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The fine structure of the spectrum of norm-normal operators

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Abstract. *In the paper, the fine structure of the spectrum of the normal operator is Banach space and the states in Taylor-Halberg sense is described. Some results on spectral properties of normal operators are obtained.*

Keywords. Banach space of normal operator, then structure of spectrum, states of operator, essential spectrum, Weyl's theorem.

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1 Introduction

In the note [1] the fine structure of the spectrum of Hermitian operators in Banach space is represented. The goal of the present paper to give description of the thin structure of the spectrum of normal operators in Banach space (see theorem 3.1). Hence we derive theorem 4.1 on states of Taylor-Halberg's normal operators in Banach space (Hilbert case [10, theorem 21]). At reflexivity of Banach space, the collection of states coincides with the case of Hilbert space. Some corollaries on spectral properties of normal operators are also obtained.

2 Preliminaries and notation

In what follows, $B(X)$ is algebra of linear bounded operators in Banach space X over the field of complex number C; $S(X)$ is a unit sphere in X; $*$is Banach conjugation. For the set M denote $\bar{M}$ closure, coM convex shell, $\overline{co}M$-convex closure. For the operator $T \in B(X)$ denote by $KerT$ and $RanT$ its kernel and image. We consider the following parts of the spectrum $\sigma(T)$ of the operator $T \in B(X)$: the point spectrum
$\sigma_p(T) = \{\lambda \in C : Ker(T - \lambda) \neq 0\}$; approximately point spectrum

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$\sigma_\pi(T) = \{\lambda \in C : \|(T-\lambda)x_n\| \to 0,\ n \to \infty,$ at some sequence $x_n \in S(X)\}$; approximately defective spectrum $\sigma_\delta(T) = \{\lambda \in C : Ran(T-\lambda) \neq X\}$; compression spectrum $\sigma_\gamma(T) = \left\{\lambda \in C : \overline{Ran}(T-\lambda) \neq X\right\}$; residual spectrum $\sigma_r(T) = \sigma_\gamma(T) - \sigma_p(T)$; Goldberg's essential spectrum: $\sigma_g(T) = \left\{\lambda \in C : Ran(T-\lambda) \neq \overline{Ran}(T-\lambda)\right\}$, is connected with other essential spectra: $\sigma_g(T) \subset \sigma_k(T) \subset \sigma_f(T) \subset \sigma_w(T) \subset \sigma_b(T)$, where $\sigma_k(T)$ is the Kato spectrum, $\sigma_f(T)$ Fredholm's spectrum, $\sigma_w(T)$ Weyl's spectrum, $\sigma_b(T)$ Brouder's spectrum (see [6, p. 305-306]). Denote by $\pi_{00}(T)$ $(\hat{\pi}_{00}(T))$ the set of all isolated eigen numbers of finite geometrical (algebraic) multiplicity [6, p. 219 and 229]. In general case, $\hat{\pi}_{00}(T) \subset \pi_{00}(T)$. $\pi_{0\nu}(T)$ denotes the set of all normally isolated eigen numbers λ of the operator T, where normality means that$X = Ker(T-\lambda) \oplus \overline{Ran}(T-\lambda)$, geometrical and algebraic multiplicity of the number λ were agreed.

It is said that for the essential spectrum $\sigma_e(T)$ $(e = k, f, w, b)$ the Weyl theorem (Weyl-type theorem) is true if $\sigma_e(T) = \sigma(T) - \pi_{00}(T)$ $(\sigma_e(T) = \sigma(T) - \hat{\pi}_{00}(T))$. The Bauer's or spatial numerical range of the operator $T \in B(X)$ is the set
$V(T) = \{\lambda \in C : \lambda = f(Tx), x \in S(X), f \in D(x, X)\}$, where
$D(x, X) = \{f \in S(X^*) : f(x) = 1,\ x \in S(X)\}$. In any Banach space X one can give a mapping $s = [,] : X \times X \to C$ such that $[x, y]$ is linear with respect to the first argument, $[x, x] = \|x\|^2$ and $|[x, y]| \leq \|x\| \cdot \|y\|$. The mapping [,] is said to be a semiinner product (sip) in X generating the norm in X. In general Banach space X such a sip is not unique and we denote their set by $\sum$. The Lumerian numerical range for $T \in B(X)$ responding to sip $[,] \in \sum$ is called $W_{[,]}(T) = \{\lambda \in C : \lambda = [Tx, x],\ x \in S(X)\}$.

$V(T) = \bigcup\limits_{[,] \in \Sigma} W_{[]}(T)$, and $\overline{co}W_{[,]}(T) = \overline{co}V(T)$ are valid for $T \in B(X)$ for any $[,] \in \sum$ (see [1]).

The operator $T \in B(X)$ is called norm-Hermitian or simply Hermitian if some $W_{[,]}(T) \subset R$ (real numbers). From the previous paragraph it follows that the Hermiticity of T is independent of the choice of sip, it depends on the norm. Therefore we speak on norm-Hermiticity.

If $T = H + iK$ where H, K are commuting Hermitian operators, then T is called norm-normal or in short normal. $r(T) = Sup\{|\lambda| : \lambda \in \sigma(T)\}$ $(v(T) = \sup\{|\lambda| : \lambda \in V(T)\})$ is called a spectral (numerical) radius of the operator $T \in B(X)$. For $T \in B(X)$ it is valid $r(T) \leq v(T) \leq \|T\|$, where both inequalities, generally speaking are strict.

3 Auxiliary lemmas.

For convenience we state the known facts that will be used later on. As $\lambda \in \sigma_j(T)$ is equivalent to $0 \in \sigma_j(T-\lambda)$ $j = p, \gamma, \pi, \delta$ for any $T \in B(X)$, then when proving lemmas 1,2 we can assume $\lambda = 0$.

Lemma 3.1 *For any $T \in B(X)$ we have*

$$\sigma_\pi(T) = \sigma_p(T) \bigcup \sigma_g(T). \tag{3.1}$$

If $0 \notin \sigma_\pi(T)$, then $0 \notin \sigma_p(T)$ is obvious, and for $0 \notin \sigma_g(T)$ it is necessary to prove $RanT = \overline{Ran}(T)$. As the preimage x_n of the fundamental sequence $y_n = Tx_n \in RanT$ is fundamental in X, then $RanT$ is complete, and consequently $RanT$ is closed in X.

If $0 \notin (\sigma_p(T) \bigcup \sigma_g(T))$, then there exists a bounded mapping of Banach spaces $T^{-1}: RanT \to X$. Therefore $\left\|T^{-1}y\right\| \le c\|y\|$ for $y = Tx \in RanT$ we have $\|x\| \le c\|Tx\|$, i.e. $0 \notin \sigma_\pi(T)$. Lemma is proved.

Lemma 3.2 *For any $T \in B(X)$ the following relations are true:*

$$\sigma_\pi(T^*) = \sigma_\delta(T) \tag{3.2}$$

$$\sigma_\pi(T) = \sigma_\delta(T^*) \tag{3.3}$$

$$\sigma_p(T^*) = \sigma_\gamma(T) \tag{3.4}$$

$$\sigma_p(T) \subset \sigma_\gamma(T^*) \tag{3.5}$$

and for reflexivity X

$$\sigma_p(T) = \sigma_\gamma(T^*). \tag{3.6}$$

Proof. (3.2) and (3.3) see [10, theorem 4,5,6]. The proof of (3.4) follows from equivalences $\overline{Ran}T = X \leftrightarrow (RanT)^\perp = 0 \leftrightarrow KerT^* = 0$ allowing for the equality $(RanT)^\perp = KerT^*$ [9, p. 112].

Show (3.6). If $0 \in \sigma_p(T)$, i.e. there will be found $x_0 \in X,\ x_0 \neq 0$ such that $Tx_0 = 0$. Then $(T^*\varphi)x_0 = \varphi(Tx_0) = 0$ for any $\varphi \in X^*$. Consequently $RanT^* \subset \{x_0\}^\perp$, where $\{x_0\}^\perp$ is a proper closed subspace in X^*. Therefore $0 \in \sigma_\gamma(T^*)$. For proving (3.6) we apply (3.4) to the operator T^*. Then $\sigma_p(T^{**}) = \sigma_\gamma(T^*)$ and because of reflexivity of X according to [2, p. 516, lemma 6] we have $T = T^{**}$.

Consequently $\sigma_p(T) = \sigma_\gamma(T^*)$. Lemma is proved.

A counterexample to importance of reflexivity X for (3.6) is in [8, p. 216], where T is the left shift in $X = l_1$, T^* is the right shift in $X^* = l_\infty$ and $\sigma_p(T) \neq \sigma_\gamma(T^*)$

Lemma 3.3 *If $H \in B(X)$ is Hermitian, then*

$$\left\|(H - i\mu)^{-1}\right\| \le |i\mu|^{-1} \ \ for\ \ \mu \neq 0,\ \ \mu \in R. \tag{3.7}$$

Proof. As for the set M of positive numbers $(\inf M)^{-1} = \sup(M)^{-1}$, then at $0 \notin \sigma(A)$ for $A \in B(X)$ it is easy to see

$$|A|^{-1} = \left\|A^{-1}\right\|. \tag{3.8}$$

Here $|A| = \inf\left\{\|Ax\| \cdot \|x\|^{-1},\ \ x \neq 0\right\}$. The Hermiticity of the operator H means $V(H) \subset R$ and by the Williams theorem $\sigma(H) \subset \overline{V}(H)$ [1a, p. 88, theorem 1]. Therefore $\sigma(H) \subset R$. For $f \in D(x, X)$ we have $|i\mu| \le |f(Hx) - i\mu| = |f[(H - i\mu)x]| \le ||(H - i\mu)x||$. From the inequality $|i\mu| \le \|(H - i\mu)x\|$ it follows $|i\mu| \le |H - i\mu|$, i.e. $|H - i\mu|^{-1} \le |i\mu|^{-1}$. Taking into account (3.8), from the previous inequality we obtain (3.7). The lemma is proved.

Lemma 3.4

$$KerT = Ker(H, K) \tag{3.9}$$

$$\overline{Ran}T = \overline{Ran}(H, K) \tag{3.10}$$

where $T = H + iK$ is a normal operator, $Ker(H, K) = KerH \bigcap KerK$, $Ran(H, K)$ is a linear chell of $RanH \bigcup RanK$.

Proof. (3.9) in [3, lemma 3]. Prove (3.10). The inclusion $\overline{RanT} \subset \overline{Ran}(H,K)$ is obvious. In order to prove $\overline{Ran}(H,K) \subset \overline{RanT}$ we assume the contrary, i.e. there exists $y \in \overline{Ran}(H,K) \backslash \overline{RanT}$. Then there exists $f \in X^*$ such that $f(y) = 1$ and $f\left(\overline{RanT}\right) = 0$ [2. p. 27]. Then $f \in KerT^* = Ker(H^*, K^*)$ according to (3.9) and normality of T^* (see the proof of theorem 3.1). Consequently, $f\left(\overline{Ran}(H,K)\right) = 0$ and this contradicts the inequality $f(y) = 1$. The lemma is proved.

The following Markov-Kakutini fixed point principle was proved in [2, p. 493] or [8, p. 172].

Lemma 3.5 *Let $X \supset K$ be a convex compact and J be the family of linear continuous mappings $K \to K$ commuting between themselves. Then there exists a point $p \in K$ such that $Tp = p$ for all $T \in J$.*

Recall that for linear subspaces Y, Z in X we say that Y is Birkhoff orthogonal to Z and write $Y \perp Z$ if $\|y\| \leq \|y+z\|$ for any $y \in Y,\ z \in Z$.

Lemma 3.6 *(Fong's theorem [3]). If $T \in B(X)$ is a normal operator, then $KerT \perp RanT$.*

The proof is based on construction of some projection. It is constructed according to power bounded operator $V \in B(X)$ by means of generalized Banach limit and projects X^* on the set of fixed points of the operator V^*. For details see [3, p. 164-167].

4 Thin structure of the spectrum

Describe the thin structure of the spectrum of normal operators in Banach spece. The case of Hermitian operator was announced in [11, p. 240]. This is shown through the codensity of numerical ranges of Hermitian operators [13, in p. 176]. For normal operators it is not so, and another approach is required.

Theorem 4.1 *For normal operators $T \in B(X)$ it is fulfilled*

$$\sigma_p(T) \subset \sigma_\gamma(T) \subset \sigma_\pi(T) \tag{4.1}$$

and for the reflexivity X

$$\sigma_p(T) = \sigma_\gamma(T) \tag{4.2}$$

The reflexivity condition X for (4.2) *may not be omitted.*

Proof. For proving $\sigma_p(T) \subset \sigma_\gamma(T)$ we give two approaches. The first approach uses the Fong theorem (lemma 2.6), and in the second proof vice versa, the Fong theorem will be a corollary [n. 5, corollary 5.2]. The second way was prompted by application of Markov.-Kakutani's theorem (see lemma 2.5) in the proof of theorem 4 from [1 b), p. 26]. For adaptation to our proof the normal operator $T = H_1 + iH_2$ should be considered as the family of Hermitian operators H_1, H_2 and estimation of the norm of the resolvent H_1, H_2(lemma 2.4).

I. The first proof $\sigma_p(T) \subset \sigma_\gamma(T)$.

At first we are convinced that for any $\lambda \in C$ the operator $T-\lambda$ is normal if T is normal. As
$T = H_1 + iH_2$, where H_1, H_2 are commuting Hermitian operators, then $T-\lambda = (H_1 - \alpha_1) + i(H_2 - \alpha_2)$, where $\lambda = \alpha_1 + i\alpha_2$. By $V(H_j - \alpha_j) = V(H_j) - \alpha_j$ the operators $H_j - \alpha_j, j = 1, 2$ are Hermitian, and they obviously commute.

According to Fong's theorem (lemma 2.6): $Ker\,(T-\lambda) \perp Ran\,(T-\lambda)$. Be convinced that
$Ker\,(T-\lambda) \perp \overline{Ran}\,(T-\lambda)$. For any $y \in \overline{Ran}\,(T-\lambda)$ there exists a sequence $y_n \in Ran\,(T-\lambda)$ such that $y_n \to y$ as $n \to \infty$. As for any $x \in Ker\,(T-\lambda)$ we have $\|x\| \leq \|x - y_n\|$ for any $n = 1, 2, 3, ...$, because of continuity of the norm in limit as $n \to \infty$ we have $\|x\| \leq \|x - y\|$. Assume that $\lambda \notin \sigma_\gamma(T)$, i.e. $\overline{Ran}\,(T-\lambda) = X$. Then $Ker\,(T-\lambda) \bigcap \overline{Ran}\,(T-\lambda) = 0$, as only the zero vector is orthogonal to itself. Consequently, $Ker\,(T-\lambda) = 0$, i.e. $\lambda \notin \sigma_p(T)$. The inclusion $\sigma_p(T) \subset \sigma_\gamma(T)$ is proved.

The second proof $\sigma_p(T) \subset \sigma_\gamma(T)$.

If suffices to prove that $0 \in \sigma_p(T)$ implies $0 \in \sigma_\gamma(T)$ for a normal $T = H_1 + iH_2$. If $0 \in \sigma_p(T)$, then $KerT \neq 0$ and by lemma 2.4 $KerT = Ker\,(H_1, H_2)$ and therefore there exists $u \in X,\ \|u\| = 1$ such that $H_j u = 0,\ j = 1, 2$. By lemma 2.3 for $H_j,\ j = 1, 2$ for any $\mu_j \in R,\ \mu_j \neq 0, j = 1, 2$ we have $\left\| i\mu_j (H_j - i\mu_j)^{-1} \right\| \leq 1$.

Then we have $u = (H_j - i\mu_j)^{-1}(H_j - i\mu_j)\,u = -i\mu_j (H_j - i\mu_j)^{-1} u$. As a result, $u = -i\mu_j (H_j - i\mu_j)^{-1} u$. Having denoted by $U_j = -i\mu_j (H_j - i\mu_j)^{-1}$, for any $u \in Ker\,(H_1, H_2)$, $\|u\| = 1$ we have $U_j u = u,\ j = 1, 2$.

Hence for $f \in D(u, X)$ it follows the equality $\left(U_j^* f\right)(u) = f(U_j u) = f(u) = 1$ and the estimation

$$\left\|U_j^* f\right\| \leq \left\|U_j^*\right\| \cdot \|f\| = \left\|U_j^*\right\| = \|U_j\| \leq 1.$$

We get a commuting collection of linear *-weakly continuous operators $U_j^* : D(u, X) \to D(u, X)$, where $D(u, X)$ is a convex *-weak compact [1 b) p. 27]. By Markov-Kakutani's theorem (lemma 2.5) there exists $g \in D(u, X)$ such that$U_j^* g = g,\ \ j = 1, 2$.

Then $\left[-i\mu_j (H_j - i\mu_j)^{-1}\right]^* g = g$ implies $i\mu_j \left[(H_j - i\mu_j)^*\right]^{-1} g = g$. Hence $i\mu_j g = = (H_j - i\mu_j)^* g$ or $i\mu_j g = H_j^* g + i\mu_j g$ or, finally $H_j^* g = 0, j = 1, 2$. By the normality of T^* (see later on a part of the proof) by lemma 2.4 we have $KerT^* = KerH_1^* \bigcap KerH_2^*$. Consequently $g \in KerT^*,\ g \neq 0$ i.e. $0 \in \sigma_p(T^*)$. Thus, by (3.4) from lemma 2.2 shows that $0 \in \sigma_\gamma(T)$.

II. Now prove the inclusion $\sigma_\gamma(T) \subset \sigma_\pi(T)$

Let $\lambda \in \sigma_\gamma(T)$. Then by (3.4) from lemma 2.2 the number $\lambda \in \sigma_p(T^*)$. It is easy to see that $T^* \in B(X^*)$ is a normal operator. Indeed, $T^* = (H_1 + iH_2)^* = H_1^* - iH_2^*$, where $H_j^*,\ j = 1, 2$ are Hermitian operators as $V\left(H_j^*\right) \subset \overline{V}(H_j)$, $j = 1, 2$ [1 b) p. 12]. Furthermore, $H_1^* \cdot H_2^* = (H_2 H_1)^* = (H_1 H_2)^* = H_2^* H_1^*$ and H_1^* commutes with H_2^*. By the normality of T^*, according to the first part of the theorem, from the inclusion $\lambda \in \sigma_p(T^*)$ it follows $\lambda \in \sigma_\gamma(T^*)$, and the more so $\lambda \in \sigma_\delta(T^*)$. By (3.3) in lemma 2.2 $\sigma_\delta(T^*) = \sigma_\pi(T)$. Consequently $\lambda \in \sigma_\pi(T)$ and the chain of inclusions (4.1) is proved.

III. Establish the equality $\sigma_p(T) = \sigma_\gamma(T)$ at reflexivity of X for a normal $T \in B(X)$ and show that the requirement of reflexivity of X is not excessive.

Taking into account part 1 of the theorem, it suffices to prove $\sigma_\gamma(T) \subset \sigma_p(T)$ for the reflexive X. If $\lambda \in \sigma_\gamma(T)$, then $\lambda \in \sigma_p(T^*)$ according to (3.4) from lemma 2.2. By the first part of the theorem, $\lambda \in \sigma_\gamma(T^*)$ for a normal T^*. The reflexivity of X by (3.6) of lemma 2.2 implies $\lambda \in \sigma_p(T)$. Equality (4.2) is proved.

Consider a nonreflexsive Banach space $X = C[0,1]$ of all continuous functions $x : [0,1] \to C$ with the norm $\|x\| = \max\{|x(t)| : t \in [0,1]\}$. X is a commutative C^* algebra by multiplication $x \cdot y = x(t) \cdot y(t)\ ,\ t \in [0,1]$, involution $x^* = \overline{x(t)}, t \in [0,1]$ and a unit $x(t) \equiv 1$.

We take an operator $L_y : X \to X,\ \ L_y x = t \cdot x(t)$ of multiplication $x \in X$ by the element
$y(t) = t,\ t \in [0,1]$ [2, p. 621].

Be convinced in Hermiticity of L_y. For that we use the equality: $V(L_y) = \mathcal{V}(y) = \{f(y) : f \in B(X)^*, \|f\| = f(e) = 1\}$ is an algebraic numerical range of the element $y \in X$ [1 a), p. 15]. Indeed, if in definition $V(L_y) = \{f(L_y x) : f \in D(x, X), x \in S(X)\}$ we take $x = e$, then $\mathcal{V}(y) \subset V(L_y)$. Conversely, if $\lambda \in V(L_y)$, then $\lambda = f(L_y x_0)$ for some $x_0 \in S(X)$. We $g(y) = f(L_y x_0)$. Then $g(e) = \|g\| = 1$ and $\lambda = g(y)$, i.e. $\lambda \in \mathcal{V}(y)$. As $y(t) = t,\ t \in [0,1]$ will be Hermitian element $y^* = y$ [1 a, p. 47], then $\mathcal{V}(y) \subset R$ and therefore $V(L_y) \subset R$.

Now describe the spectrum $\sigma(L_y)$. Taking into account $\sigma(L_y) = \sigma(y)$ [9, p. 281], from the Gelfond theory of commutative Banach algebras for $y \in C[0,1]$ we have $\sigma(y) = [0,1]$ [9, p. 304]. Therefore $\sigma(L_y) = [0,1]$.

Finally show that $\sigma_p(L_y)$ is empty and $\sigma_\gamma(L_y) = [0,1]$.

For $\lambda \in \sigma(L_y)$ from $(L_y - \lambda)x = (t - \lambda)x(t) = 0$ for $t \in [0,1]$ it follows $x(t) \equiv 0$, i.e. $\lambda \notin \sigma_p(L_y)$. Therefore there exists an operator $(L_y - \lambda)^{-1} : Ran(L_y - \lambda) \to X$, where $Ran(L_y - \lambda) = \{x(t) : x(t) = (t - \lambda)x(t)\} = \{x(t) : x(\lambda) = 0\}$. Therefore $\overline{Ran}(L_y - \lambda) \neq X$, i.e. $\lambda \in \sigma_\gamma(L_y) = \sigma_r(L_y) = [0,1]$. Thus, there exist normal operators $T \in B(X)$ in nonreflexive X, for which $\sigma_p(T) \neq \sigma_\gamma(T)$. Theorem is proved

Remark 4.1 Ignoring the theory of Banach algebras, we can determine the equality $\sigma(L_y) = [0,1]$ by elementary way from the inclusions $\sigma_\pi(L_y) \subset [0,1] \subset \sigma(L_y)$.

5 The state of normal operators.

By considering such fundamental issues in theory of operators (as solvability of operator equations, condition of uniqueness of their solutions, existence of the inverse operator and its continuity) the relations between the spectra (of their parts) of the operator $T \in B(X)$ and its conjugated T^* is used.

In the paper [10] the notion of the state of operators $T \in B(X)$ is introduced. Interrelations between the properties of the operators T and T^* are analyzed by its means. The results of this analysis are applied to consideration of the spectra of the operators T and T^* and the states of normal operators in Hilbert space are described [10, theorem 21].

Before describing the Taylor-Halberg states of normal operators $T \in B(X)$ we give a brief information on the states of an arbitrary operator $T \in B(X)$. One can see details in the paper [10].

For any operator $T \in B(X)$ in the normed space X with respect to the image $RanT = \{Tx : x \in X\}$ the following cases are possible: $I.\, RanT = X$; $II.\, RanT \neq X$; but $\overline{Ran}T = X$; $III.\, \overline{Ran}T \neq X$.

With respect to the inverse operator T^{-1} the following cases are possible:
1. T^{-1} exists and is continuous; 2. T^{-1} exists and is discontinuous; 3. T^{-1} doesn't exist.

If the operator satisfies the condition I, we write $T \in I$ (similarly for the conditions II and III). If $T \in I$ satisfies condition 1, then we write $T \in I_1$ (similarly for conditions 2 and 3). The notation $T \in I_1$ means that $T \in I$ and $T \in 1$, i.e. conditions I and 1 are fulfilled simultaneously (the similar notation is used for other combinations of I, II, III with conditions 1,2,3).

Thus, we obtained 9 possible conditions for the operator T responding to all possible combinations of conditions I, II, III with conditions 1,2,3.

This classification scheme is applicable also to the conjugate operator $T^* \in B(X^*)$. If we take an ordered pair (T, T^*), then conditions on T and conditions on T^* give ordered pair of conditions that is called the state of the pair (T, T^*). For example, the notation $(T, T^*) \in (II_3, III_1)$ means that the state of the pair (T, T^*) is (II_3, III_1), i.e. $T \in II_3$ and $T^* \in III_1$.

The states give a more exact information on the relation of the parts of the spectrum $\sigma(T)$ with the parts of the spectrum $\sigma(T^*)$. At usual rough partition, $\sigma(T) = \sigma_p(T) \bigcup \sigma_r(T) \bigcup \sigma_c(T)$ [2, p. 620], the finer partition $\sigma(T)$ is ignored. The use of states divides $\sigma_p(T)$ into three more parts, while $\sigma_r(T)$ into two more parts. More exactly, $\lambda \in \sigma_p(T) \leftrightarrow (T - \lambda) \in (I_3 \bigcup II_3 \bigcup III_3)$, $\lambda \in \sigma_r(T) \leftrightarrow (T - \lambda) \in (III_1 \bigcup III_2)$, $\lambda \in \sigma_c(T) \leftrightarrow (T - \lambda) \in II_2$.

Using theorem 3.1 on fine structure of the spectrum of normal operator, we describe the states of normal operators in Banach space. The following theorem generalizes into Banach space the Hilbert space, represented in [10]. In general case they differ.

Theorem 5.1 *The normal operator $T \in B(X)$ in arbitrary Banach space has the following possible states of the pair (T, T^*):*

$$(I_1, I_1)\ ,\ (II_2, II_2)\ ,\ (III_3, III_3)\ ,\ (II_2, III_2)\ ,\ (III_2, III_3)\ , \tag{5.1}$$

If X is a reflexive Banach space, then the normal operator $T \in B(X)$ may have the following states of the pair (T, T^*):

$$(I_1, I_1)\ ,\ (II_2, II_2)\ ,\ (III_3, III_3)\,. \tag{5.2}$$

Proof. As shown in [10], from 81 a priori possible states of (T, T^*) 16 states may be really possible, if on the normed space X or on $T \in B(X)$ additional restritions are not imposed. When X is Banach (restriction of completeness on the normed space X), then for an arbitrary operator $T \in B(X)$ the amount of really possible states of the pair (T, T^*) equals 9:

$$(I_1, I_1)\ ,\ (II_2, II_2)\ ,\ (III_3, III_3)\ ,\ (I_3, III_1)\ ,\ (III_1, I_3)\ , \tag{5.3}$$
$$(II_3, III_2)\ ,\ (III_2, II_3)\ ,\ (II_2, III_2)\ ,\ (III_2, III_3)\,.$$

Furthermore if the Banach space X is reflexive, then in this case the states (II_2, III_2) (III_2, III_3) are impossible. Therefore, in reflexive Banach space X the number of really possible states of the pair (T, T^*) in (5.3) will descrease up 7:

$$(I_1, I_1), (II_2, II_2), (III_3, III_3), (I_3, III_1), (III_1, I_3),$$
$$(II_3, III_2), (III_2, II_3). \tag{5.4}$$

Consider how theorem 3.1 on fine structure of the spectrum of a normal operator$T \in B(X)$ influences on the collections (5.3) and (5.4) of the states of the pair (T, T^*).

By theorem 3.1 for the normal operator $T \in B(X)$ we have the inclusion $\sigma_p(T) \subset \sigma_\gamma(T)$. Therefore the state I_3 indicating $0 \notin \sigma_\delta(T)$ and $0 \in \sigma_p(T)$ is excluded, as the inclusion $\sigma_\gamma(T) \subset \sigma_\delta(T)$ for any $T \in B(X)$ contradicts I_3.

In the same way, the inclusion $\sigma_p(T) \subset \sigma_\gamma(T)$ for the normal operator T excludes the state II_3 that indicates $0 \notin \sigma_\gamma(T)$ and $0 \in \sigma_p(T)$.

Furthermore, from theorem 3.1, for the normal operator T the inclusion $\sigma_\gamma(T) \subset \sigma_\pi(T)$ is true. Then the state III_1 indicating $0 \in \sigma_\gamma(T)$ and $0 \notin \sigma_\pi(T)$, is impossible.

Thus, for a normal operator $T \in B(X)$ in any space X the states I_1, II_2, III_2, III_3 are possible. Consequently, from the collection of states (5.3) the pairs (T, T^*) for a normal T in arbitrary Banach space only the states of pairs from (5.1) are possible.

Now if X is a reflexive Banach space and T is normal, then by theorem 3.1 we have the equality $\sigma_p(T) = \sigma_\gamma(T)$. This equality contradicts the state III_2 that is expressed by the relations $0 \in \sigma_\gamma(T)$ and $0 \notin \sigma_p(T)$. Therefore, for normal operators $T \in B(X)$ in reflexive Banach space X three pairs from (5.2) are possible. Theorem 4.1 is proved

In Hilbert space, from the second part of theorem 4.1 we obtain theorem 21 from [10]. We once more underline that in general Banach space, a normal operator has five pairs of possible states instead of three pairs in Hilbert space. Coincidence with the pairs of states in Hilbert space happens at reflexivity of Banach space.

Remark 5.1 For Hermitian operator $T \in B(X)$ in any X the possible states of pairs (T, T^*) coincide with the collection (5.1) while at reflexivity of X with the collection of realized states will be (5.2).

This remark correct typographical misprints in [13, in p. 176, theorem 4.1].

6 Corollary of theorem 3.1

Now we focus on a number of conclusions from the theorem on fine structure of the spectrum of normal operators $T \in B(X)$.

The first corollary is connected with H. Weyl's classic criterion on the spectrum of Hermitian operators in Hilbert space.

Corollary 6.1 *For a normal operator $T \in B(X)$ we have*

$$\sigma(T) = \sigma_\pi(T) = \sigma_\delta(T). \tag{6.1}$$

In the reflexive X the residual spectrum of the normal $T \in B(X)$ is absent

$$\sigma_r(T) = \emptyset. \tag{6.2}$$

Proof. For any $T \in B(X)$ we have $\sigma(T) = \sigma_\gamma(T) \bigcup \sigma_\pi(T)$ and by theorem 3.1 for a normal T the inclusion $\sigma_\gamma(T) \subset \sigma_\pi(T)$ is true. Consequently, the left equality in (6.1) is valid. As the spectrum of any operator is invariant with respect to Banach conjugation, and as shown in theorem 3.1, the operator T^* is normal in X^*, then $\sigma(T) = \sigma_\pi(T^*)$. By lemma 3.2 for any $T \in B(X)$ it is valid $\sigma_\pi(T^*) = \sigma_\delta(T)$. Consequently, the equality $\sigma(T) = \sigma_\delta(T)$ for a normal T is proved. The emptiness $\sigma_r(T)$ for a normal T in reflexive X immediately follows from the equality $\sigma_p(T) = \sigma_\gamma(T)$ in theorem 3.1. Corollary is proved.

In particular, the well known result in Hilbert space [9, p. 365, 18 a], proved by the spectral expansion of a normal operator is obtained. At Hilbert hermiticity this reduces to H. Weyl's classic result [8, p. 217].

From theorem 3.1 it follows the main result of the paper [3, theorem A] proved there by the generalized Banach limit.

Corollary 6.2 *For a normal* $T \in B(X)$ *we have* $KerT \perp \overline{Ran}T$.

Proof. By theorem 3.1 $g(H_j x) = \left(H_j^* g\right) x = 0$ for all $x \in X$, where $g \in D(u, X)$, i.e. $g(u) = \|g\| = \|u\| = 1$. Then $g(Ran(H_1, H_2)) = 0$.

Therefore for $u \in Ker(H_1, H_2)$, $\|u\| = 1$ and $y \in Ran(H_1, H_2)$ we have

$$\|u\| = |g(u + y)| \le \|g\| \cdot \|u + y\| .$$

This means Birkhoff orthogonality $Ker(H_1, H_2) \perp Ran(H_1, H_2)$ that implies $Ker(H_1, H_2) \perp \overline{Ran}(H_1, H_2)$.

Applying (3.9) and (3.10) from lemma 2.4 to previous orthogonality, we have $KerT \perp \overline{Ran}T$.

Theorem 3.1 admits to optimize localization of the spectrum and the compression spectrum of a normal operator $T \in B(X)$ by numerical ranges of the operator T. For any operator $T \in B(X)$ by the Williams localization theorem [1 a, p. 88 theorem 1], we have $\sigma(T) \subset \overline{V}(T)$. As shown by the author in [13 a] in reflexive Banach space X it is fulfilled $\sigma_\gamma(T) \subset V(T)$. Both localization relations, generally speaking are not valid for Lumerian numerical ranges $W_s(T)$. We can replace in these two inclusions $V(T)$ by $W_s(T)$ under additional smoothness condition of X. For normal operators $T \in B(X)$ theorem 3.1 enables such a replacement without restriction on geometry of space X.

Corollary 6.3 *For a normal operator* $T \in B(X)$ *in any Banach space* X *the following localization inclusions are valid:*

$$\sigma(T) \subset \bigcap_s \overline{W}_s(T) \tag{6.3}$$

and at reflexivity

$$\sigma_\gamma(T) \subset \bigcap_s W_s(T) \tag{6.4}$$

where in both cases the intersection is taken on all possible semi inner products agreed with the norm in X.

Proof. (6.3) follows from Winter-Lumer's localization theorem $\sigma_\pi(T) \subset \overline{W}_s(T)$ [1 a, p. 89 Theorem 5] for an arbitrary $T \in B(X)$ in any X for any $s \in \Sigma$ and equality $\sigma_\pi(T) = \sigma(T)$ for a normal $T \in B(X)$ (see Corollary 5.1).

(6.4) follows from the easily provable localization relation $\sigma_p(T) \subset \bigcap_s W_s(T)$ for any $T \in B(X)$ in arbitrary X and equality $\sigma_p(T) = \sigma_\gamma(T)$ from theorem 3.1 for a normal T in reflexive X.

The next corollary is connected with essential spectra of normal operator $T \in B(X)$. Its first part is used in Corollary 5.5, and the second part gives another proof of the completed result from [5, p. 1056, theorem 4.10].

Corollary 6.4 *For a normal operator $T \in B(X)$ the following equalities are true*

$$a)\sigma(T) = \sigma_g(T) \bigcup \sigma_p(T) \tag{6.5}$$

$$b)\sigma_w(T) = \sigma_b(T) = \sigma(T) - \hat{\pi}_{00}(T) \tag{6.6}$$

Proof. Proof (6.5) follows from (6.1) of corollary 5.1 and lemma 2.1 of point 2. For proving (6.6) recall that T - λ normal for any $\lambda \in C$ (see the proof of theorem 3.1) and by corollary 2 $Ker(T-\lambda) \perp Ran(T-\lambda)$. Then $Ker(T-\lambda)^2 = Ker(T-\lambda)$ [1 b, lemma 20.2], i.e. the ascent of the operator equals 0 or 1. Therefore by Werner's theorem [14, p. 469] for $\sigma_w(T)$ (6.6) is true. Coincidence $\sigma_w(T) = \sigma_b(T)$ follows from the characterization $\sigma_b(T) = \sigma(T) - \hat{\pi}_{00}(T)$ of the spectrum $\sigma_b(T)$ for any $T \in B(X)$ [4, p. 775]. The corollary is proved.

Remark 6.1 As in the case of Hermitian $T \in B(X)$ [13 b p. 177, theorem 4.3 and p. 178 corollary 4.4] we can make precise and complement corollary 5.4. More exactly, $\sigma_p(T)$ in (6.5) may be replaced by $\pi_{0v}(T)$. In equality (6.6) the Kato, Fredholm, Weyl, Browder spectra coincide, and $\hat{\pi}_{00}(T)$ may be replaced by $\pi_{00}(T)$.

It is known that a compact normal operator T in Hilbert space possesses an eigen value achieving $\|T\|$ [9, p. 352, theorem 12,31]. The proof of this fact uses spectral expansion of a normal operator not suitable in Banach space. We give appropriate result for normal compact operators in Banach space with another proof.

Corollary 6.5 *For a nonzero normal compact operator $T \in B(X)$ there exists $\lambda_0 \in \sigma_p(X)$ such that $|\lambda| = v(T)$, where $v(T)$ is a numerical radius of T.*

Proof. As for a normal $T \in B(X)$ we have $r(T) = v(T)$ [7, p. 389], then there exists $\lambda_0 \in \sigma(T)$ such that $|\lambda_0| = v(T)$. From $T \neq 0$ it follows $v(T) \neq 0$ and therefore $\lambda_0 \neq 0$. By the compactness of T for $\lambda_0 \neq 0$ is closed $Ran(T-\lambda_0)$ [9, p. 121, theorem 4.23], i.e. $\lambda_0 \notin \sigma_g(T)$. By (6.5) from corollary 5.4 it follows $\lambda_0 \in \sigma_p(T)$. The corollary is proved.

Remark 6.2 There exist normal operators $T \in B(X)$ for which $r(T) < \|T\|$ [1 a, p. 125]. Therefore, by passing to Banach space, in corollary 5.5 the equality $|\lambda| = \|T\|$ is replaced by $|\lambda| = v(T)$. As for Hermitian $T \in B(X)$ it is valid $r(T) = \|T\|$ [1 b), p. 72], then we have the exact analog of the Hilbert case.

As in the case of Hermitian $T \in B(X)$ [13 b, p. 178, corollary 4.7] in corollary 5.5 we can replace $\lambda_0 \in \sigma_p(T)$ by $\lambda_0 \in \pi_{0v}(T)$.

Finally, show the influence of theorem 3.1 on the structure of invariant subspaces. By applying theorem 3.1, we distinguish normal operator in reflexive Banach space whose invariant subspaces are regular. The invariant subspace M with respect to $T \in B(X)$ is called regular if $\overline{T(M)} = M$. The operator $T \in B(X)$ is called invariant normal if for any T-invariant space M its contraction T_M is normal.

Corollary 6.6 *If $T \in B(X)$ is an injective invariantly normal operator, in reflexive X, then any T-invariant subspace is regular.*

Proof. Let T_M be contraction of T -on any T is invariant subspace M. As T injective, then T_M is injective, and by the condition T_M is normal. The heredity of reflexivity by Pettis's theorem [2, p. 79, theorem 23] implies reflexivity of M. According to (4.2) from theorem 3.1, applied to T_M gives $\overline{T(M)} = M$ and the regularity of M is shown.

Remark 6.3 Analogs of corollary 5.6 are the component while solving the problems of approximation of the inverse operators for different classes of operator [12, p. 75, theorem 4.2].

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