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On asymptotic distribution of negative eigen values of second order equation with operator coefficients on a semi-axis

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Abstract. *In the paper, spectral properties of a second order operator-differential equation on a semi-axis are studied. The discreteness conditions of the negative part of the spectrum is found and asymptotic formula for the number of negative eigen values of the operator less than the given number* ε $(\varepsilon > 0)$ *is obtained.*

Keywords. operator-differential equation, spectrum, eigenvalue, eigenfunction, boundary conditions, Hilbert space.

Mathematics Subject Classification (2010): 35J25 · 47A10 · 58J40

1 Introduction

Spectral theory of operator-differential equations is one of the rapidly developing fields of mathematics. Investigation of spectral properties of operator-differential equations get started with F.S. Roffe-Beketov's paper [11], where he obtained a formula of expansion in eigen functions of second order differential operator with a self-adjoint operator coefficient. In the paper [9], A.G. Kostyuchenko and B.M. Levitan first distinguished a class of Sturm-Liouville type differential equations with an unbounded self-adjoint operator coefficient with discrete spectrum and found asymptotic formula for the number of eigen values less than the given number Numerous studies on this theme followed this paper, among them the studies of E. Abdukadirov [1], M. Bayramoglu [5], V. I. Gorbachuk, M.L. Gorbachuk [8], V.M. Mikhailets [10], H.I. Aslanov [3], etc. The present paper is devoted to study of discreteness of the negative spectrum and obtaining asymptotic formula for the number of negative eigen values of the given operator. Asymptotic formulas for negative eigen values of scalar differential operators were obtained in the papers of N. Rozenfeld [13], B. Ya. Skachek [14, 15], G.I. Rozenbloom [12], while for differential operators with operator coefficients of negative spectrum in the papers of M.G. Gasymov, V.V. Zhikov, B.M. Levitan [7], D.R. Yafaev [16], A.A. Adygezelov [2], J.A. Zeynalov [17], A.M. Bayramov [6] and others.

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Let H be a separable Hilbert space. In separable Hilbert space $H = L_2(H;\ [0,\infty))$ we consider the operator L generated by the differential expression

$$l(y) = -\left(p(x)\ y'\right) - Q(x)\ y \tag{1.1}$$

and the boundary condition

$$y'\ (0) = 0 \tag{1.2}$$

It is supposed that the function $p(x)$ and the operator function $Q(x)$ satisfy the following condition:
1). $p(x)$ is a scalar continuous function with bounded derivatives. The function $p(x)$ does not decrease and there exist positive constants c_1 and c_2 such that the following inequalities are fulfilled $c_1 \leq p(x) \leq c_2$.
2). For every $x \in [0,\infty)$ the operator function $Q(x)$ is a completely continuous, positive, monotone decreasing operator, $\|Q(x)\|_H$ is a continuous function, and $\lim\limits_{x\to\infty} \|Q(x)\|_H = 0$.

Subject to conditions 1), 2), it is proved that the operator L generated by expression (1.1) and boundary condition (1.2) is lower bounded and the negative part of its spectrum is discrete. Every eigen value has finite multiplicity. The set of eigen values may have a unique limit point at zero. Denote by $\alpha_1(x) \geq \alpha_2(x) \geq ... \geq \alpha_n(x) \geq ...$ eigen values of the operator $Q(x)$ in space H. Since $Q(x) > 0$ for all $x \in [0,\infty)$, we obtain that $\alpha_j(x) > 0\ \ (j = 1, 2, ...)$. It is known [see [17]] that $\alpha_1(x) = \sup\limits_{\|f\|=1} (Q(x)\ f\ ,\ f)$ and $\|Q(x)\| = \sup\limits_{\|f\|=1} |Q(x)\ f\ ,\ f|$. Hence we get that $\alpha_1(x) = \|Q(x)\|$. The function $\alpha_1(x)$ is a continuous function on the interval $[0,\infty)$. As the operator function $Q(x)$ is monotonically decreasing, the functions $\alpha_1(x)\ , \alpha_2(x)\ , ...\alpha_n(x)\ ...$ are also monotonically decreasing in the interval $[0,\infty)$. By condition $\lim\limits_{x\to\infty} \alpha_1(x) = 0$. Therefore the interval $(0, \alpha_1\ (0))$ is the image of the function $\alpha_1(x)$. In the interval $(0, \alpha_1\ (0))$ the function $\alpha_1(x)$ has a continuous inverse. We denote this function by $\psi_1(x)$.

2 Asymptotic formula for the number of negative eigen functions

Let $\varepsilon > 0$ be an arbitrary number. Denote by $N(\varepsilon)$ the number of negative eigen values less than ε, i.e. assume $N(\varepsilon) = \sum\limits_{\lambda_n < -\varepsilon} 1$. Here we establish asymptotic formulas for the number of negative eigen values of the operator L as $\varepsilon \to 0$.

At first assume that the first eigen value of the operator $Q(x)$ satisfies the condition:
3). For any $\eta > 0$

$$\lim_{x\to\infty} \alpha_1(x)\ x^{k_0-\eta} = \lim_{x\to\infty} \left[\alpha_1(x)\ X^{k_0+\eta}\right]^{-1} = 0$$

where k_0 is a number from interval $(0, 2)$. Let for some value of $m \in (0,\infty)$ the series

$$\sum_{j=1}^{\infty} [\alpha_j(0)]^m \tag{2.1}$$

converge.

From this condition it follows that

$$const \geq \sum_{\alpha_j(o)\geq\varepsilon} [\alpha_j(0)]^m \geq \sum_{\alpha_j(0)\geq\varepsilon} \varepsilon^m = \varepsilon^m \cdot l_\varepsilon .$$

Hence it follows that
$$l_\varepsilon \le const \cdot \varepsilon^{-m}. \tag{2.2}$$
The following theorem holds.

Theorem 2.1 *Let conditions 1), 2), 3) be fulfilled. Additionally, let for some* m*, satisfying the condition* $0 < m < \frac{(2-k_0)^2}{8k_0-2k_0^2}$ *the series* $\sum\limits_{j=1}^{\infty}[\alpha_j(0)]^m$ *converge. Then as* $\varepsilon \to 0$ *the following asymptotic formula holds*
$$N(\varepsilon) = \frac{1}{\pi}\left[1 + O\left(\varepsilon^{t_0}\right)\right]\sum_j \int\limits_{\alpha_j(x)\ge\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx. \tag{2.3}$$
Here t_0 *is some positive number.*

Proof. In the paper [4] we have proved the inequality (theorems 4 and 7) from which the following estimation follows
$$\left| N(\varepsilon) - \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx \right| < cons\left[l_\varepsilon\cdot\int\limits_0^{\delta}\sqrt{\alpha_1(x)}\,dx + l_\varepsilon\cdot\psi_1^k(\varepsilon)\right]. \tag{2.4}$$
Here $\tilde{\psi}_j(\varepsilon) = \sup E_{j,\varepsilon}$, where $E_{j,\varepsilon}\{x\,|x\in[0,\infty]\,;\ \alpha_j(x)\ge\varepsilon\}$ and $\delta = \frac{\psi_1(\varepsilon)}{\left[\psi_1^k(\varepsilon)\right]} + 1$, $\left[\psi_1^k(\varepsilon)\right]$ is the entire part of the number $\psi_1^k(\varepsilon)$.

Since $\alpha_1(x)$ is monotonically decreasing on the interval $[0,\ \psi_1\ (2\varepsilon)]$, we get
$$\alpha_1(x) \ge \alpha_1(\psi_1(2\varepsilon)) = 2\varepsilon. \tag{2.5}$$
Using the estimations $\alpha_1(x)\ge 2\varepsilon$ and $c_1 \le p(x) \le c_2$, we have:
$$\sum_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > \int\limits_0^{\psi_1(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > \int\limits_0^{\psi_1(2\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx$$
$$> \sqrt{\varepsilon}\int\limits_0^{\psi_1(2\varepsilon)}\sqrt{\frac{1}{p(x)}}\,dx > \sqrt{c_2^{-1}\cdot\varepsilon\cdot\psi_1(2\varepsilon)}. \tag{2.6}$$
From (2.5) and the condition $\lim\limits_{\varepsilon\to\infty}\psi_1(\varepsilon)=\infty$ it follows that
$$\lim_{\varepsilon\to\infty}\left[\alpha_1(\psi_1(2\varepsilon))\,(\psi_1(2\varepsilon))^{k_o+\eta}\right]^{-1} = 0.$$
Hence for small values of ε we get
$$\psi_1(2\varepsilon) > \varepsilon^{-\frac{1}{k_0+\eta}}.$$
Using this inequality, from (2.6) we get
$$\sum_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > c_3\cdot\varepsilon^{\frac{k_0+\eta-2}{2(k_0+\eta)}}. \tag{2.7}$$

Estimate now the integral $\int_0^\delta \sqrt{\alpha_1(x)}\,dx$ standing in the right hand side of inequality (2.4). From condition (2.1) it follows that

$$\alpha_1(x) \le c_4 \cdot x^{\eta - x_0}\,(\eta < k_0,\ \ 0 < x < \infty)\,. \tag{2.8}$$

Then we get

$$\int\limits_0^\delta \sqrt{\alpha_1(x)}\,dx \le \sqrt{c_4}\int\limits_0^\delta \sqrt{x^{\eta - x_0}}\,dx \le c_5 \delta^{\frac{1}{2}(2-k_0-\eta)}. \tag{2.9}$$

From the inequality

$$\delta = \frac{\psi_1(\varepsilon)}{\left(\left[\psi_1^k(\varepsilon)\right]+1\right)} < \psi_1^{1-k}(\varepsilon) \tag{2.10}$$

and inequality (2.8), allowing for $x = \psi_1(\varepsilon)$ we get $\alpha_1(\psi_1(\varepsilon)) \le c_4 \psi_1^{\eta - k_0}\quad (\eta - k_0)$ or

$$\psi_1(\varepsilon) \le c_5 \cdot \varepsilon^{-\frac{1}{k_0-\eta}}. \tag{2.11}$$

(2.9), (2.10) and (2.11) yield

$$\int\limits_0^\delta \sqrt{\alpha_1(x)}\,dx < c_6 \cdot \varepsilon^{-\frac{(1-k)(2-k_0+\eta)}{2(k_0-\eta)}},\ \ c_6 > 0. \tag{2.12}$$

From (2.2), (2.11) and (2.12) we get

$$l \cdot \int\limits_0^\delta \sqrt{\alpha_1(x)}\,dx < c_7 \cdot \varepsilon^{-m-\frac{(1-k)(2-k_0+\eta)}{2(k_0-\eta)}} \tag{2.13}$$

or

$$l_3 \cdot \psi_1^k(\varepsilon) < c_7 \cdot \varepsilon^{-\frac{m(k_0-\eta)+k}{k_0-\eta}}. \tag{2.14}$$

From (2.7), (2.13), (2.14) we get

$$\frac{l \cdot \int\limits_0^\delta \sqrt{\alpha_1(x)}\,dx}{\sum\limits_{j=1}^{l_\varepsilon} \int\limits_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx} < c_8 \cdot \varepsilon^{-m-\frac{(1-k)(2-k_0+\eta)}{2(k_0-\eta)}} + \frac{2-k_0-\eta}{2(k_0+\eta)}. \tag{2.15}$$

Hence we have

$$\frac{l_3 \cdot \psi_1^k(\varepsilon)}{\sum\limits_{j=1}^{l_\varepsilon} \int\limits_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx} < c_9 \cdot \varepsilon^{-\frac{m(k-\eta_0)+k}{k_0-\eta}+\frac{2-k_0-\eta}{2(k_0+\eta)}}. \tag{2.16}$$

For every $t > 0$ one can find a number ω such that for $0 < \eta < \omega$ the inequality

$$m - \frac{(1-k)(2-k_0+\eta)}{2(k_0-\eta)} + \frac{2-k_0-\eta}{2(k_0+\eta)} > \frac{2k - kk_0 - 2mk_0}{2k_0} > t \tag{2.17}$$

is fulfilled.

Hence

$$-\frac{m(k_0-\eta)+k}{k_0-\eta}+\frac{2-k_0-\eta}{2(k_0+\eta)}>\frac{2-k_0-2mk_0-2k}{2k_0}-t. \tag{2.18}$$

Note that up to now the number k was an arbitrary number from the interval (0,1). Now we take

$$k=\frac{(2-k_0)^2+2mk_0^2}{4(2-k_0)}$$

and

$$t=t_0=\frac{(2-k_0)^2-8mk_0+2mk_0^2}{16k_0}.$$

Under such a choice of the numbers k and t, inequalities (2.17) and (2.18) take the form

$$-m-\frac{(1-k)(2-k_0+\eta)}{2(k_0-\eta)}+\frac{2-k_0-\eta}{2(k_0+\eta)}>\frac{(2-k_0)^2-8mk_0+2mk_0^2}{8k_0}-t_0$$

$$=\frac{(2-k_0)^2-8mk_0+2mk_0^2}{16k_0}>t_0 \tag{2.19}$$

$$-\frac{m(k_0-\eta)+k}{k_0-\eta}+\frac{2-k_0-\eta}{2(k_0+\eta)}>\frac{(2-k_0)^2-8mk_0+2mk_0^2}{8k_0-4k_0^2}-t>t_0. \tag{2.20}$$

As for the number m the inequality

$$0<m<\frac{(2-k_0)^2}{8k_0-2k_0^2}$$

is fulfilled, we get $k\in(0,1)$ and $t_0>0$.

From inequalities (2.15), (2.16), (2.17) and (2.18) we get

$$\frac{l_3\int\limits_0^\delta\sqrt{\alpha_1(x)}\,dx}{\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx}<c_9\cdot\varepsilon^{t_0}. \tag{2.21}$$

Hence

$$\frac{l_3\cdot\psi_1^k(\varepsilon)}{\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx}<c_9\cdot t^{t_0}. \tag{2.22}$$

For small values of ε, from (2.6), (2.21), (2.22) we get

$$\left|\frac{N(\varepsilon)}{\frac{1}{\pi}\sum\limits_{j=1}^{l_\varepsilon}\int\limits_0^{\tilde{\psi}_j(\varepsilon)}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}\,dx}-1\right|<c_{10}\cdot\varepsilon^{t_0}. \tag{2.23}$$

As $\varepsilon\to 0$, from this inequality we get the asymptotic formula

$$N(\varepsilon)=\frac{1}{\pi}\left[1+0\left(\varepsilon^{t_0}\right)\right]\sum_j\int\limits_{j_j(x)\geq\varepsilon}\sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx.$$

Theorem 1 is proved.

3 Auxiliary lemma and proof of the main theorem

Following the paper [14] denote by $l_{n_s}x \quad (s = 0, 1, 2, ...)$ the following functions

$$\ln_0 x = x, \quad \ln_s x = \ln\left(\ln_{s-1} x\right).$$

Suppose that the function $\alpha_1(x) = \|Q(x)\|$ satisfies the condition:
4). For some $\xi > 0, \quad s \geq 1$ and for $x \in [a, \infty) \ (a > 0)$ the function $\alpha_1(x) - (\ln_s x)^{-\xi}$ is nonnegative and does not increase monotonically.

By the mathematical induction method we can show that for large values of x the inequality

$$\ln_s\left(\frac{x}{\ln x}\right) < \quad \ln_s x - \ln_x^{1-s} \tag{3.1}$$

is fulfilled.

Using this inequality, we can prove the following lemma.

Lemma 3.1 *Let the operator function $Q(x)$ satisfy conditions 2) and 4). Then for small values of $\varepsilon > 0$ the inequality*

$$\alpha_1\left(\frac{\psi_1(\varepsilon)}{\ln\psi_1(\varepsilon)}\right) - \varepsilon > (\ln\psi_1(\varepsilon))^{-(\xi+1)(s+1)} \tag{3.2}$$

is valid.

Proof. Take a natural number $k \geq 1$. By inequality (3.1), for large values of x we have

$$\ln_s^k\left(\frac{x}{\ln x}\right) < \quad \left(\ln x - \ln^{1-s} x\right)^k \leq \ln_s^k - (\ln x)^{k(1-s)}.$$

Hence

$$\ln_s^{-k}\left(\frac{x}{\ln x}\right) - \quad \ln_s^{-k} x > \frac{1}{\ln_s^k x - (\ln x)^{k-(1-s)}} - \ln_s^{-k} x$$

$$> \frac{1}{\ln_s^k x \cdot \ln^{k(1-s)} x \cdot \ln_s^k x} > (\ln x)^{-k(s+1)}. \tag{3.3}$$

By assumption, for $x \in [a, +\infty) \ (a > 0)$ the function $\alpha_1 \ (x) - \ln_s^{-\xi} x$ does not increase monotonically. In this case show that for any $k_1 > \xi$ the function $\alpha_1(x) - \ln_s^{-k} x$ monotonically decreasing on some interval $[b, +\infty) \ (b > 0)$. For that, represent the function $\alpha_1 \ (x) - \ln_s^{-k_1} x$ in the form

$$\alpha_1 \ (x) - \ln_s^{-k_1} x = u(x) + \vartheta(x) \tag{3.4}$$

where

$$u(x) = \alpha_1(x) - \ln_s^{-\xi} x$$

$$\vartheta(x) = \ln_s^{-\xi} x - \ln_s^{-k} x.$$

Show that the function $\vartheta(x)$ monotonically decreases on some interval $[b, +\infty) \ (b > 0)$.

Indeed,

$$\vartheta'(x) = \xi_1 \ln_s^{-\xi-1} x \cdot (\ln_s x)' + k_1 \ln_s^{-k_1--1} x \cdot (\ln_s x)'$$

$$= (\ln_s x)' \cdot \frac{k}{\ln_s^{k_1+1} x}\left[1 - \frac{\xi \ \ln_s^{k_1-\xi} x}{k_1}\right]. \tag{3.5}$$

Since for large x $(\ln_s x)' > 0$, then for $k_1 > \xi$ from (3.5) it follows that on some interval $[b, +\infty)$ $(b > a)$ $\vartheta'(x) < 0$. Thus, it is shown that the function $v(x)$ monotonically decreases on some interval $[b, +\infty)$. On the other hand, the function $u(x)$ is monotonically non-increasing on the interval $[b, +\infty)$. Then from (3.4) it follows that the function $\alpha_1(x) - \ln_s^{-k_1} x$ monotonically decreases on the interval $[b, +\infty)$. In inequality (3.3) take $k = [|\xi|] + 1$ ($[|\xi|]$ as the entire part of the number $|\xi|$). Repeating the above reasonings, we can show that $\alpha_1(x) - \ln_s^{-k} x$ is a monotonically decreasing function on the interval $[b, +\infty)$. Therefore, for all values of x the following inequality is valid

$$\alpha_1\left(\frac{x}{\ln x}\right) - \left(\ln_s\left(\frac{x}{\ln x}\right)\right)^{-k} > \alpha_1(x) - \ln_s^{-k} x. \tag{3.6}$$

From (3.3) and (3.6) we get

$$\alpha_1\left(\frac{x}{\ln x}\right) - \alpha(x) > \left(\ln_s\left(\frac{x}{\ln x}\right)\right)^{-k} - \ln_s^{-k} x > (\ln x)^{-k(s+1)} . \tag{3.7}$$

From conditions $\lim\limits_{x\to\infty} \alpha_1(x) = 0$ we get $\lim\limits_{\varepsilon\to 0} \psi_1(\varepsilon) = \infty$. Therefore, for rather small positive ε, inequality (3.7) is valid for $x = \psi_1(\varepsilon)$.

Therefore we get

$$\alpha_1\left(\frac{\psi_1(\varepsilon)}{\ln \psi_1(\varepsilon)}\right) - \alpha_1(\psi_1(\varepsilon)) > (\ln \psi_1(\varepsilon))^{-k(s+1)} .$$

Since $k = [|\xi|] + 1 \le \xi + 1$, we get

$$\alpha_1\left(\frac{\psi_1(\varepsilon)}{\ln \psi_1(\varepsilon)}\right) - \varepsilon > (\ln \psi_1(\varepsilon))^{-(\xi+1)(s+1)} .$$

The lemma is proved.

The following main theorem holds.

Theorem 3.1 *Let conditions 1), 2), 4) be fulfilled, and for some $m \in (0, \infty)$ the series $\sum\limits_{j=1}^{\infty} [\alpha_j(0)]^m$ converge. Then as $\varepsilon \to 0$, the following asymptotic formula is true:*

$$N(\varepsilon) = \frac{1}{\pi}\left[1 + \mathrm{O}\left(e^{-\varepsilon^{-\beta}}\right)\right] \sum_{j=1}^{\infty} \int\limits_{\alpha_j(x)\ge\varepsilon} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx.$$

Here β is a positive number.

Proof. Earlier in the paper [4] we have proved theorem (theorem 3.5) from which the following estimation follows

$$\left| N(\varepsilon) - \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon} \int\limits_0^{\tilde\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx \right| < const \cdot \left[l_\varepsilon \cdot \delta + l_\varepsilon \cdot \psi_1^k(\varepsilon)\right].$$

If in this inequality we accept $k = \frac{1}{2}$ and take into account the equality $\delta = \frac{\psi_1(\varepsilon)}{[\psi_1^k(\varepsilon)]+1}$, we get:

$$\left| N(\varepsilon) - \frac{1}{\pi}\sum_{j=1}^{l_\varepsilon} \int\limits_0^{\tilde\psi_j(\varepsilon)} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} dx \right| < const \cdot l_3 \cdot \sqrt{\psi(\varepsilon)}. \tag{3.8}$$

Estimate the sum in the left hand side of this inequality. Assume $f(\varepsilon) = \psi_1(\varepsilon)\ (\ln\psi_1(\varepsilon))^{-1}$. Using the property of monotone decrease of the operator function $Q(x)$ and condition 2) imposed on the function $p(x)$, we get

$$\sum_{j=1}^{l_\varepsilon} \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > \int_0^{f(\varepsilon)} \sqrt{\frac{\alpha_1(x)-\varepsilon}{p(x)}}\, dx$$

$$\sqrt{c_2^{-1}} \int^{f(\varepsilon)} \sqrt{\alpha_1(x)-\varepsilon}\, dx > \frac{f(\varepsilon)}{2\sqrt{c_2}} \sqrt{\alpha_1(f(c))-\varepsilon}. \tag{3.9}$$

From the lemma and inequality (3.9) we get the following inequality

$$\sum_{j=1}^{l_\varepsilon} \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}dx > \frac{\psi_1(\varepsilon)}{2\sqrt{c_2}\ln\psi_1(\varepsilon)} (\ln\psi_1(\varepsilon))^{-\frac{1}{2}(\xi+1)(s+1)} > c_{11}\psi_1^{\frac{3}{4}}(\varepsilon). \tag{3.10}$$

As the function $\alpha_1(x)$ satisfies condition 4), then for rather small ε it holds the inequality

$$\varepsilon = \alpha_1(\psi_1(\varepsilon)) \geq (\ln_s \psi_1(\varepsilon))^{-\xi}.$$

Hence it follows that

$$\psi_1(\varepsilon) > e^{\varepsilon^{-\frac{1}{\xi}}}. \tag{3.11}$$

From (2.2), (3.10) and (3.11) we get

$$\left| \frac{N(\varepsilon)}{\frac{1}{\pi}\sum\limits_{j=1}^{l_\varepsilon} \int_0^{\tilde{\psi}_j(\varepsilon)} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}}} - 1 \right| < c_{12}\cdot\varepsilon^m \cdot e^{\frac{1}{4}\varepsilon^{-\frac{1}{4}}} < e^{\varepsilon^{-\beta}}.$$

As $\xi \to 0$ from this inequality it follows the following asymptotic formula

$$N(\varepsilon) = \frac{1}{\pi}\left[1 + \mathrm{O}\left(e^{-\varepsilon^{-\beta}}\right)\right] \sum_{j=1}^{\infty} \int\limits_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{px}}dx.$$

Theorem 3.1 is proved.

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