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Derivatives with respect to horizontal and vertical lifts of the modified Riemannian extension $\tilde{g}_{\nabla,c}$ on Cotangent Bundle

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Abstract. *In this paper, we define the modified Riemannian extension $\tilde{g}_{\nabla,c}$ in the cotangent bundle T^*M, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the modified Riemannian extension with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.*

Keywords. Covarient derivative · Lie derivative · modified Riemannian extension · Horizontal lift · Vertical lift

1 Introduction

1.1 The Cotangent Bundle

Let M be an $n-$dimensional smooth manifold and denote by $\pi : T^*M \to M$ its cotangent bundle whose fibres are cotangent spaces to M. Then T^*M is a $2n-$dimensional smooth manifold and some local charts induced naturally from local charts on M can be used. Namely, a system of local coordinates $(U, x^i), i = 1, ..., n$ in M induces on T^*M a system of local coordinate $(\pi^{-1}(U), x^i, x^{\bar{\imath}} = p_i), \bar{\imath} = n+i = n+1, ..., 2n$, where $x^{\bar{\imath}} = p_i$ are the components of covectors p in each cotangent space T^*_xM, $x \in U$ with respect to the natural coframe $\{dx^i\}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be the local expressions in U of a vector field X and a covector (1-form) field ω on M, respectively. Then the vertical lift ${}^V\omega$ of ω, the horizontal lift ${}^HX$ and the complete lift ${}^CX$ of X are given, with respect to the induced coordinates, by

$$\omega^V = \omega_i \partial_{\bar{\imath}}, \tag{1.1}$$

$$X^H = X^i \partial_i + p_h \Gamma^h_{ij} X^j \partial_{\bar{\imath}} \tag{1.2}$$

and

$$X^C = X^i \partial_i - p_h \partial_i X^h \partial_{\bar{\imath}}, \tag{1.3}$$

where $\partial_i = \frac{\partial}{\partial x^i}, \partial_{\bar{\imath}} = \frac{\partial}{\partial x^{\bar{\imath}}}$ and Γ^h_{ij} are the coefficients of a symmetric (torsion-free) affine connection ∇ in M.

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Definition 1.1 *The Lie bracket operation of vertical and horizontal vector fields on T^*M is given by the formulas*

$$\left[{}^{H}X,{}^{H}Y\right]={}^{H}[X,Y]+{}^{V}(p\circ R(X,Y)) \tag{1.4}$$

$$\left[{}^{H}X,{}^{V}\omega\right]={}^{V}(\nabla_{X}\omega)$$

$$\left[{}^{V}\theta,{}^{V}\omega\right]=0$$

for any $X,Y\in\Im_0^1(M)$ and $\theta,\omega\in\Im_1^0(M)$, where R is the curvature tensor of the symmetric connection ∇ defined by $R(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ [13](for details,see [33]).

1.2 Modified Riemannian Extension

Let M be an $n-$dimensional differentiable manifold and T^*M be its cotangent bundle. There is a well-known natural construction which yields, for any affine connection ∇ on M, a pseudo-Riemannian metric $\tilde{g}\nabla$ on T^*M. The metric $\tilde{g}\nabla$ is called the Riemannian extension of ∇. Riemannian extensions were originally defined by Patterson and Walker [23] and further studied by Afifi [4], thus relating pseudo-Riemannian properties of T^*M with the affine structure of the base manifold (M,∇). Moreover, Riemannian extensions were also considered by Garcia-Rio et al. in [12] in relation to Osserman manifolds (see also Derdzinski [9]). Since Riemannian extensions provide a link between affine and pseudo-Riemannian geometries, some properties of the affine connection ∇ can be investigated by means of the corresponding properties of the Riemannian extension $\tilde{g}\nabla$(see also almost product Riemannian manifolds [2,3,14,15]). For instance, ∇ is projectively flat if and only if $\tilde{g}\nabla$ is locally conformally flat [4]. For Riemannian extensions, also see [1, 11, 16, 19, 20,26,28,30,31]. In [5,6], the authors introduced a modification of the usual Riemannian extensions which is called the modified Riemannian extension.

Let M_{2k} be a $2k-$dimensional differentiable manifold endowed with an almost complex structure J and a pseudo-Riemannian metric g of signature (k,k) such that $g(JX,Y)=g(X,JY)$ for arbitrary vector fields X and Y on M_{2k}. Then the metric g is called the Norden metric. Norden metrics are referred to as anti-Hermitian metrics or B-metrics. The study of such manifolds is interesting because there exists a difference between the geometry of a $2k-$dimensional almost complex manifold with Hermitian metric and the geometry of a $2k-$dimensional almost complex manifold with Norden metric. A notable difference between Norden metrics and Hermitian metrics is that $G(X,Y)=g(X,JY)$ is another Norden metric, rather than a differential 2-form. Some authors considered almost complex Norden structures on the cotangent bundle [10,21,22].

In this paper, we will use a deformation of the Riemannian extension on the cotangent bundle T^*M over (M,∇) by means of a symmetric tensor field c on M, where ∇ is a symmetric affine connection on M. The metric is the socalled modified Riemannian extenson. The article is constructed as follows, firstly, we define the modified Riemannian extension $\tilde{g}_{\nabla,c}$ in the cotangent bundle T^*M, which is completely determined by its action on vector fields of type X^H and ω^V. Later, we obtain the covarient and Lie derivatives applied to the modified Riemannian extension with respect to the horizontal and vertical lifts of vector and kovector fields, respectively.

For a given symmetric connection ∇ on an $n-$dimensional manifold M, the cotangent bundle T^*M can be equipped with a pseudo-Riemannian metric $\tilde{g}\nabla$ of signature (n,n) : the Riemannian extension of ∇ [23], given by

$$\tilde{g}_{\nabla}\left({}^{C}X,{}^{C}Y\right)=-\gamma\left(\nabla_XY+\nabla_YX\right),$$

where ${}^{C}X, {}^{C}Y$ denote the complete lifts to T^*M of vector fields X, Y on M. Moreover, for any $Z \in \Im_0^1(M)$, $Z = Z^i \partial_i, \partial Z$ is the function on T^*M defined by $\gamma Z = p_i Z^i$ [13]. The Riemannian extension is expressed by

$$\tilde{g}_\nabla = \begin{pmatrix} -2p_h \Gamma_{ij}^h & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix}$$

with respect to the natural frame.

Now we give a deformation of the Riemannian extension above by means of a symmetric $(0,2)$-tensor field c on M whose metric is called the modified Riemannian extension. The modified Riemannian extension is expressed by

$$\tilde{g}_{\nabla,c} = g_\nabla + \pi^* c = \begin{pmatrix} -2p_h \Gamma_{ij}^h + c_{ij} & \delta_j^i \\ \delta_i^j & 0 \end{pmatrix} \tag{1.5}$$

with respect to the natural frame. It follows that the signature of $\tilde{g}_{\nabla,c}$ is (n,n) [13].

Denote by ∇ the Levi-Civita connection of a semi-Riemannian metric g. In this section, we will consider T^*M equipped with the modified Riemannian extension $\tilde{g}_{\nabla,c}$ over a pseudo-Riemannian manifold (M,g). Since the vector fields ${}^{H}X$ and ${}^{V}\omega$ span the module of vector fields on T^*M, any tensor field is determined on T^*M by their actions on ${}^{H}X$ and ${}^{V}\omega$. The modified Riemannian extension $\tilde{g}_{\nabla,c}$ has the following properties [13]:

$$\tilde{g}_{\nabla,c}\left({}^{H}X, {}^{H}Y\right) = c\left(X,Y\right) \tag{1.6}$$
$$\tilde{g}_{\nabla,c}\left({}^{H}X, {}^{V}\omega\right) = g_{\nabla,c}\left({}^{V}\omega, {}^{H}X\right) = \omega\left(X\right)$$
$$\tilde{g}_{\nabla,c}\left({}^{V}\omega, {}^{V}\theta\right) = 0$$

for all $X, Y \in \Im_0^1(M)$ and $\omega, \theta \in \Im_1^0(M)$, which characterize $\tilde{g}_{\nabla,c}$.

We know see, from (1.2) and (1.3), that the complete lift X^C of $X \in \Im_0^1(M)$ is expressed by

$$X^C = X^H - (p(\nabla X))^V, \tag{1.7}$$

where $p(\nabla X) = p_i(\nabla_h X^i)dx^h$.

Using (1.6) and (1.7), we have

$$\tilde{g}_{\nabla,c}(X^C, Y^C) = c(X,Y) - p(\nabla Y)(X) - p(\nabla X)(Y). \tag{1.8}$$

Since the tensor field $\tilde{g}_{\nabla,c} \in \Im_2^0(T^*M)$ is completely determined also by its action on vector fields type X^C and Y^C, we have an alternative characterization of $\tilde{g}_{\nabla,c}$ on T^*M : $\tilde{g}_{\nabla,c}$ is completely determined by the condition (1.8). Similarly, we get the following results

$$\begin{aligned}
i)\tilde{g}_{\nabla,c}(X^C, Y^C) &= (X^H - (p(\nabla X))^V, Y^H - (p(\nabla Y))^V) \\
&= \tilde{g}_{\nabla,c}(X^H, Y^H) - \tilde{g}_{\nabla,c}(X^H, (p(\nabla Y))^V) \\
&\quad - \tilde{g}_{\nabla,c}((p(\nabla X))^V, Y^H) - \tilde{g}_{\nabla,c}((p(\nabla X))^V, (p(\nabla Y))^V) \\
&= c(X,Y) - p(\nabla Y)(X) - p(\nabla X)(Y) \\
ii)\tilde{g}_{\nabla,c}(X^C, \omega^V) &= \tilde{g}_{\nabla,c}(X^H - (p(\nabla X))^V, \omega^V) \\
&= \tilde{g}_{\nabla,c}(X^H, \omega^V) - \tilde{g}((p(\nabla X))^V, \omega^V) \\
&= \omega(X) \\
iii)\tilde{g}_{\nabla,c}(\omega^V, Y^C) &= \tilde{g}_{\nabla,c}(\omega^V, Y^H - (p(\nabla Y))^V) \\
&= \tilde{g}_{\nabla,c}(\omega^V, Y^H) - \tilde{g}_{\nabla,c}(\omega^V, (p(\nabla Y))^V) \\
&= \omega(Y) \\
iv)\tilde{g}_{\nabla,c}(\omega^V, \theta^V) &= 0
\end{aligned}$$

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^{∞}. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(T^*M)$ the corresponding set on the cotangent bundle T^*M.

2 Main Results

Definition 2.1 *Let M^n be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra $T(M^n)$, defined by*

$$D = \nabla_X : T(M^n) \to T(M^n)\,, X \in \Im_0^1(M^n)$$

is called as covariant derivation with respect to vector field X if

$$\nabla_{fX+gY} t = f\nabla_X t + g\nabla_Y t, \tag{2.1}$$

$$\nabla_X f = Xf,$$

where $\forall f,g \in \Im_0^0(M^n)$, $\forall X,Y \in \Im_0^1(M^n)$, $\forall t \in \Im(M^n)$ (see [18], p.123).
On the other hand, a transformation defined by

$$\nabla : \Im_0^1(M^n) \times \Im_0^1(M^n) \to \Im_0^1(M^n)$$

is called as an affin connection (see for details [18, 25]).

Proposition 2.1 *Covarient differentiation with respect to the horizontal lift ∇^H of a symetric affine connection ∇ in M^n to $T^*(M^n)$ satisfies*

$$\nabla^H_{X^H} Y^H = (\nabla_X Y)^H, \nabla^H_{\theta^V} \omega^V = 0, \tag{2.2}$$

$$\nabla^H_{X^H} \omega^V = (\nabla_X \omega)^V, \nabla^H_{\theta^V} Y^H = 0,$$

for any $X,Y \in \Im_0^1(M^n)$, $\theta,\omega \in \Im_1^0(M^n)$ [33].

Theorem 2.1 *Let $\tilde{g}_{\nabla,c}$ be the modified Riemannian extension, is defined by (1.6) and the horizontal lift ∇^H of a symetric affine connection ∇ in M^n to $T^*(M^n)$. From Proposition (2.1) and Definition 2.1, we get the following results*

$$\begin{aligned}
i)(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= 0,\\
ii)(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= 0,\\
iii)(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= 0,\\
iv)(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= 0,\\
v)(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= 0,\\
vi)(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= 0,\\
vii)\left(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c}\right)(Y^H,Z^H) &= 0,\\
viii)(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(Y^H,Z^H) &= ((\nabla_X c)(Y,Z))^V,
\end{aligned}$$

where the vertical lift $\omega^V \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$, the horizontal and complete lifts $X^H, X^C, \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$, defined by (1.1),(1.2),(1.3), respectively.

Proof. $i)$

$$\begin{aligned}(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= \nabla^H_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^H_{\omega^V}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^H_{\omega^V}\xi^V)\\ &= 0\end{aligned}$$

$ii)$

$$\begin{aligned}(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= \nabla^H_{X^H}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^H_{X^H}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^H_{X^H}\xi^V)\\ &= -\tilde{g}_{\nabla,c}((\nabla_X\theta)^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,(\nabla_X\xi)^V)\\ &= 0\end{aligned}$$

$iii)$

$$\begin{aligned}(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= \nabla^H_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,Z^H) - \tilde{g}_{\nabla,c}(\nabla^H_{\omega^V}\theta^V,Z^H) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^H_{\omega^V}Z^H)\\ &= \nabla^H_{\omega^V}\theta(Z)\\ &= 0\end{aligned}$$

$iv)$

$$\begin{aligned}(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= \nabla^H_{X^H}\tilde{g}_{\nabla,c}(\theta^V,Z^H) - \tilde{g}_{\nabla,c}(\nabla^H_{X^H}\theta^V,Z^H) - \tilde{g}_{\nabla,c}(\theta^V,\nabla^H_{X^H}Z^H)\\ &= X^H(\theta(Z)) - \tilde{g}_{\nabla,c}((\nabla_X\theta)^V,Z^H) - \tilde{g}_{\nabla,c}(\theta^V,(\nabla_X Z)^H)\\ &= ((\nabla_X\theta(Z)) - ((\nabla_X\theta)(Z)) - (\theta(\nabla_X Z)))^V\\ &= 0\end{aligned}$$

$v)$

$$\begin{aligned}(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= \nabla^H_{\omega^V}\tilde{g}_{\nabla,c}(Y^H,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^H_{\omega^V}Y^H,\xi^V) - \tilde{g}_{\nabla,c}(Y^H,\nabla^H_{\omega^V}\xi^V)\\ &= \nabla^H_{\omega^V}\xi\,(Y)\\ &= 0\end{aligned}$$

$vi)$

$$\begin{aligned}(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= \nabla^H_{X^H}\tilde{g}_{\nabla,c}(Y^H,\xi^V) - \tilde{g}_{\nabla,c}(\nabla^H_{X^H}Y^H,\xi^V) - \tilde{g}_{\nabla,c}(Y^H,\nabla^H_{X^H}\xi^V)\\ &= X^H\left(\xi\left(Y\right)\right) - \tilde{g}_{\nabla,c}\left(\nabla_X Y\right)^H,\xi^V) - \tilde{g}_{\nabla,c}(Y^H,(\nabla_X\xi)^V)\\ &= ((\nabla_X\xi(Y)) - \xi(\nabla_X Y) - (\nabla_X\xi)\left(Y\right))^V\\ &= 0\end{aligned}$$

$vii)$

$$\begin{aligned}\left(\nabla^H_{\omega^V}\tilde{g}_{\nabla,c}\right)(Y^H,Z^H) &= \nabla^H_{\omega^V}\tilde{g}_{\nabla,c}(Y^H,Z^H) - \tilde{g}_{\nabla,c}(\nabla^H_{\omega^V}Y^H,Z^H) - \tilde{g}_{\nabla,c}(Y^H,\nabla^H_{\omega^V}Z^H)\\ &= \omega^V c(Y,Z)\\ &= 0\end{aligned}$$

$viii)$

$$\begin{aligned}(\nabla^H_{X^H}\tilde{g}_{\nabla,c})(Y^H,Z^H) &= \nabla^H_{X^H}\tilde{g}_{\nabla,c}(Y^H,Z^H) - \tilde{g}_{\nabla,c}(\nabla^H_{X^H}Y^H,Z^H) - \tilde{g}_{\nabla,c}(Y^H,\nabla^H_{X^H}Z^H)\\ &= X^H c(Y,Z) - \tilde{g}_{\nabla,c}((\nabla_X Y)^H,Z^H) - \tilde{g}_{\nabla,c}(Y^H,(\nabla_X Z)^H)\\ &= (\nabla_X c(Y,Z) - c((\nabla_X Y),Z) - c(Y,(\nabla_X Z)))^V\\ &= ((\nabla_X c)(Y,Z))^V\end{aligned}$$

where $f^V\left(\tilde{P}\right) = f(P)$ ($P, \tilde{P}$ are points of M and $T^*(M)$ respectively), $c(Y,Z)$ and $\omega(Y)$ are function on $T^*(M)$, $X,Y,Z \in \Im_0^1(M)$, $\omega,\theta,\xi \in \Im_1^0(M)$, $f \in \Im_0^0(M)$ and $\nabla_X\theta(Z) = (\nabla_X\theta)\,Z + \theta\,(\nabla_X Z)$.

Definition 2.2 *Let M^n be an $n-$dimensional diferentiable manifold. Differential transformation $D = L_X$ is called as Lie derivation with respect to vector field $X \in \Im_0^1(M^n)$ if*

$$L_X f = Xf, \forall f \in \Im_0^0(M^n), \tag{2.3}$$

$$L_X Y = [X,Y], \forall X,Y \in \Im_0^1(M^n).$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [7, 8, 33]

$$(L_X F)Y = [X,FY] - F[X,Y]. \tag{2.4}$$

Theorem 2.2 *Let $\tilde{g}_{\nabla,c}$ be the modified Riemannian extension, is defined by (1.6) and L_X the operator Lie derivation with respect to X. From (1.6), Definition (1.1) and Definition (2.2), we get the following results*

$$\begin{aligned}
i)(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= 0,\\
ii)(L_{X^H}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= 0,\\
iii)(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= 0,\\
iv)(L_{X^H}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= ((L_X\theta)Z - (\nabla_X\theta)(Z))^V,\\
v)(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= 0,\\
vi)(L_{X^H}\tilde{g}_{\nabla,c})(Y^H,\xi^V) &= (L_X\xi)Y - (\nabla_X\xi)(Y),\\
vii)(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^H,Z^H) &= (\nabla_Y\omega)(Z) + (\nabla_Z\omega)(Y),\\
viii)(L_{X^H}\tilde{g}_{\nabla,c})(Y^H,Z^H) &= ((L_Xc)(Y,Z))^V - pR(X,Y)(Z) - pR(X,Z)(Y),
\end{aligned}$$

where the vertical lift $\omega^V \in \Im_0^1(T^*M^n)$ of $\omega \in \Im_1^0(M^n)$, the horizontal and complete lifts $X^H, X^C, \in \Im_0^1(T^*M^n)$ of $X \in \Im_0^1(M^n)$, defined by (1.1),(1.2),(1.3), respectively.

Proof. $i)$

$$\begin{aligned}
(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= L_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(L_{\omega^V}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,L_{\omega^V}\xi^V)\\
&= 0
\end{aligned}$$

$ii)$

$$\begin{aligned}
(L_{X^H}\tilde{g}_{\nabla,c})(\theta^V,\xi^V) &= L_{X^H}\tilde{g}_{\nabla,c}(\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(L_{X^H}\theta^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,L_{X^H}\xi^V)\\
&= -\tilde{g}_{\nabla,c}((\nabla_X\theta)^V,\xi^V) - \tilde{g}_{\nabla,c}(\theta^V,(\nabla_X\xi)^V)\\
&= 0
\end{aligned}$$

$iii)$

$$\begin{aligned}
(L_{\omega^V}\tilde{g}_{\nabla,c})(\theta^V,Z^H) &= L_{\omega^V}\tilde{g}_{\nabla,c}(\theta^V,Z^H) - \tilde{g}_{\nabla,c}(L_{\omega^V}\theta^V,Z^H) - \tilde{g}_{\nabla,c}(\theta^V,L_{\omega^V}Z^H)\\
&= L_{\omega^V}\theta(Z) + \tilde{g}_{\nabla,c}(\theta^V,(\nabla_Z\omega)^V)\\
&= 0
\end{aligned}$$

$iv)$

$$
\begin{aligned}
(L_{X^H}\tilde{g}_{\nabla,c})(\theta^V, Z^H) &= L_{X^H}\tilde{g}_{\nabla,c}(\theta^V, Z^H) - \tilde{g}_{\nabla,c}(L_{X^H}\theta^V, Z^H) - \tilde{g}_{\nabla,c}(\theta^V, L_{X^H}Z^H) \\
&= L_{X^H}\theta(Z) - \tilde{g}_{\nabla,c}((\nabla_X\theta)^V, Z^H) \\
&\quad -\tilde{g}_{\nabla,c}(\theta^V, [X,Z]^H + (pR(X,Z))^V) \\
&= (L_X\theta(Z)) - (\nabla_X\theta)(Z) - \tilde{g}_{\nabla,c}(\theta^V, [X,Z]^H) \\
&\quad -\tilde{g}_{\nabla,c}(\theta^V, (pR(X,Z))^V) \\
&= ((L_X\theta(Z)) - (\nabla_X\theta)(Z) - \theta(L_XZ))^V \\
&= ((L_X\theta)Z - (\nabla_X\theta)(Z))^V
\end{aligned}
$$

$v)$

$$
\begin{aligned}
(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^H, \xi^V) &= L_{\omega^V}\tilde{g}_{\nabla,c}(Y^H, \xi^V) - \tilde{g}_{\nabla,c}(L_{\omega^V}Y^H, \xi^V) - \tilde{g}_{\nabla,c}(Y^H, L_{\omega^V}\xi^V) \\
&= L_{\omega^V}\xi(Y) + \tilde{g}_{\nabla,c}((\nabla_Y\omega)^V, \xi^V) \\
&= 0
\end{aligned}
$$

$vi)$

$$
\begin{aligned}
(L_{X^H}\tilde{g}_{\nabla,c})(Y^H, \xi^V) &= L_{X^H}\tilde{g}_{\nabla,c}(Y^H, \xi^V) - \tilde{g}_{\nabla,c}(L_{X^H}Y^H, \xi^V) - \tilde{g}_{\nabla,c}(Y^H, L_{X^H}\xi^V) \\
&= X^H\xi(Y) - \tilde{g}_{\nabla,c}([X,Y]^H + (pR(X,Y))^V, \xi^V) \\
&\quad -\tilde{g}_{\nabla,c}(Y^H, (\nabla_X\xi)^V) \\
&= (L_X\xi(Y))^V - \tilde{g}_{\nabla,c}([X,Y]^H, \xi^V) \\
&\quad -\tilde{g}_{\nabla,c}(pR(X,Y))^V, \xi^V) - (\nabla_X\xi)(Y) \\
&= ((L_X\xi(Y)) - \xi(L_XY) - (\nabla_X\xi)(Y)) \\
&= (L_X\xi)Y - (\nabla_X\xi)(Y)
\end{aligned}
$$

$vii)$

$$
\begin{aligned}
(L_{\omega^V}\tilde{g}_{\nabla,c})(Y^H, Z^H) &= L_{\omega^V}\tilde{g}_{\nabla,c}(Y^H, Z^H) - \tilde{g}_{\nabla,c}(L_{\omega^V}Y^H, Z^H) - \tilde{g}_{\nabla,c}(Y^H, L_{\omega^V}Z^H) \\
&= \omega^V c(Y,Z) + \tilde{g}_{\nabla,c}((\nabla_Y\omega)^V, Z^H) + \tilde{g}_{\nabla,c}(Y^H, (\nabla_Z\omega)^V) \\
&= (\nabla_Y\omega)(Z) + (\nabla_Z\omega)(Y)
\end{aligned}
$$

$viii)$

$$
\begin{aligned}
(L_{X^H}\tilde{g}_{\nabla,c})(Y^H, Z^H) &= L_{X^H}\tilde{g}_{\nabla,c}(Y^H, Z^H) - \tilde{g}_{\nabla,c}(L_{X^H}Y^H, Z^H) - \tilde{g}_{\nabla,c}(Y^H, L_{X^H}Z^H) \\
&= X^H c(Y,Z) - \tilde{g}_{\nabla,c}([X,Y]^H + (pR(X,Y))^V, Z^H) \\
&\quad -\tilde{g}_{\nabla,c}(Y^H, [X,Z]^H + (pR(X,Z))^V \\
&= (Xc(Y,Z))^V - \tilde{g}_{\nabla,c}([X,Y]^H, Z^H) - \tilde{g}_{\nabla,c}((pR(X,Y))^V, Z^H) \\
&\quad -\tilde{g}_{\nabla,c}(Y^H, [X,Z]^H - \tilde{g}_{\nabla,c}(Y^H, (pR(X,Z))^V) \\
&= ((L_Xc(Y,Z)) - c(L_XY,Z) - c(Y,L_XZ))^V \\
&\quad -pR(X,Y)(Z) - pR(X,Z)(Y) \\
&= ((L_Xc)(Y,Z))^V - pR(X,Y)(Z) - pR(X,Z)(Y)
\end{aligned}
$$

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