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Some characterization of functions from Lizorkin-Triebel-Morrey type spaces with the many groups variables.

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Abstract. In this paper, we introduce a new function space $F^{<l>}_{p,\theta,a,\varkappa,\tau}(s,G)$ with the parameters of many groups of variables of type Lizorkin-Triebel -Morrey. In view of the embedding theorems we study some properties of the functions, which are belonging to these spaces.

Keywords. Spaces Lizorkin-Triebel -Morrey type, many groups of variables, integral representations, embedding theorems.

Mathematics Subject Classification (2010): 4235; 46E30; 46E35

1 Introduction

In this paper we shall introduce and study the new function space

$$F^{<l>}_{p,\theta,a,\varkappa,\tau}(s,G) \tag{1.1}$$

of several groups of variables of Lizorkin- Trieble-Morrey type, where the analysis is based on a setting space, related methods of the integral representation and differential properties of some classes of such function. Note that, when s=1, then this space (1.1) is equivalent to the space Lizorkin-Triebel-Morrey type $F^{<l>}_{p,\theta,a,\varkappa,\tau}(G)$, which are studied in [5, 12], and when $s = n$, then this space is equivalent to the space Lizorkin-Triebel-Morrey type with dominant mixed derivatives $S^{<l>}_{p,\theta,a,\varkappa,\tau}$*F(G)*, which are studied in [13, 14], when $\tau = \infty, a = 0$, then this space is equivalent to the space Lizorkin-Triebel-Morrey type with many groups variables, which are studied in [9] and case $s = 1$ is studied in [1, 6, 21, 22].

Let $G \subset R^n$ $1 \le s \le n$; s, n be naturals and $n_1 + \cdots + n_s = n$ and we consider the sufficient smooth function f(x), where the point $x = (x_1, ..., x_s) \in R^n$ has coordinates $x_k =$

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$(x_{k,1};\cdots;x_{k,n_k})\in R^{n_k}$ $(k\in e_s=\{1,\cdots,s\})$. Furthermore $R^n=R^{n_1}\times R^{n_2}\times...\times R^{n_s}$. Let $l=(l_1,\cdots,l_s)$ be given non-negative vectors, such that $l_k=(l_{k,1};\ldots;l_{k,n_k})$, $(k\subset e_s)$ that is, $l_{k,j}>0$, $(j=1,...,n_k)$ for every $k\in e_s$ and we shall denote by Q the set of vectors $i=(i_1,\cdots,i_s)$, where $i_k=1,2,\cdots,n_k$ for all $k\in e_s$. The number of the set Q is equal to: $|Q|=\prod_{k=1}^{s}(1+n_k)$. Therefore, to the vectors $i=(i_1,...,i_s)\in Q$ we let correspond the vector $l^i=(l_1^{i_1};...;l_s^{i_s})$. It means that, the vector $l=(l_1,\cdots,l_s)$ defines as $l^0=(0,0,\cdots,0)$, $l_k^1=(l_{k,1},0,\cdots,0),\cdots,l_k^{n_k}=(0,0,\cdots,l_{k,n_k})$ for all $k\in e_s$. And to the vector e^i we correspond the vector $\overline{l}^i=\left(\overline{l}_1^{i_1},\overline{l}_2^{i_2},\cdots,\ \overline{l}_s^{i_s}\right)$, which are associated with coordinates of vectors $\overline{l}_k^{i_k}=\left(\overline{l}_{k,1}^{i_1},\overline{l}_{k,2}^{i_2},\cdots,\ \overline{l}_{k,n_k}^{i_k}\right)$ $(k\in e_s)$, where the largest number $\overline{l}_{k,j}^{i_k}$ is less than $l_{k,j}^{i_k}$ for every $l_{k,j}^{i_k}>0$, when $l_{k,j}^{i_k}=0$, then we must assume that, $\overline{l}_{k,j}^{i_k}=0$ for all $k\in e_s$.

Definition 1.1 *We denote by $F_{p,\theta,a,\varkappa,\tau}^{<l>}(s,G)$ normed Lizorkin-Triebel-Morrey space of function f on G, with many groups variables, with finite norm*

$$\|f\|_{F_{p,\theta,a,\varkappa,\tau}^{<l>}(G,s)}=\sum_{i\in Q}\|f\|_{L_{p,\theta,a,\varkappa,\tau}^{<l^i>}(G)},\tag{1.2}$$

$$\|f\|_{L_{p,\theta,a,\varkappa,\tau}^{<l^i>}(G)}=\left\|\left\{\int_0^{t_{0,1}^i}\int_0^{t_{0,s}^i}\left[\frac{\Delta^{2\omega}(t,G)\,D^{\overline{l}^i}f}{\prod_{k\in e^i}t_k^{\left|\beta_k^{i_k}\right|}}\right]^{\theta}\prod_{k\in e^i}\frac{dt_k}{t_k}\right\}^{1/\theta}\right\|_{p,a,\varkappa,\tau},\tag{1.3}$$

and

$$\|f\|_{p,a,\varkappa,\tau:G}=\sup_{x\in G}\left\{\int_0^{\infty}\cdots\int_0^{\infty}\left[\prod_{k\in e_s}[t_k]_1^{\frac{-|\varkappa_k|a}{p}}\|f\|_{p,G_{t^\varkappa}(x)}\right]^{\tau}\prod_{k\in e_s}\frac{dt_k}{t_k}\right\}^{1/\tau},\tag{1.4}$$

where $D^{\overline{l}^i}f=D_1^{\overline{l}_1^{i_1}}\cdots D_s^{\overline{l}_s^{i_s}}f$; $D_k^{\overline{l}_k^{i_k}}f=D_{k,1}^{\overline{l}_k^{i_k}}\cdots D_{k,i_k}^{\overline{l}_k^{i_k}}f$; *and* $G_{t^\varkappa}=G\cap I_{t^\varkappa}$; $I_{t^\varkappa}=I_{t_1^{\varkappa_1}}\times I_{t_2^{\varkappa_2}}\times\cdots\times I_{t_s^{\varkappa_s}}$; $I_{t_k^{\varkappa_k}}=\left\{x_k:|x_k|<\frac{1}{2}t_k^{|\varkappa_k|},\ k\in e_s\right\}$. *Where* $\left|\beta_k^{i_k}\right|=\sum_{j=1}^{n_k}\beta_{k,j}^{i_k}$, $0<\beta_{k,j}^{i_k}=l_{k,j}^{i_k}-\overline{l}_{k,j}^{i_k}\leq 1$ *for* $\beta_{k,j}^{i_k}>0$, *when* $l_{k,j}^{i_k}=0$, *then* $\beta_{k,j}^{i_k}=0$. *Here* $t=(t_1,\cdots,t_s)$, $t_k=(t_{k,1};\cdots;t_{k,n_k})$, $\omega=(\omega_1,\cdots,\omega_s)$, $\omega_k=(\omega_{k,1};\cdots;\omega_{k,n_k})$ *and when* $\omega_{k,j}=1$ *for* $k\in e^i$ *or* $\omega_{k,j}=0$ *for* $k\in e_s/e^i$; $e^i=supp\,l^i=supp\overline{l}^i=supp\,\omega$, $t_0=(t_{0,1},\cdots,t_{0,s})$, $t_{0,k}=(t_{0,k,1},\cdots,t_{0,k,n_k})$ *are fixed non negative vectors,* $\varkappa\in(0,\infty)^n$ *a* $\in[0,1]$, $\tau\in[1,\infty]$, $[t_k]_1=min\{1,t_k\}$, $k\in e_s$.

We must note that, the analysis is based on a Sobolev's function space, with parameters $W_p^{(l)}(G)$, which was first studied by Morrey [10, 11]. Futher these results were generalized and developed by V. I. Ilyin [7] and R. Ross [20], Y. V. Netrusov [19], I. Mazzucato [8], V. I. Burenkov and H. V. Guliyev [2], H. V. Guliyev [3, 5], Y. Sawano [7], V. S. Guliyev., Y. Sawano [4] and others [15-18].

Let us note some properties of the normed function space $F_{p,\theta,a,\varkappa,\tau}^{<l>}(G,s)$.

1)For any $\varkappa > 0,\ 1 \le \tau \le \infty,\ c > 0$, we obtain

$$\|f\|_{F^{<l>}_{p,\theta}(G,s)} \le \|f\|_{F^{<l>}_{p,\theta,a,\varkappa}(G,s)} \le C\,\|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s)}. \tag{1.5}$$

2)The function space (1.1) is complete.

3)For any real number $c > 0$

$$\|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s)} = \frac{1}{c^{\frac{1}{\tau}}}\|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s)}.$$

4)

$$\|f\|_{F^{<l>}_{p,\theta,0,\varkappa,\infty}(G,s)} = \|f\|_{F^{<l>}_{p,\theta}(G,s)}.$$

5) Let $1 < \theta \le r \le s \le \sigma < \infty$ and $\theta \le p \le \sigma$, then

$$B^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s) \subset_{>} F^{<l>}_{p,r,a,\varkappa,\tau}(G,s) \subset_{>} F^{<l>}_{p,s,a,\varkappa,\tau}(G,s) \subset_{>} B^{<l>}_{p,\sigma,a,\varkappa,\tau}(G,s)$$

and when $l \in N^n,\ \ r = s = 2$ then

$$B^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s) \subset_{>} F^{<l>}_{p,2,a,\varkappa,\tau}(G,s) = W^{<l>}_{p,a,\varkappa,\tau}(G,s) \subset_{>} B^{<l>}_{p,\sigma,a,\varkappa,\tau}(G,s)$$

In this case when $p = \theta$, then $F^{<l>}_{p,p,a,\varkappa,\tau}(G,s) = B^{<l>}_{p,p,a,\varkappa,\tau}(G,s)$.

Here the space $W^{<l>}_{p,a,\varkappa,\tau}(G,s)$ and $B^{<l>}_{p,\theta,a,\varkappa,\tau}(G,s)$ are defined and studied in [18] and [12].

Here, it is said that, sub domain $U \subset G \subset R^n$ calls domain satisfying the condition "σ -semi-horn", if the vector $\sigma = (\sigma_1, \cdots, \sigma_s)$ is such that,

$$x + V(\sigma) = x + \bigcup_{0<t_k \le T_k} \left\{ y : \frac{y}{(t^\sigma + T^\sigma)^i} \in S(\Psi_{i,\delta}) \right\} \subset G,$$

$$(t^\sigma + T^\sigma)^i = t_j^{\sigma_j},\ j \in e^i; (t^\sigma + T^\sigma)^i = T_j^{\sigma_j},\ j \in e_s/e^i$$

for all $x \in U$ (U- is an arbitrary open set, which belonging to the domain G and $U + V(\sigma) \subset G$ satisfying the condition "σ -semi-horn"),

It is clear, that $V(\sigma) \subset I_{T^\sigma}, U-$ is an open set, which belonging to the domain G and $U \subset G \subset R^n$. Here it is said that, the sub domain $U \subset G \subset R^n$ calls domain satisfying the condition "σ -semi-horn", if the vector $\sigma = (\sigma_1, \cdots, \sigma_s)$ is such that, $G \subset A(T^\sigma)$, if we have finite sub domains $G_1, \cdots, G_N \subset G$, satisfying the condition "σ -semi-horn" and surfacing the domain G, that is,

$$G \subset \bigcup_{j=1}^{N} G_j. \tag{1.6}$$

But we suppose $G \in A_\epsilon(T^\sigma)$ ($\epsilon > 0$), if we substitute the condition (1.6) into the condition $G \subset \bigcup_{j=1}^{N} G_{j,\epsilon}$. Note that,

$$G_{j,\epsilon} = \{x : x \in G_j :\ \rho(x, G/G_j) > \epsilon\}\ .$$

2 Preliminaries

Here we must note that, $\Psi_{i,\delta}(y,z) \in C^\infty(R^n \times R^n), \Psi_i(\cdot,z) \in C_0^\infty(R^n)$ and $S(\Psi_i(\cdot,z)) = \operatorname{supp}\Psi_i \subset I_1 = \left\{y : |y_j| < \frac{1}{2}; j = 1,\cdots,n\right\}$. Let $T = (T_1,\cdots,T_s)$, $T_k = (T_{k,1},\cdots,T_{k,n_k}),\ 0 < T_{k,j} \le 1,\ k \in e_s,\ j = 1,\cdots,\ n_k$. It is clear, that $V(\sigma) \subset I_{T^\sigma}$.

Lemma 2.1 *Let* $1 \le p \le q \le r \le \infty$;$0 < |\varkappa_k| < |\sigma_k|;\ 0 \le \eta_{k,j} \le T_{k,j} \le 1;$ $(k \in e_s,\ j = 1,2,\cdots,n_k),\ v = (v_1,\cdots,v_s),\ v_{k,j} \ge 0$ *integers* $(j = 1,2,\cdots,n)$ *and* $\rho = (\rho_1,\cdots,\rho_s);0 < \rho_{k,j} < \infty$*;* $j = 1,\cdots,n_k; k \in e_s$; *let* $\Delta^{2\omega}(t)\, D^{\bar{l}^i} f \in L_{p,a,\varkappa,\tau}(G)$ *and*

$$\mu_{k,i_k} = l_{k,i_k}\sigma_{k,i_k} - (v_k,\sigma_k) - (|\sigma_k| - |\varkappa_k|\, a)\left(\frac{1}{p} - \frac{1}{q}\right), \tag{2.1}$$

$$F_\eta^i(x) = \prod_{k\in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k - \sigma_k)}$$

$$\times \int_0^{\eta^i} \cdots \int_0^{\eta^i} \Phi_i(x,t) \prod_{k\in e^i} \frac{dt_k}{t_k^{1+|\sigma_k| - \sigma_{k,i_k} l_{k,i_k} + (v_k,\sigma_k)}}, \tag{2.2}$$

$$F_{\eta T}^i(x) = \prod_{k\in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k}\bar{l}_{k,i_k} - (v_k,\sigma_k)}$$

$$\times \int_{\eta^i}^{T^i} \cdots \int_{\eta^i}^{T^i} \Phi_i(x,t) \prod_{k\in e^i} \frac{dt_k}{t_k^{1+|\sigma_k| - \sigma_{k,i_k} l_{k,i_k} + (v_k,\sigma_k)}}. \tag{2.3}$$

Where $\eta_k^i = \eta_k,\ T_k^i = T_k\ (k \in e^i),\ \eta_k^i = 0, T_k^i = 0\ (k \in e_s/e^i)$ and $(v_k,\sigma_k) = \sum_{j=1}^{n_k} v_{k,j}\sigma_{k,j},\ |\sigma_k| = \sum_{j=1}^{n_k} \sigma_{k,j}, |\varkappa_k| = \sum_{j=1}^{n_k} \varkappa_{k,j}$.

Here

$$\Phi_i(x,t) = \int_{R^{|e^i|}} \int_{R^n} \left\{ \Delta^{2\omega}(u)\, D^{\bar{l}^i} f(x+y) \Psi_{i,\sigma}\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i} \right) \right\} dy du, \tag{2.4}$$

Then the inequality takes place:

$$\sup_{\overline{x}\in U} \left\| F_\eta^i \right\|_{q,U_{\rho^\varkappa}(\overline{x})} \le C_1 \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u)\, D^{\bar{l}^i} f \right\|_{p,a,\varkappa,\tau;G}$$

$$\times \prod_{k=1}^{s} [\rho_k]_1^{\frac{|\varkappa_k| a}{p}} \prod_{k\in e_s/e^i} T_k^{\mu_{k,i_k}} \times \prod_{k\epsilon e^i} \eta_k^{\mu_{k,i_k}}, (\mu_{k,i_k} > 0), \tag{2.5}$$

$$\sup_{\overline{x}\in U} \left\| F_{\eta T}^i \right\|_{q,U_\rho^\varkappa}(\overline{x}) \le C_2 \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u)\, D^{\bar{l}^i} f \right\|_{p,a,\varkappa,\tau;G} \prod_{k=1}^{s} [\rho_k]_1^{\frac{|\varkappa_k| a}{p}}$$

$$\times \prod_{k\in e_s/e^i} T_k^{\mu_{k,i_k}} \begin{cases} \prod_{k\epsilon e^i} T_k^{\mu_{k,i_k}}; \mu_{k,i_k} > 0 \\ \prod_{k\epsilon e^i} ln\frac{T_k}{\eta_k};\ \mu_{k,i_k} = 0 \\ \prod_{k\epsilon e^i} \eta_k^{\mu_{k,i_k}}; \mu_{k,i_k} < 0, \end{cases} \tag{2.6}$$

where C_1 and C_2 are constants independent of f, ρ, η, T and $U_{\rho^{\varkappa}}(\overline{x}) = \left\{ y : |y_j - \overline{x}_j| < \frac{\rho^{\varkappa_j}}{2} : j = 1, 2, \cdots, n \right\}$.

Proof. Using the generalized Minkowski's inequality for any $\overline{x} \in U$, we have:

$$sup_{\overline{x}\in U}\left\|F_\eta^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C \prod_{k\in e_s/e^i} T_k^{-|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}-(v_k,\sigma_k)}$$

$$\times \int_0^{\eta^i} \cdots \int_0^{\eta^i} \|\Phi_i(\cdot,t)\|_{q,U_{\rho^{\varkappa}(\overline{x})}} \prod_{k\in e^i} t_k^{-1-|\sigma_k|+\sigma_{k,i_k}\bar{l}_{k,i_k}+(v_k,\sigma_k)} dt_k. \tag{2.7}$$

We must estimate $\|\Phi_i(\cdot,t)\|_{q,U_{\rho^{\varkappa}(\overline{x})}}$, from the Holder's inequality $(q \le r)$, we have:

$$\|\Phi_i(\cdot,t)\|_{q,U_{\rho^{\varkappa}(\overline{x})}} \le \|\Phi_i(\cdot,t)\|_{r,U_{\rho^{\varkappa}}(\overline{x})} \prod_{k\in e_s} \rho_k^{\frac{1}{q}-\frac{1}{r}|\varkappa_k|}. \tag{2.8}$$

Once again, using Holder's inequality for $|\Phi_i(x,t)|$, then we get $(1 \le p \le q \le r \le \infty;$ $s \le r, \frac{1}{s} = 1 - \frac{1}{p} + \frac{1}{q})$

$$\|\Phi_i(\cdot,t)\|_{r,U_{\rho^{\varkappa}}(\overline{x})}$$

$$\le sup_{x\in U_{\rho^{\varkappa}}(\overline{x})}\left(\int_{R^n}\int_{R^{|e^i|}}\left|\Delta^{2\omega}(u)\,D^{\bar{l}^i}f(x+y)\,du\right|^p X\left(\frac{y}{t^\sigma}\right)dy\right)^{1/p-1/r}$$

$$\times sup_{y\in V}\left(\int_{U_{\rho^{\varkappa}}(\overline{x})}\left|\int_{R^{|e^i|}}\Delta^{2\omega}(u)\,D^{\bar{l}^i}f(x+y)\,du\right|^p dx\right)^{1/r}$$

$$\times\left(\int_{R^n}\left|\Psi_{i,\sigma}\left(\frac{y}{(t^\sigma+T^\sigma)^i},\ \frac{u}{(t^\sigma+T^\sigma)^i}\right)^s\right|dy\right)^{1/s}. \tag{2.9}$$

Here X is a characteristic function of the set $S(\Psi_{i,\sigma})$. Because $G_{(t^\sigma+T^\sigma)^i}(x) \subset G_{(t^{\varkappa}+T^{\varkappa})^i}(x)$ for any $x \in U$ and $0 < t_j \le T_j \le 1,\ \ 0 < \varkappa_j < \sigma_j$, $j = 1, 2, ..., n$, we have:

$$\int_{R^n}\left|\int_{R^{|e^i|}}\Delta^{2\omega}(u)D^{\bar{l}^i}f(x+y)du\right|^p X\left(\frac{y}{(t^\sigma+T^\sigma)^i}\right)dy$$

$$\le \int_{G_{(t^\sigma+T^\sigma)^i}(x)}\left|\int_{R^{|e^i|}}\Delta^{2\omega}(u)\,D^{\bar{l}^i}f(y)du\right|^p dy$$

$$\le \int_{G_{(t^{\varkappa}+T^{\varkappa})^i}(x)}\left|\int_{R^{|e^i|}}\Delta^{2\omega}(u)\,D^{\bar{l}^i}f(y)du\right|^p dy$$

$$\le \prod_{k\in e^i} t_k^{\left|\beta_k^{i_k}\right|p} \times \left\|\prod_{k\epsilon e^i} t_k^{-\left|\beta_k^{i_k}\right|}\Delta^{2\omega}(u)\,D^{\bar{l}^i}f\right\|_{p,G_{(t^{\varkappa}+T^{\varkappa})^i}(x)}$$

$$\leq \prod_{k\in e^i} t_k^{\left|\beta_k^{i_k}\right|p} \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f \right\|_{p,a,\varkappa,\tau:G} \prod_{k\in e^i} t_k^{\mu_{k,i_k}} \prod_{k\in e_s/e^i} T_k^{\mu_{k,i_k}}. \qquad (2.10)$$

Next, for $y \in V$

$$\int_{U_{\rho^\varkappa}(\overline{x})} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f(x+y)\, du \right|^p dx$$

$$\leq \int_{G_{\rho^\varkappa}(\overline{x})} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f(x+y)\, du \right|^p dx$$

$$\leq \int_{G_{\rho^\varkappa}(\overline{x}+y)} \left| \int_{R^{|e^i|}} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f(x+y)\, du \right|^p dx$$

$$\leq \prod_{k\in e^i} t_k^{\left|\beta_k^{i_k}\right|p} \prod_{k\in e_s} [\rho_k]_1^{|\varkappa_k| a} \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f \right\|_{p,a,\ \varkappa,r;\ G}$$

$$\times \prod_{k\in e^i} t_k^{|\varkappa_k| a} \prod_{k\in e_s/e^i} T_k^{|\varkappa_k| a}, \qquad (2.11)$$

$$\int_{R^n} \left| \Psi_{i,\sigma}\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i} \right) \right|^s dy = \prod_{k\in e^i} t_k^{|\sigma_k|} \prod_{k\in e_s/e^i} T_k^{|\sigma_k|} \|\Psi_{i,\sigma}\|_s^s. \qquad (2.12)$$

From (2.8)-(2.12) and using the inequality (2.7) for $1 \leq \tau \leq \infty$, we get (2.5). Similarly, we can prove the inequality (2.6).

Lemma 2.2 *Let* $1 \leq p \leq q < \infty; 0 < |\varkappa_k| < |\sigma_k|; 0 \leq T_j \leq 1 \quad (j = 1, \ldots, n); 1 \leq \tau_1 \leq \tau_2 \leq \infty$ *;* $\mu_{k,i_k} > 0$ *and*

$$\mu_{k,i_k,0} = l_{k,i_k}\sigma_{k,i_k} - (v_k, \sigma_k) - (|\sigma_k| - |\varkappa_k|\, a)\frac{1}{p}. \qquad (2.13)$$

Then the following inequality holds for the function $F_T^i(x)$*, which is defined by inequality (2.2):*

$$\left\|F_T^i\right\|_{q,b,\varkappa,\ \tau_2;U} \leq C^1 \left\| \prod_{k\in e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u)\, D^{\overline{l}^i} f \right\|_{p,a,\varkappa,\ \tau_1;G} \qquad (2.14)$$

where $b = (b_1, b_2, \ldots, b_n)$*,* b_i *is an arbitrary number, satisfying the following condition:*

$$0 \leq b \leq 1, \ \text{if } \mu_{k,i_k,0} > 0;$$

$$0 \leq b < 1, \ \text{if } \mu_{k,i_k,0} = 0;$$

$$0 \leq b < a + \frac{\mu_{k,i_k} q (1-a)}{|\sigma_k| - |\varkappa_k|\, a} \ \text{ if } \ \mu_{k,i_k,0} < 0, \qquad (2.15)$$

where C^1 *is a constant independent of* f*.*

Proof. The idea of the proof is like that of lemma 1. Taking $\overline{x}\in U, 0<\rho_j<\infty$, (j=1,2,..,n) we must remark $\left\|F_T^i(x)\right\|_{q,\,U_{\rho^{\varkappa}}(\overline{x})}.(i=1,2,..,n)$ First as before, assume that, $0<\rho_j<T_j$. Then

$$\left\|F_T^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})}\leq\left\|F_\rho^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})}+\left\|F_{\rho T}^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})}\ . \tag{2.16}$$

Using (2.5) for $\eta=\rho$, we get

$$\left\|F_\rho^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})}\leq C^1\left\|\prod_{k\in e^i}t_k^{-\left|\beta_k^{i_k}\right|}\Delta^{2\omega}(u)\,D^{\overline{l}^i}f\right\|_{p,a,\varkappa,\tau;G}\prod_{k\in e^i}\rho_k^{\mu_{k,i_k}+\frac{|\varkappa_k|a}{p}}, \tag{2.17}$$

and using (2.6) for $\tau=\infty$, we get

$$\left\|F_{\rho T}^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})}\leq C^2\left\|\prod_{k\in e^i}t_k^{-\left|\beta_k^{i_k}\right|}\Delta^{2\omega}(u)\,D^{\overline{l}^i}f\right\|_{p,a,\ \varkappa}\varphi(\rho,\ T;r), \tag{2.18}$$

where $\varphi(\rho,\ T;r)=\prod_{k\in e_s}\rho_k^{\delta_k(r)}\prod_{k\in e_s/e^i}T_k^{\mu_{k,i_k}}\int_{\rho^i}^{T^i}\cdots\int_{\rho^i}^{T^i}\prod_{k\in e^i}t_k^{\mu_{k,i_k}(r)-1}dt_k$,

$\delta_k(r)=\frac{|\varkappa_k|}{q}-\frac{|\varkappa_k|}{r}(1-a)\,,\ \mu_{k,i_k}(r)=l_{k,i_k}\sigma_{k,i_k}-(v_k,\sigma_k)-(|\sigma_k|-|\varkappa_k|\,a)\left(\frac{1}{p}-\frac{1}{q}\right)$,
C^2 is a constant independent of f and ρ.

The estimation (2.17) is real for all r $(q\leq r\leq\infty)$. One can chose r such that, the indication of this remark would be maximal for its ρ order. To do it, we must note that, the parameters $\delta_k(r)$, however continue to decrease (monotone) in $[q,\infty]$, and

$$\mu_{k,i_k}(q)=\mu_{k,i_k},\mu_{k,i_k}(\infty)=\mu_{k,i_k,0}.$$

We consider two cases: $\mu_{k,i_k,0}\geq 0$ and $\mu_{k,i_k,0}<0$. If $\mu_{k,i_k,0}\geq 0$, then we can get the largest indication for maximal indication for $r=\infty$, for in To do it, we must note that, $\delta_k(\infty)=\frac{|\varkappa_k|}{q}$, then we get

$$\Psi(\rho,\ T;\infty)=\begin{cases}\prod_{k\in e_s}\mu_{k,i_k,0}\rho_k^{\frac{|\varkappa_k|}{q}}\prod_{k\in e^i}\left(T_k^{\mu_{k,i_k,0}}-\rho_k^{\mu_{k,i_k,0}}\right),\ \text{if}\ \mu_{k,i_k,0}>0,\\ \prod_{k\in e_s}\rho_k^{\frac{|\varkappa_k|}{q}}\prod_{k\in e^i}ln\frac{T_k}{\rho_k},\ \text{if}\ \mu_{k,i_k,0}=0.\end{cases}$$

Let $\mu_{k,i_k,0}<0$. Because $\mu_{k,i_k}(q)>0$, and $\mu_{k,i_k}(\infty)=\mu_{k,i_k,0}<0$, then for any r_0, $q<r_0<\infty$, we get $\mu_{k,i_k}(r_0)=0$. Then we can get the largest indication for maximal indication for hence, in addition, if we put $r=r_0$ in $\varphi(\rho,T;r)$, then we have:

$$\varphi(\rho,T;r_0)=\prod_{k\in e_s}\rho_k^{\delta_k(r_0)}\prod_{k\in e^i}\ln\frac{T_k}{\rho_k},$$

$$\delta(r_0)=\frac{|\varkappa_k|}{q}\left(1+\frac{\mu_{k,i_k,0}q(1-a)}{|\sigma_k|-|\varkappa_k|\,a}\right)=\frac{|\varkappa_k|}{q}\left(a+\frac{\mu_{k,i_k}q(1-a)}{|\sigma_k|-|\varkappa_k|\,a}\right)$$

Taking (2.15)-(2.17) for $\tau=\infty$, we find

$$\left\|F_T^i(x)\right\|_{p,U_{\rho^{\varkappa}}(\overline{x})} \le C^3 \left\| \prod_{k\in e_s/e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u) D^{\overline{l}^i} f \right\|_{p,a,\ \varkappa} \prod_{k\in e_s/e^i} T_k^{\frac{|\varkappa_k|a}{q}}$$

$$\times \begin{cases} \prod_{k\epsilon e_s} \rho_k^{\frac{|\varkappa_k|}{q}} \prod_{k\in e^i} T_k^{\mu_{k,i_k,0}}; \text{ if } \mu_{k,i_k,0} > 0 \\ \prod_{k\epsilon e_s} \rho_k^{\frac{|\varkappa_k|}{q}} \prod_{k\in e^i} \frac{T_k}{\rho_k}; \text{ if } \mu_{k,i_k,0} = 0. \\ \prod_{k\in e_s} \rho_k^{\delta_k(r_0)} \prod_{k\epsilon e^i} \frac{T_k}{\rho_k}\ ; \text{ if } \mu_{k,i_k,0} < 0 \end{cases} \tag{2.19}$$

Now let $\rho_j \ge T_j$, $j = 1,2,..,n$. Using (2.5) $(\eta_j = T_j,\ j = 1,2,..,n,\ \tau = \infty)$, we have

$$\left\|F_T^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C^4 \left\| \prod_{k\in e_s/e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u) D^{\overline{l}^i} f \right\|_{p,a,\varkappa}$$

$$\times \begin{cases} \prod_{k\in e_s} \rho_k^{\frac{|\varkappa_k|a}{q} + \mu_{k,i_k}}, \text{ if } T_j \le \rho_j \le 1,\ j = 1,2,..,n \\ \prod_{k\in e_s} T_k^{\mu_{k,i_k}},\ \ if\ 1 < \rho_j < \infty\ (j = 1,2,..,n) \end{cases} \tag{2.20}$$

From (2.19)-(2.20) for any $\overline{x} \in U$, ρ, $0 < \rho_j < \infty$,$(j = 1,2\cdots,n)$ we get

$$\left\|F_T^i(x)\right\|_{q,U_{\rho^{\varkappa}}(\overline{x})} \le C^5 \left\| \prod_{k\in e_s/e^i} t_k^{-\left|\beta_k^{i_k}\right|} \Delta^{2\omega}(u) D^{\overline{l}^i} f \right\|_{p,a,\ \varkappa} \prod_{k\in e_s} [\rho_k]_1^{\frac{|\varkappa_k|b}{q}},$$

where b are numbers, satisfying condition (2.15), and C^5 is constant independent of f, ρ and $\overline{x}$. From the last inequality and for $1 \le \tau_1 \le \tau_2 \le \infty$ follows (2.14). The lemma 2 justifies.

3 Main results.

Theorem 3.1 *Let $G \subset R^n$ be a domain with "σ-semi-horn" condition, that is, $G \in A(T^{\sigma})$, $1 \le p \le q \le \infty$, $1 < \theta < \infty$; $v = (v_1,\cdots,v_s)$; $v_i \ge 0$ are integers, $i = 1,2,\cdots,n$ and*

a) $v_{k,j} \ge l_{k,j}^0$ (j=$1,2,..,n_k,\ k\in e_s$);

b)$v_{k,j} \ge l_{k,j}^{i_k} + 1$; $(j \ne i_k)$ $v_{k,i_k} < l_{k,i_k}^{i_k} + 1$. Where $|\varkappa_k| < |\sigma_k|\,(k\in e_s)$; $1 \le \tau_1 \le \tau_2 \le \infty$, $f \in F_{p,\theta,a,\varkappa,\tau_1}^{<l>}(G;s)$ and let

$$\mu_{k,i_k} = l_{k,i_k}\sigma_{k,i_k} - (v_k,\sigma_k) - (|\sigma_k| - |\varkappa_k|\,a)\left(\frac{1}{p} - \frac{1}{q}\right) > 0 (i_k = 1,2,\cdots,n_k; k\in e_s).$$

Then there exits the generalized derivatives $D^v f$ on the domain G, which satisfying the inequality:

$$\|D^v f\|_{q,\ G} \le C^6 \sum_{i=(i_1,\cdots,i_s)\in Q} \left(\prod_{k\in e_s} T_k^{\mu_{k,i_k}} \|f\|_{F_{p,\theta,a,\varkappa,\tau_1}^{<l>}(G,s)} \right), \tag{3.1}$$

$$\|D^v f\|_{q,b,\varkappa,\tau_2:G} \le C^7 \|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau_1}(G:s)}, (p \le q < \infty). \quad (3.2)$$

In particular, if $\mu_{ki_k,0} > 0$, $(i_k = 1, 2, \cdots, n_k,\ \ k \in e_s)$ then the function $D^v f$ continuous on G and in addition

$$\sup_{x\in Q} |D^v f(x)| \le C^8 \sum_{i=(i_1,..,i_s)\in Q} \prod_{k\in e_s} T_k^{\mu_{k,i_k,0}} \|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau_1}(G,s)}. \quad (3.3)$$

Where $\mathrm{T}_k \in (0, \min(1, t_{0,k})]$, $(k \in e_s)$, $t_0 = (t_{0,1}, \ldots, t_{0,k})$ is a fixed, positive vector, b is an arbitrary number satisfying condition (2.15), C^6 and C^7 are constants independent of f, and C^6 independent of T.

The proof of theorem. Obviously, the generalized derivative $D^v f$ exits. It is clear that, if $\mu_{k,i_k} > 0$ $(i_k = 1, \cdots, n_k; k \in e_s)$, because of $p \le q, |\varkappa_k| < |\sigma_k|$ $(k \in e_s)$, $a \in [0, 1]$ then we can get $f \in F^{<L>}_{p,\theta,a,\varkappa,\tau_1}(G, s)$ then $F^{<l>}_{p,\theta}(G, s)$. It means that, the generalized derivative $D^v f$ exits, for $D^v f \in L_p(G)$ and for almost every point of $x \in G$, the integral representation is valid:

$$D^v f = \sum_{i=(i_1,\cdots,i_s)\in Q} (-1)^{|l^i - v|} C_i \prod_{k\in e_s/e^i} T_k^{-|\sigma_k| + \sigma_{k,i_k} \bar{l}_{k,i_k} - (v_k,\sigma_k)}$$

$$\times \int_0^{T^i} \cdots \int_0^{T^i} \prod_{k\in e^i} \frac{dt_k}{t_k^{1+|\sigma_k| - \sigma_{k,i_k}\bar{l}_{k,i_k} + (v_k,\sigma_k)}}$$

$$\times \int_{R^{|e^i|}} \int_{R^n} \Delta^{2\omega}(u) D^{\bar{l}^i} f(x+y) \Psi_{i,\sigma}\left(\frac{y}{(t^\sigma + T^\sigma)^i}, \frac{u}{(t^\sigma + T^\sigma)^i}\right) du dy. \quad (3.4)$$

Here $T^i = (T^i_1, \cdots, T^i_n)$, $T^i_j = T_j$, for $j \in e^i$, and $T^i_j = 0$, for $j \in e_s/e^i$, $|l^i - v| = \sum_{k\in e_s} |l^i_k - v_k|$, $\Psi_{i,\sigma}(\cdot, z) \in C_0^\infty(R^n)$. Henceforth, their carries are in $x + V(\sigma) \subset G$, from the Minkowski's inequality, we have:

$$\|D^v f\|_{q,G} \le C_1 \sum_{i=(i_1,\ldots,i_s)\in Q} \|F^i_T\|_{q;G}. \quad (3.5)$$

From (2.5) for $u = t$, $U = G$, $\eta = T, \rho \to \infty$ and for $p \le \theta$ we can obtain (3.1). Similarly, using (2.14), we can get (3.2).

Now let $\mu_{k,i_k,0} > 0 (i_k = 1, \cdots, n_k; k \in e^i)$. Now we must show that, $D^v f$ is continuous on G. From the inequality (3.4), taking the inequality (2.5), for $q = \infty, \mu_{k,i_k}(\infty) = \mu_{k,i_k,0} > 0, (i_k = 1, \cdots, n_k; k \in e_s)$ we can get:

$$\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} \le \sum_{i\in Q} \prod_{k\in e_s} T_k^{\mu_{k,i_k,0}}$$

$$\le \sum_{\substack{i = (i_1, \ldots, i_s) \in Q \\ i_k \ne 0}} \prod_{k\in e_s} T_k^{\mu_{k,i_k,0}} \left\{ \int_0^{t_0^i} \cdots \int_0^{t_0^i} \left\| \frac{\Delta^{2\omega}(t,\ G) D^{\bar{l}^i} f}{\prod_{k\in e^i} t_k^{|\beta_k^{i_k}|}} \right\|^\theta_{p,a,\varkappa,\tau_1} \prod_{k\in e_s} \frac{dt_k}{t_k} \right\}^{1/\theta}.$$

For $p \leq \theta$ we obtain

$$\|D^v f - D^v f_{T^\sigma}\|_{\infty,G} \leq \sum_{\substack{i=(i_1,\ldots,i_s)\in Q \\ i_k \neq 0}} \prod_{k\in e_s} T_k^{\mu_{k,i_k,0}}$$

$$\times \left\| \left\{ \int_0^{t_0^i} \cdots \int_0^{t_0^i} \left[\frac{\Delta^{2\omega}(t,G)\, D^{\overline{l}^i} f}{\prod_{k\in e^i} t_k^{\left|\beta_k^{i_k}\right|}} \right]^\theta \prod_{k\in e^i} \frac{dt_k}{t_k} \right\}^{1/\theta} \right\|_{p,a,\varkappa,\tau_1}.$$

It follows that, the left part of the last inequality tends to zero (vanishes), as $T_k \to 0$ $(k \in e_s)$. Because the function $D^v f_{T^\sigma}$ is continuous on G, then the convergence of $L_\infty(G)$ coincides with the absolutely convergence, and the limit function $D^v f$ is continuous on G. This completes the proof.

Let $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ be n dimensional vectors.

Theorem 3.2 *Let all conditions of theorem 1 be satisfied, besides $G \in A_\epsilon(T^\sigma)$. Then for $\mu_{k,i_k} > 0$, $(i_k = 1, 2, .., n_k, k \in e_s)$ the derivative $D^v f$ satisfies the Holder condition on the domain G, for metric of L_q, with indication ε. More precisely,*

$$\|\Delta(\gamma, G)\, D^v f\|_{q,G} \leq C_2 \|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau_1}(G:s)} \prod_{k\in e_s} |\gamma_k|^{\varepsilon_k}, \tag{3.6}$$

where ε is an arbitrary number satisfying the condition:

$$0 \leq \varepsilon_k \leq 1, \ \textit{if} \ \frac{\mu_k^0}{\sigma_k^0} > 1; \ (k \in e_s)$$

$$0 \leq \varepsilon_k < 1, \ \textit{if} \ \frac{\mu_k^0}{\sigma_k^0} = 1, (k \in e_s) \tag{3.7}$$

$$0 \leq \varepsilon_k < \frac{\mu_k^0}{\sigma_k^0}, \ \textit{if} \ \frac{\mu_k^0}{\sigma_k^0} < 1, (k \in e_s),$$

where $\mu_k^0 = \min \mu_{k,i_k}$ $(i_k = 1, 2, .., n_k,\ k \in e_s)$, $\sigma_k^0 = max\sigma_k$ $(k \in e_s)$.

In particular, if $\mu_{k,i_k,0} > 0, (i_k = 1, 2, \ldots, n_k,\ k \in e_s)$, then

$$sup_{x\in G} |\Delta(\gamma, G)\, D^v f(x)| \leq C_3 \|f\|_{F^{<l>}_{p,\theta,a,\varkappa,\tau}(G:\,s)} \prod_{k\in e_s} |\gamma_k|^{\varepsilon_k{}^0}, \tag{3.8}$$

where ε_k^0 satisfies the same condition, but we must substitute $\mu_{k,i_k,0}$ into μ_{k,i_k}. Here C_1, C_2 are constants independent of f and $\gamma(\gamma = \gamma_1, \ldots, \gamma_n)$.

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