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On approximation theorems for Bernstein-Chlodovsky polynomials

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Abstract. *In this paper considered analog of Bernstein operator given by some sequence converges to infinity. For the generalized Bernstein polynomial Bernstein type approximation theorems is proved for derivative of given function.*

Keywords. generalized Bernstein operator, approximation theorem, analog of Bernstein type theorem, derivatie of function

Mathematics Subject Classification (2010): 41A17, 41A35.

1 Introduction

In the paper we prove approximation theorems for Bernstein-Chlodovsky polynomial.

The polynomial constructed by Bernstein in 1912 for a continuous function has the form

$$\overline{B_n}\left(f;x\right)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)C_n^k\left(x\right)^k\left(1-x\right)^{n-k},0\leq x\leq 1,k=0,1,...n.$$

In 1932, Bernstein's follower Chlodovsky had constructed the increasing sequences of polynomials

$$B_n\left(f;x\right)=\sum_{k=0}^{n}f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k},\ 0\leq x\leq b_n, \tag{1.1}$$

that was named as Bernstein-Chlodovsky polynomials some later.
Here $\lim\limits_{n\to\infty}b_n=\infty$, $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$.

Note that, the condition imposed on b_n in the definition of classic Bernstein-Khladovsky operators doesn't provide continuous convergence of $B_n\left(f;x\right)$ polynomials to the function $f\left(x\right)$. We henceforth expect b_n is monotone non-negative increasing sequence satisfy the condition $\lim\limits_{n\to\infty}b_n=\infty$.

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2 Preliminaries and auxilary results

For the function $f(t) = t^2$ we have,

$$B_n(f;x) - x^2 = \frac{x(b_n - x)}{n}.$$

Therefore,

$$\sup_{x\in[0.b_n]} \left|B_n\left(t^2;x\right) - x^2\right| = \frac{b_n^2}{4n}.$$

Then for convergence to zero as $n \to \infty$ of the right hand part of this equality does not suffice the fulfillment of condition $\frac{b_n}{n} \to \infty$.
We impose the stronger condition on b_n

$$\lim_{n\to\infty} M_n(f)\frac{b_n^2}{n} = 0, \tag{2.1}$$

where $M_n(f) = \sup\limits_{x\in[0,\,b_n]} |f(x)|$.

Note that in the papers [1], [2], [3], related directly to the Bernstein-Chlodowsky polynomials and their generalizations and the paper [5], generalizing the results of the paper [6] on infinitely increasing interval from which we can also conclude a number of theorems on these polynomials. Some main proportions of the polynomials (1.1) were stated in the monograph [4].

Lemma 2.1 *For Bernstein-Chlodovski polynom* (1.1) *the following properties*

$$B_n(1;x) = 1,$$

$$B_n(t;x) = x,$$

$$B_n\left(t^2;x\right) = x^2 + \frac{x(b_n - x)}{n}$$

is hold.

Theorem 2.1 *Let the function f' be uniformly continuous on the semiaxis and $B_n(f;x)$ be Bernstein-Chlodowsky polynomials of order n for the function f and $\lim\limits_{n\to\infty} \frac{b_n}{\sqrt{n}} = 0$. Then for any $x \in [0, b_n]$*

$$|B_n(f;x) - f(x)| \le \frac{3}{2}\omega\left(f;\frac{b_n}{\sqrt{n}}\right).$$

Theorem 2.2 *Let $f \in C^2[0, b_n]$ and $B_n(f;x)$ be the Bernstein-Chlodowsky polynomials for f. Then for every $x \in [0, b_n]$ the asymptotical equality*

$$B_n(f;x) = f(x) + \frac{f''(x)}{2n}x(b_n - x) + \frac{\rho_n}{n},\ \rho_n \to 0,\ n \to \infty$$

is exact.

Remark 2.1 Note that Theorem 2.1 and Theorem 2.2 was proved in [1].

Theorem 2.3 *Let f is a function defined on $[0,\infty)$, uniformly continuous and differentiable. Let the sequence $\{b_n\}$ satisfy condition (2.1).*

Then for any $x \in [0, A]$, $A \geq 0$

$$\lim_{n\to\infty} B_n'(f;x) = f'(x),$$

where $B_n'(f;x) = \frac{d}{dx}B_n(f;x)$.

Proof. Let $0 < x < b_n$. Differentiating $B_n(f;x)$ we find

$$B_n'(f;x) = \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)\times$$

$$C_n^k\left\{\frac{k}{b_n}\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-k} - \frac{(n-k)}{b_n}\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k-1}\right\}$$

$$= \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-k-1}\left\{\frac{k}{b_n}\left(1-\frac{x}{b_n}\right) - \frac{(n-k)}{b_n}\left(\frac{x}{b_n}\right)\right\}$$

$$= \sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-k-1}\left\{\frac{k}{b_n} - \frac{kx}{b_n^2} - \frac{nx}{b_n^2} + \frac{kx}{b_n^2}\right\}$$

$$= \frac{1}{b_n^2}\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)(kb_n - nx)\,C_n^k\left(\frac{x}{b_n}\right)^{k-1}\left(1-\frac{x}{b_n}\right)^{n-k-1}.$$

Since

$$B_n'(f;x) = \frac{1}{x(b_n-x)}\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k}(kb_n - nx)$$

$$= \frac{n}{x(b_n-x)}\sum_{k=0}^{n} f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k}\left(\frac{kb_n}{n} - x\right).$$

From Lemma 2.1, we have

$$\sum_{k=0}^{n}\left(\frac{kb_n}{n} - x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^{k}\left(1-\frac{x}{b_n}\right)^{n-k} = \frac{x(b_n-x)}{n}.$$

We denote

$$\lambda\left(\frac{kb_n}{n}\right) = \begin{cases}\frac{f\left(\frac{kb_n}{n}\right)-f(x)}{\frac{kb_n}{n}-x} - f'(x), & \text{for } x \neq \frac{kb_n}{n},\ k=0,1,\ldots,n,\\ 0, & \text{for } \ x = \frac{kb_n}{n},\ k=1,\ldots,n-1.\end{cases}$$

Hence we get

$$f\left(\frac{kb_n}{n}\right) = f(x) + \left(f'(x) + \lambda\left(\frac{kb_n}{n}\right)\right)\left(\frac{kb_n}{n} - x\right),\quad k=0,1,\ldots,n.$$

By virtue of Lemma 2.1, we have

$$B_n'(f;x) = f'(x) +$$

$$\frac{n}{x\left(b_n-x\right)}\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}. \tag{2.2}$$

We represent (2.2) as

$$B_n'\left(f;x\right)=f'\left(x\right)+$$

$$\frac{n}{x\left(b_n-x\right)}\sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$+\frac{n}{x\left(b_n-x\right)}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}. \tag{2.3}$$

(2.3) implies that

$$B_n'\left(f;x\right)-f'\left(x\right)=$$

$$\frac{n}{x\left(b_n-x\right)}\sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$+\frac{n}{x\left(b_n-x\right)}\sum_{\left|\frac{kb_n}{n}-x\right|\geq\delta}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$=\frac{n}{x\left(b_n-x\right)}\left(I_1+I_2\right).$$

We estimate I_1. By virtue of differentiability of function f in point x, for arbitrary $\varepsilon>0$ we may choose $\delta_\varepsilon>0$ such that

$$\left|\lambda\left(\frac{kb_n}{n}\right)\right|<\frac{\varepsilon}{2},\quad \left|\frac{kb_n}{n}-x\right|<\delta_\varepsilon.$$

In particular, we may take $\delta_\varepsilon=\delta$. Therefore , we have

$$\left|\mathrm{I}_1\right|\leq\sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\left|\lambda\left(\frac{kb_n}{n}\right)\right|\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$<\frac{\varepsilon}{2}\sum_{\left|\frac{kb_n}{n}-x\right|<\delta}\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

$$<\frac{\varepsilon}{2}\sum_{k=0}^{n}\left(\frac{kb_n}{n}-x\right)^2 C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}=\frac{x\left(b_n-x\right)}{n}\frac{\varepsilon}{2}.$$

Hence

$$\frac{n}{x\left(b_n-x\right)}\left|I_1\right|<\frac{\varepsilon}{2}. \tag{2.4}$$

Now we estimate I_2. Further, we have

$$\left|\lambda\left(\frac{kb_n}{n}\right)\right|\leq\frac{\left|f\left(\frac{kb_n}{n}\right)-f\left(x\right)\right|}{\left|\frac{kb_n}{n}-x\right|}+\left|f'\left(x\right)\right|\leq\frac{2M_n(f)}{\delta}+\left|f'\left(x\right)\right|.$$

We get

$$|I_2| \le \sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta} \left|\lambda\left(\frac{kb_n}{n}\right)\right| \left(\frac{kb_n}{n}-x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\le \left(\frac{2M_n(f)}{\delta}+\left|f'(x)\right|\right) \sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta} \left(\frac{kb_n}{n}-x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}. \quad (2.5)$$

Using (2.5), we get

$$\delta^2 |I_2| \le \left(\frac{2M_n(f)}{\delta}+\left|f'(x)\right|\right) \sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta} \delta^2 \left(\frac{kb_n}{n}-x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\le \left(\frac{2M_n(f)}{\delta}+\left|f'(x)\right|\right) \sum_{\left|\frac{kb_n}{n}-x\right|\ge\delta} \left(\frac{kb_n}{n}-x\right)^4 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}$$

$$\le \left(\frac{2M_n(f)}{\delta}+\left|f'(x)\right|\right) \sum_{k=0}^{n} \left(\frac{kb_n}{n}-x\right)^4 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k}. \quad (2.6)$$

It is known that [1]

$$\sum_{k=0}^{n} \left(\frac{kb_n}{n}-x\right)^4 C_n^k \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k} = x\left(b_n-x\right) b_n^2 \frac{(3n-6)\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{n^3}.$$

Replacing last formulae in (2.6), we have

$$\delta^2 |I_2| \le \left(\frac{2M_n(f)}{\delta}+\left|f'(x)\right|\right) x\left(b_n-x\right) b_n^2 \frac{(3n-6)\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{n^3}. \quad (2.7)$$

(2.7) implies that

$$\frac{n}{x\left(b_n-x\right)} |I_2| \le \left(\frac{2M_n(f)+\delta\left|f'(x)\right|}{\delta^3}\right) b_n^2 \frac{(3n-6)\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{n^2}.$$

It is obvious that for $x \in [0, b_n]$

$$\frac{(3n-6)\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{n^2} \le \frac{3n-6+4}{4n^2} = \frac{3n-2}{4n^2} \le \frac{3}{4n}.$$

Thus

$$\frac{n}{x\left(b_n-x\right)} |I_2| \le \frac{6M_n(f)+3\delta\left|f'(x)\right|}{4\delta^3} \frac{b_n^2}{n}.$$

For this δ, we can choose large $n \ge n_0$, such that

$$\frac{6M_n(f)+3\delta\left|f'(x)\right|}{4\delta^3} \frac{b_n^2}{n} < \frac{\varepsilon}{2}.$$

Thus

$$\frac{n}{x\left(b_n-x\right)} |I_2| < \frac{\varepsilon}{2}. \quad (2.8)$$

Finally, combining inequalities (2.4) and (2.8), we have

$$\left|B_n'(f;x)-f'(x)\right|\le\frac{n}{x(b_n-x)}|I_1|+\frac{n}{x(b_n-x)}|I_2|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

It is obvious that from condition (2.1) implies that $\lim\limits_{n\to\infty}\frac{b_n}{n}=0$. Therefore for $x=0$, we have

$$B_n'(f;0)=\frac{n}{b_n}\left(f\left(\frac{b_n}{n}\right)-f(0)\right)=\frac{f\left(\frac{b_n}{n}\right)-f(0)}{\frac{b_n}{n}}\to f'(0),\quad n\to\infty.$$

Analogously, for $x=b_n$, we have

$$B_n'(f;b_n)=\frac{f(b_n)-f\left(\frac{n-1}{n}b_n\right)}{\frac{b_n}{n}}\to f'(\infty),\quad n\to\infty.$$

This complete the proof of Theorem 2.3.

Theorem 2.4 *Let f is defined in $[0,\infty)$ and is continuously differentiable function in every finite interval $[0,b]\subset(0,\infty)$. Let the sequence $\{b_n\}$ satisfy condition $\lim\limits_{n\to\infty}M_n(f')\frac{b_n^2}{n}=0$.*

Then $B_n'(f;x)$ is uniformly converges to $f'(x)$ on $[0,\infty)$.

Proof. We have

$$B_n'(f;x)=\frac{n}{x(b_n-x)}\sum_{k=0}^{n}f\left(\frac{kb_n}{n}\right)C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left(\frac{kb_n}{n}-x\right).$$

We denote

$$\lambda\left(\frac{kb_n}{n}\right)=\begin{cases}\frac{f\left(\frac{kb_n}{n}\right)-f(x)}{\frac{kb_n}{n}-x}-f'(x), & \text{for } x\ne\frac{kb_n}{n},\ k=0,1,\dots,n,\\ 0, & \text{for } x=\frac{kb_n}{n},\ k=1,\dots,n-1.\end{cases}$$

Hence by mean value theorem, we have

$$\lambda\left(\frac{kb_n}{n}\right)=\frac{f\left(\frac{kb_n}{n}\right)-f(x)}{\frac{kb_n}{n}-x}-f'(x)=\frac{f'(z_k)\left(\frac{kb_n}{n}-x\right)}{\frac{kb_n}{n}-x}-f'(x)=f'(z_k)-f'(x),$$

where z_k is in the open interval with endpoints $\frac{kb_n}{n}$ and x, when $x\ne\frac{kb_n}{n}$. In the case $x=\frac{kb_n}{n}$, we may choose $f'(z_k)$ arbitrary, and we choose x as z_k. We have

$$B_n'(f;x)=f'(x)+\frac{n}{x(b_n-x)}\sum_{k=0}^{n}\lambda\left(\frac{kb_n}{n}\right)\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}$$

and hence

$$B_n'(f;x)-f'(x)$$
$$=\frac{n}{x(b_n-x)}\sum_{k=0}^{n}\left(\frac{kb_n}{n}-x\right)^2C_n^k\left(\frac{x}{b_n}\right)^k\left(1-\frac{x}{b_n}\right)^{n-k}\left[f'(z_k)-f'(x)\right].\qquad(2.9)$$

We represent (2.9) as

$$B_n'(f;x)-f'(x)$$

$$= \frac{n}{x\left(b_n - x\right)} \sum_{\left|\frac{kb_n}{n} - x\right| < \delta} \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \left[f'\left(z_k\right) - f'\left(x\right)\right]$$

$$+ \frac{n}{x\left(b_n - x\right)} \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} \left[f'\left(z_k\right) - f'\left(x\right)\right]$$

$$= \frac{n}{x\left(b_n - x\right)} \left(\mathrm{J}_1 + J_2\right), \tag{2.10}$$

where δ same small positive number.
Obviously, $|x - z_k| = \left|\frac{kb_n}{n} - x\right| - \left|\frac{kb_n}{n} - z_k\right| < \delta - \left|\frac{kb_n}{n} - z_k\right| < \delta$. Therefore by virtue of uniform continuity of f' on $[0,\ b_n]$ for this $\delta > 0$ implies that $\left|f'\left(z_k\right) - f'\left(x\right)\right| < \frac{\varepsilon}{2}$.
We estimate J_1. We get

$$|J_1| \leq \sum_{\left|\frac{kb_n}{n} - x\right| < \delta} \left|f'\left(z_k\right) - f'\left(x\right)\right| \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$< \frac{\varepsilon}{2} \sum_{\left|\frac{kb_n}{n} - x\right| < \delta} \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\leq \frac{\varepsilon}{2} \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k} = \frac{x\left(b_n - x\right)}{n} \frac{\varepsilon}{2}.$$

Hence

$$\frac{n}{x\left(b_n - x\right)} |J_1| < \frac{\varepsilon}{2}. \tag{2.11}$$

Now we estimate J_2. Let $M_n(f') = \max\limits_{x \in [0,\, b_n]} \left|f'\left(x\right)\right|$. Further, we have

$$|J_2| \leq \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left|f'\left(z_k\right) - f'\left(x\right)\right| \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\leq 2M_n(f') \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}. \tag{2.12}$$

Using (2.12), we get

$$\delta^2 |J_2| \leq 2M_n(f') \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \delta^2 \left(\frac{kb_n}{n} - x\right)^2 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\leq 2M_n(f') \sum_{\left|\frac{kb_n}{n} - x\right| \geq \delta} \left(\frac{kb_n}{n} - x\right)^4 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$\leq 2M_n(f') \sum_{k=0}^{n} \left(\frac{kb_n}{n} - x\right)^4 C_n^k \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)^{n-k}$$

$$= 2M_n(f')x\,(b_n - x)\; b_n^2 \frac{(3n-6)\,\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{n^3}. \tag{2.13}$$

(2.13) implies

$$\frac{n}{x\,(b_n - x)}\,|J_2| \le 2M_n(f')\frac{(3n-6)\,\frac{x}{b_n}\left(1-\frac{x}{b_n}\right)+1}{\delta^2 n}\,\frac{b_n^2}{n} \le \frac{3M_n(f')}{2\delta^2}\,\frac{b_n^2}{n}.$$

For this δ, we can choose large $n \ge n_0$, such that

$$\frac{3M_n(f')}{2\delta^2}\,\frac{b_n^2}{n} < \frac{\varepsilon}{2}.$$

Thus

$$\frac{n}{x\,(b_n - x)}\,|J_2| < \frac{\varepsilon}{2}. \tag{2.14}$$

Finally, combining inequalities (2.11) and (2.14), we have

$$\left|B_n'\,(f;x) - f'\,(x)\right| \le \frac{n}{x\,(b_n - x)}\,|J_1| + \frac{n}{x\,(b_n - x)}\,|J_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus

$$\max_{x\in[0,b_n]}\left|B_n'\,(f;x) - f'\,(x)\right| < \varepsilon.$$

Passing to the limit, we have

$$\lim_{n\to\infty} B_n'\,(f;x) = f'(x), \text{ for all } x \in [0,\infty).$$

This complete the proof of Theorem 2.4.

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