,m}(G)}\,, \quad \nu \ge 2; \tag{1.10}$$

$$\|\sigma_\nu(\cdot, f) - f\|_{C[0,1]} = o\left(\nu^{-\frac{1}{2}}\right), \;\; \nu \to +\infty, \tag{1.11}$$

where $const$ is independent of $f(x)$, the symbol "o" is dependent on $f(x)$.

Note that similar results for the Sturm-Liouville operator for $m = 1$ were established in the papers [7], [2], while for the Schrodinger operator in the paper [3].

2 Proof of the results

Before starting the proof of theorem 1.1 we note that conditions A provide Riesz basicity of each of the systems $\left\{\psi_k(x)\|\psi_k(x)\|_{2,m}^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\varphi_k(x)\|\varphi_k(x)\|_{2,m}^{-1}\right\}_{k=1}^{\infty}$ in $L_2^m(G)$ (see [4], [5]). Therefore, for these systems the Bessel inequality is valid in $L_2^m(G)$.

Let the conditions of the theorem be fulfilled. Prove uniform convergence of the series

$$\sum_{k=1}^{\infty}|f_k|\,|\psi_k(x)|\,,\quad x\in\overline{G}. \tag{2.1}$$

For that we give some auxiliary lemmas

Lemma 2.1 *For the Fourier coefficient f_k of the vector-function $f(x)\in W_{2,m}^1(G)$ the following representation is valid:*

$$f_k=(f,\varphi_k)=-\frac{1}{\lambda_k}\sum_{i=0}^{m_k}\frac{\left.<f(x),\varphi_{k+i}^{(3)}(x)>\right|_0^1}{\lambda_k^i}$$

$$+\frac{1}{\lambda_k}\sum_{i=0}^{m_k}\frac{\left(f',\varphi_{k+i}^{(3)}(x)\right)}{\lambda_k^i}+\frac{1}{\lambda_k}\sum_{i=0}^{m_k}\frac{\left(Q_2f,\varphi_{k+i}^{(3)}(x)\right)}{\lambda_k^i}+$$

$$+\frac{1}{\lambda_k}\sum_{i=0}^{m_k}\frac{\left(Q_3f,\varphi_{k+i}^{(1)}(x)\right)}{\lambda_k^i}+\frac{1}{\lambda_k}\sum_{i=0}^{m_k}\frac{(Q_4f,\varphi_{k+i})}{\lambda_k^i},\quad \lambda_k\neq 0, \tag{2.2}$$

where $Q_2=-U_2$; $Q_3=U_3-2(U_2)'$; $Q_4=(U_3)'-(U_2)''-U_4$; m_k is the order of the associated vector-function $\varphi_k(x)$.

Proof. By definition of the function $\varphi_k(x)$ the equality $\varphi_k=-\frac{L^*\varphi_k}{\overline{\lambda}_k}+\frac{\theta_{k+1}\varphi_{k+1}}{\overline{\lambda}_k}$ holds for $\lambda_k\neq 0$.

Taking this into account, we get

$$(\varphi_k,f)=-\frac{1}{\overline{\lambda}_k}(L^*\varphi_k,f)+\frac{\theta_{k+1}}{\overline{\lambda}_k}(\varphi_{k+1},f)=-\frac{1}{\overline{\lambda}_k}\left(\varphi_k^{(4)},f\right)$$

$$-\frac{1}{\overline{\lambda}_k}\left(\left(U_2^*\varphi_k\right)^{(2)},f\right)+\frac{1}{\overline{\lambda}_k}\left(\left(U_3^*\varphi_k\right)',f\right)-\frac{1}{\overline{\lambda}_k}\left(U_4^*\varphi_k,f\right)+$$

$$+\frac{\theta_{k+1}}{\overline{\lambda}_k}(\varphi_{k+1},f)=-\frac{1}{\overline{\lambda}_k}\left(\varphi_k^{(4)},f\right)+\frac{1}{\overline{\lambda}_k}\left(\varphi_k'',Q_2f\right)+\frac{1}{\overline{\lambda}_k}\left(\varphi_k',Q_3f\right)+$$

$$+\frac{1}{\overline{\lambda}_k}(\varphi_k,Q_4f)+\frac{\theta_{k+1}}{\overline{\lambda}_k}(\varphi_{k+1},f).$$

From this recurrent relation, with regard to $\theta_{k+1}=\theta_{k+2}=...=\theta_{k+m_k}=1,\theta_{k+m_k+1}=0$ we find

$$(\varphi_k,f)=-\frac{1}{\overline{\lambda}_k}\sum_{i=0}^{m_k}\frac{\left(\varphi_{k+i}^{(4)},f\right)}{\overline{\lambda}_k^i}+\frac{1}{\overline{\lambda}_k}\sum_{i=0}^{m_k}\frac{\left(\varphi_{k+i}'',Q_2f\right)}{\overline{\lambda}_k^i}+$$

$$+\frac{1}{\overline{\lambda}_k}\sum_{i=0}^{m_k}\frac{\left(\varphi'_{k+i},Q_3f\right)}{\overline{\lambda}_k^i}+\frac{1}{\overline{\lambda}_k}\sum_{i=0}^{m_k}\frac{\left(\varphi_{k+i},Q_4f\right)}{\overline{\lambda}_k^i}.$$

At first we conduct integration by parts in the expression $\left(\varphi_{k+i}^{(4)},f\right)$, and then having taken the complex conjugation we get formula (2.2). Lemma 2.1 is proved.

Lemma 2.2 *For any $\mu\geq 2$ the following estimations are valid:*

$$\sum_{Re\mu_k\geq\mu}\frac{\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}}{\mu_k^{1+\delta}}\leq C_2(\delta)\,\mu^{-\delta}, \tag{2.3}$$

$$\sum_{Re\mu_k\geq\mu}\frac{\|\varphi_k\|_{\infty,m}^2\|\varphi_k\|_{2,m}^{-2}}{\mu_k^{1+\delta}}\leq C_3(\delta)\,\mu^{-\delta}, \tag{2.4}$$

where $\delta>0,\ C_2(\delta)$, $C_3(\delta)$ are constants independent of μ.

Proof. Prove estimation, (2.3). By conditions (1.1), (1.2), (1.6) and Abel's transformation for any fixed $\ell\in N$ we have

$$\sum_{\mu\leq\rho_k\leq[\mu]+\ell}\frac{\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}}{\mu_k^{1+\delta}}\leq\sum_{[\mu]\leq\rho_k\leq[\mu]+\ell}\frac{\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}}{\rho_k^{1+\delta}}\leq$$

$$\leq\sum_{n=[\mu]}^{[\mu]+\ell}\frac{1}{n^{1+\delta}}\sum_{n\leq\rho_k\leq n+1}\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}\leq\sum_{n=[\mu]}^{[\mu]+\ell-1}\left(\sum_{1\leq\rho_k\leq n+1}\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}\right)\cdot$$

$$\cdot\left(\frac{1}{n^{1+\delta}}-\frac{1}{(n+1)^{1+\delta}}\right)+\left(\sum_{1\leq\rho_k\leq[\mu]+\ell}\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}\right)([\mu]+\ell)^{-1-\delta}+$$

$$+\left(\sum_{1\leq\rho_k\leq[\mu]-1}\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}\right)[\mu]^{-1-\delta}\leq const\sum_{n=[\mu]}^{[\mu]+\ell-1}(n+1)\frac{(1+\delta)(n+1)^{\delta}}{(n(n+1))^{1+\delta}}+$$

$$+const\,([\mu]+\ell)\,([\mu]+\ell)^{-1-\delta}+const\,([\mu]-1)\,[\mu]^{-1-\delta}\leq$$

$$\leq const\left((1+\delta)\sum_{n=[\mu]}^{\infty}n^{-1-\delta}+[\mu]^{-\delta}\right)\leq C_2(\delta)\,\mu^{-\delta},$$

where $C_2(\delta)=const\left(1+\frac{1}{\delta}\right)$.

Hence, by arbitrariness of the natural number l we get estimation (2.3). Estimation (2.4) is established in the same way. Lemma 2.2 is proved.

Lemma 2.3 *(see [6]) Subject to conditions A, the systems $\left\{\varphi_k^{(\ell)}(x)\,\|\varphi_k\|_{2,m}^{-1}\,\mu_k^{-\ell}\right\}$, $\mu_k\neq 0$, $\ell=\overline{1,3}$ are Bessel in $L_2^m(G)$ i.e. for an arbitrary $f(x)\in L_2^m(G)$ the following estimation is valid:*

$$\left(\sum_{\mu_k\neq 0}\left|\left(f,\varphi_k^{(\ell)}\,\|\varphi_k\|_{2,m}^{-1}\,\mu_k^{-\ell}\right)\right|^2\right)^{\frac{1}{2}}\leq const\,\|f\|_{2,m}\,. \tag{2.5}$$

Lemma 2.4 *Subjected to conditions A, the following estimations* ($\mu \geq 2$) *are valid:*

$$J(r,\mu) = \sum_{\rho_k \geq \mu} \left| \lambda_k^{-1} \sum_{i=0}^{m_k} \lambda_k^{-i} \left(Q_r f, \varphi_{k+i}^{(4-r)} \right) \right| |\psi_k(x)| \leq$$

$$\leq const \frac{\|Q_r f\|_{2,m}}{\mu^{r-\frac{1}{2}}}, \quad r = 2, 3; \tag{2.6}$$

$$J(4,\mu) = \sum_{\rho_k \geq \mu} \left| \lambda_k^{-1} \sum_{i=0}^{m_k} \lambda_k^{-i} (Q_4 f, \varphi_{k+i}) \right| |\psi_k(x)| \leq const \frac{\|Q_4 f\|_{1,m}}{\mu^3}. \tag{2.7}$$

Proof. Applying antiapriori estimation (1.4) and estimation (1.5) we get

$$J(r,\mu) = \sum_{\rho_k \geq \mu} \left| \lambda_k^{-1} \sum_{i=0}^{m_k} \lambda_k^{-i} \left(Q_r f, \varphi_{k+i}^{(4-r)} \right) \right| |\psi_k(x)| \leq$$

$$\leq \sum_{\rho_k \geq \mu} \frac{\|\psi_k(x)\|_{\infty,m}}{|\mu_k|^r} \sum_{i=0}^{m_k} |\mu_k|^{-4i} \left| \left(Q_r f, \varphi_{k+i}^{(4-r)} \|\varphi_{k+i}\|_{2,m}^{-1} \mu_k^{r-4} \right) \right| \|\varphi_{k+i}\|_{2,m} \leq$$

$$\leq const \sum_{\rho_k \geq \mu} \frac{\|\psi_k\|_{\infty,m} \|\varphi_k\|_{2,m}}{|\mu_k|^r} \sum_{i=0}^{m_k} \left| \left(Q_r f, \varphi_{k+i}^{(4-r)} \|\varphi_{k+i}\|_{2,m}^{-1} \mu_k^{r-4} \right) \right| \leq$$

$$\leq const \sum_{\rho_k \geq \mu} \frac{\|\psi_k\|_{\infty,m} \|\psi_k\|_{2,m}^{-1}}{|\mu_k|^r} \sum_{i=0}^{m_k} \left| \left(Q_r f, \varphi_{k+i}^{(4-r)} \|\varphi_{k+i}\|_{2,m}^{-1} \mu_k^{r-4} \right) \right|.$$

At first we apply the Cauchy-Bunyakovskii inequality for the sum, and then take into account
$\sup_k m_k < \infty$ (this follows from condition (1.2)) and having used lemma 2.2 and 2.3 we get

$$J(r,\mu) \leq const \left(\sum_{\rho_k \geq \mu} \frac{\|\psi_k\|_{\infty,m}^2 \|\psi_k\|_{2,m}^{-2}}{|\mu_k|^{2r}} \right)^{1/2} \times$$

$$\times \left(\sum_{\rho_k \geq \mu} \left(\sum_{i=0}^{m_k} \left| \left(Q_r f, \varphi_{k+i}^{(4-r)} \|\varphi_{k+i}\|_{2,m}^{-1} \mu_k^{r-4} \right) \right| \right)^2 \right)^{1/2} \leq$$

$$\leq const \mu^{\frac{1}{2}-r} \left(\sup_k m_k \right) \|Q_r f\|_{2,m} \leq const \mu^{\frac{1}{2}-r} \|Q_r f\|_{2,m}, \quad r = 2, 3.$$

Estimation (2.6) is established

For proving estimation (2.7) at first we use the Holder inequality, then take into account $\sup_k m_k < \infty$ and apply antiapriori estimation (see [7])

$$\theta_{k+1} \|\varphi_{k+1}\|_{\infty,m} \leq const (1 + |\mu_k|)^4 \|\varphi_k\|_{\infty,m}. \tag{2.8}$$

As a result we have

$$J(4,\mu)=\sum_{\rho_k\geq\mu}\left|\lambda_k^{-1}\sum_{i=0}^{m}\lambda_k^i\left(Q_4f,\varphi_{k+i}\right)\right|\left|\psi_k(x)\right|\leq$$

$$\leq\sum_{\rho_k\geq\mu}\frac{\|\psi_k\|_{\infty,m}}{|\mu_k|}\left(\sum_{i=0}^{m}|\mu_k|^{-4i}\|\varphi_{k+i}\|_{\infty,m}\right)\|Q_4f\|_{1,m}\leq$$

$$\leq const\left(\sum_{\rho_k\geq\mu}\frac{\|\psi_k\|_{\infty,m}\|\varphi_k\|_{\infty,m}}{|\mu_k^4|}\right)\|Q_4f\|_{1,m}.$$

Hence by the Cauchy-Bunyakovskii inequality, condition (1.5) and lemma 2.2 it follows that

$$J(4,\mu)\leq\left(\sum_{\rho_k\geq\mu}\frac{\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{\infty,m}^{-2}}{|\mu_k^4|}\right)^{1/2}\times$$

$$\times\left(\sum_{\rho_k\geq\mu}\frac{\|\varphi_k\|_{\infty,m}^2\|\varphi_k\|_{\infty,m}^{-2}}{|\mu_k^4|}\right)^{1/2}\|Q_4f\|_{1,m}\leq const\mu^{-3}\|Q_4f\|_{1,m}.$$

Lemma 2.4 is proved.

Now we prove uniform convergence on $\overline{G}=[0,1]$ of series (2.1). For that we represent it in the form

$$\sum_{k=1}^{\infty}|f_k|\left|\psi_k(x)\right|=\sum_{0\leq\rho_k<2}|f_k|\left|\psi_k(x)\right|+\sum_{\rho_k\geq2}|f_k|\left|\psi_k(x)\right|=I_1+I_2.$$

By condition (1.2) for the sum I_1, it is fulfilled the estimation $I_1\leq const\|f\|_{1,m}$ and by representation (2.2)

$$I_2\leq\sum_{\rho_k\geq2}|\mu_k|^{-4}\left(\sum_{i=0}^{m_k}|\mu_k|^{-4i}\left|<f(x),\varphi_{k+i}^{(3)}(x)>\right|_0^1\right)\left|\psi_k(x)\right|+$$

$$+\sum_{\rho_k\geq2}|\mu_k|^{-4}\left(\sum_{i=0}^{m_k}\left|\mu_k^{-4i}\left(f',\varphi_{k+i}^{(3)}\right)\right|\right)\left|\psi_k(x)\right|+J(2,2)+J(3,2)+J(4,2)=$$

$$=J_1(2)+J_2(2)+J(2,2)+J(3,2)+J(4,2).$$

Uniform convergence of series $J(r,2)$, $r=\overline{2,4}$ follows from lemma 2.4 for $\mu=2$. Prove uniform convergence of the series $J_1(2)$ and $J_2(2)$. From condition (1.8) it follows that

$$J_1(2)\leq C_1(f)\sum_{\rho_k\geq2}|\mu_k|^{\alpha-4}\|\psi_k\|_{\infty,m}\left(\sum_{i=0}^{m_k}|\mu_k|^{-4i}\|\varphi_{k+i}\|_{\infty,m}\right).$$

Apply estimation (2.9), take into account $\sup_k m_k<\infty$ and condition (1.5), we get

$$J_1(2)\leq constC_1(f)\sum_{\rho_k\geq2}\frac{\|\psi_k\|_{\infty,m}\|\varphi_k\|_{\infty,m}}{|\mu_k|^{4-\alpha}}\leq$$

$$\leq constC_1(f)\sum_{\rho_k\geq 2}\frac{\|\psi_k\|_{\infty,m}\|\psi_k\|_{2,m}^{-1}}{|\mu_k|^{(4-\alpha)/2}}\frac{\|\varphi_k\|_{\infty,m}\|\varphi_k\|_{2,m}^{-1}}{|\mu_k|^{(4-\alpha)/2}}.$$

Hence by the Cauchy -Bunyakovskii inequality and lemma 2.2 (for $\mu = 2$) we find

$$J_1(2)\leq constC_1(f)\sqrt{C_2(3-\alpha)C_3(3-\alpha)}\leq constC_1(f)\left(1+\frac{1}{3-\alpha}\right)2^{-(3-\alpha)},$$

where $C_2(3-\alpha)$, $C_3(3-\alpha)$ are constants from lemma 22 and don't exceed $const\left(1+\frac{1}{3-\alpha}\right)$. Uniform convergence of the series $J_1(2)$ is established.

Transform the series $J_2(2)$ in the following form

$$J_2(2)=\sum_{\rho_k\geq 2}|\mu_k|^{-1}\left(\sum_{i=0}^{m_k}\left|\left(f',\varphi_{k+i}^{(3)}\|\varphi_{k+i}\|_{2,m}^{-1}\mu_k^{-3}\right)\right|\frac{\|\varphi_{k+i}\|_{2,m}}{|\mu_k|^{4i}}\right)|\psi_k(x)|.$$

Having applied estimation (1.4) and condition (1.5)

$$J_2(2)\leq const\sum_{\rho_k\geq 2}\|\psi_{k+i}\|_{\infty,m}|\mu_k|^{-1}\times$$

$$\times\left(\sum_{i=0}^{m_k}\left|\left(f',\varphi_{k+i}^{(3)}\|\varphi_{k+i}\|_{2,m}^{-1}\mu_k^{-3}\right)\right|\|\varphi_k\|_{2,m}\right)\leq$$

$$\leq const\sum_{\rho_k\geq 2}\|\psi_k\|_{\infty,m}\|\psi_k\|_{2,m}^{-1}|\mu_k|^{-1}\left(\sum_{i=0}^{m_k}\left|\left(f',\varphi_{k+i}^{(3)}\|\varphi_{k+i}\|_{2,m}^{-1}\mu_k^{-3}\right)\right|\right),$$

Hence by the Cauchy-Bunyakovskii inequality, lemma 2.2 and lemma 2.3 we get

$$J_2(2)\leq const\left(\sum_{\rho_k\geq 2}\|\psi_k\|_{\infty,m}^2\|\psi_k\|_{2,m}^{-2}|\mu_k|^{-1}\right)^{1/2}\times$$

$$\times\left(\sum_{\rho_k\geq 2}\left|\sum_{i=0}^{m_k}\left(f',\mu_k^{-3}\|\varphi_{k+i}\|_{2,m}^{-1}\mu_{k+i}\right)\right|^2\right)^{\frac{1}{2}}\leq$$

$$\leq const2^{-\frac{1}{2}}\left(\sum_{\rho_k\geq 2}m_k\sum_{i=0}^{m_k}\left|\left(f',\mu_k^{-3}\|\varphi_{k+i}\|_{2,m}^{-1}\varphi_{k+i}\right)\right|^2\right)^{\frac{1}{2}}\leq$$

$$\leq const\left(\sup_k m_k\right)^2 2^{-\frac{1}{2}}\left\|f'\right\|_{2,m}<\infty.$$

Uniform convergence of the series $J_2(2)$ is established. Consequently, series (2.1) uniformly converges on $\overline{G}=[0,1]$. Hence, it follows uniform convergence of biorthoginal series itself. By the completeness of the system $\{\varphi_k(x)\}_{k=1}^{\infty}$ in $L_2^m(G)$ and absolute continuity of $f(x)$ on $\overline{G}$, we get that biorthogonal series of the vector-function $f(x)$ converges uniformly just to $f(x)$ i.e. in the metric $G[0,1]$ it holds

$$f(x)=\sum_{k=1}^{\infty}f_k\psi_k(x),\qquad x\in\overline{G}. \tag{2.9}$$

Now establish estimation (1.9). By equality (2.9)

$$|\sigma_\nu(x,f)-f(x)| = \left|\sum_{\rho_k>\nu} f_k\psi_k(x)\right| \le \sum_{\rho_k>\nu} |f_k|\,|\psi_k(x)|\,.$$

Taking into account the expression of the coefficient f_k from lemma 2.1 we have

$$|\sigma_\nu(x,f)-f(x)| \le \sum_{\rho_k>\nu} |\mu_k|^{-4}\left(\sum_{i=0}^{m_k}|\mu_k|^{-4i}\left| < f(x),\varphi_{k+i}^{(3)}(x) > \Big|_0^1\right|\right)|\psi_k(x)| +$$

$$+\sum_{\rho_k\ge\nu}|\mu_k|^{-4}\left(\sum_{i=0}^{m_k}\left|\mu_k^{-4i}\left(f',\varphi_{k+i}^{(3)}\right)\right|\right)|\psi_k(x)| + J(2,\nu)+J(3,\nu)+J(4,\nu) =$$

$$= J_1(\nu)+J_2(\nu)+J(2,\nu)+J(3,\nu)+J(4,\nu)\,. \tag{2.10}$$

Obviously, for $J_1(\nu)$, $J_2(\nu)$ the following estimations (see the estimations of expressions $J_1(2)$ and $J_2(2)$) are fulfilled:

$$J_1(\nu) \le constC_1(f)\left(1+\frac{1}{3-\alpha}\right)\nu^{-(3-\alpha)}, \tag{2.11}$$

$$J_1(\nu) \le const\nu^{-\frac{1}{2}}\left\|f'\right\|_{2,m}. \tag{2.12}$$

For the sum $J(r,\nu)$, $r=\overline{2,4}$ by lemma 2.4 the following estimations are fulfilled

$$J(r,\nu) \le const\nu^{-\left(r-\frac{1}{2}\right)}\|Q_rf\|_{2,m},\, r=2,3; \tag{2.13}$$

$$J(4,\nu) \le const\nu^{-3}\|Q_4f\|_{1,m}. \tag{2.14}$$

From (2.10) allowing for estimations (2.11)-(2.14) it follows that

$$\max_{x\in\overline{G}}|\sigma_\nu(x,f)-f(x)| \le const\left\{C_1(f)\left(1+\frac{1}{3-\alpha}\right)\nu^{-(3-\alpha)}+\nu^{-\frac{1}{2}}\left\|f'\right\|_{2,m}+\right.$$

$$\left.+\nu^{-\frac{3}{2}}\|Q_2f\|_{2,m}+\nu^{-\frac{5}{2}}\|Q_3f\|_{2,m}+\nu^{-3}\|Q_4f\|_{1,m}\right\}.$$

Theorem 1.1 is completely proved.

Estimation (1.10) directly follows from estimation (1.9) if we take into account $C_1(f)=0$,
$\|Q_rf\|_{2,m} \le \|Q_r\|_2\|f\|_{\infty,m}$, $r=2,3$; $\|Q_4f\|_{1,m} \le \|Q_4\|_1\|f\|_{\infty,m}$ and for $f(x)\in W_{2,m}^1(G)$, $f(0)=f(1)=0$ it holds $\|f\|_{\infty,m} \le \|f'\|_{1,m} \le \|f'\|_{2,m}$.

For justification of estimation (1.11) we should pay attention to the estimation of the series $J_2(\nu)$. For it $J_2(\nu) \le const\nu^{-\frac{1}{2}}\left(\sum\limits_{\rho_k\ge\nu} m_k\sum\limits_{i=0}^{m_k}\left|\left(f',\mu_k^{-3}\|\varphi_{k+i}\|_{2,m}^{-1}\varphi_{k+i}^{(3)}\right)\right|^2\right)^{1/2} =$ $const\nu^{-\frac{1}{2}}o(1)$, is valid for $\nu\to+\infty$, as $\sup\limits_k m_k<\infty$ and the system $\left\{\mu_k^{-3}\|\varphi_{k+i}\|_{2,m}^{-i}\varphi_{k+i}^{(3)}\right\}_{\rho_k>0}$ is Bessel in $L_2^m(G)$.

References

1. Il'in, V.A.: *Componentwise equiconvergence with the trigonometric series of expansions in root vector functions of the Schrodinger operator with a matrix non-Hermitian potential, all elements of which are only integrable*. Differ. Uravn. **27** (11), 1862–1878 (1991).
2. Kurbanov, V.M., Safarov, R.A.: *On uniform convergence of orthogonal expansions in eigenfunctions of Strum-Liouville operator*. Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Mat. Sci. **24** (1), 161–168 (2004).
3. Kurbanov, V.M., Garaeva, A.T.: *Absolute and uniforrn convergence of expansions in the root vector functions of the Schrodeniger operator with a matrix potential*. Dokl. Math., **87** (3), 304-306 (2013) (in Russian), Translated from Dokl. Acad. Nauk, **450**(3), 268-270 (2013).
4. Kerimov, N.B.: *On the inconditional basis property of a system of eigenfunctions and associated functions for a fourth-order differential operator*. Dokl. Acad. Nauk SSSR, **286** (4), 803-806 (1986) (in Russian).
5. Kurbanov, V.M.: *On the eigenvalue distribution and a Bessel property criterion of root functions of a differential operator: I, (II)*. Differ. Equ., **41** (4), 489-505 (2005), Differ. Equ., **41**(5), 649-659 (2005).
6. Kurbanov, V.M.: *Analog of Riesz theorem and the basis property in L_p of a system of root functions of a differential operator: I, (II)*. Differ. Equ., **49** (1), 7-19 (2013), Differ. Equ., **49** (4), 437-449 (2013).
7. Kurbanov, V.M.: *Hausdorf-Young inequalities for system of root vector functions of n th-order differential operator*. Differ. Equ., **33** (3), 355-365 (1997).
8. Lazetic, N.L.: *On uniform convergence on closed intervals of spectral expansions and their derivatives, for functions from $W_p^{(1)}$*. Mat. Vesnik, **56** (3-4), 91-104 (2004).