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A priori estimates for the solutions to a kind of degenerate elliptic-parabolic equations

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Abstract. *We obtain a priori estimates for the solutions to some degenerate elliptic-parabolic equations.*

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1 Introduction and Main Results

Let Ω be is a bounded open set in R^n and $Q_T = \Omega \times (0,T)\,, T > 0$. We consider the following initial boundary value problems

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\left(x,t\right)\frac{\partial u}{\partial x_j}\right) - \Psi\left(x,t\right)\frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} b_i\left(x,t\right)\frac{\partial u}{\partial x_i} + c\left(x,t\right)u = 0,$$
$$(x,t) \in Q_T \tag{1.1}$$

$$u\left(x,t\right) = f\left(x,t\right), \quad (x,t) \in \Gamma = (0,T) \times \partial\Omega, \tag{1.2}$$

$$u\left(0,x\right) = h\left(x\right), \quad x \in \Omega. \tag{1.3}$$

Problems of the form (1.1)-(1.3) arise as mathematical models of various applied problems, for instance reaction-drift-diffusion processes of electrically charged species phase transition processes and transport processes in porous media. Investigations of boundary value problems for second order degenerate elliptic-parabolic equations ascend to the work by Keldysh [7], where correct statements for boundary value problems were considered for the case of one space variable as well as existence and uniqueness of solutions. In the work by Fichera [2] boundary value problems were given for multidimentional case. He proved existence of generalized solutions to these boundary value problems.

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The equation (1.1) is degenerate and d the function $\Psi(x,t)$ can tend to zero. Initial boundary problems for degenerate parabolic equations have been studied by many authors (see for example [1], [2], [3], [5]). But the structure of the equation (1.1) is different from that one considered in these papers. Boundary value problems for the degenerate equation also were studied in the stationary case in [6] and in the nonstationary case in [7].

We consider problem (1.1)-(1.3) under standard conditions for the functions $a_{ij}(x,t)$ and some conditions for the functions $b_i(x,t), c(x,t)$.

We formulate our assumptions in section 2. In Section 3 we give a priori estimate for the solution.We assume following regularity condition on the boundary $\partial\Omega$ of the set Ω. There exist positive numbers $\chi, R_0,$ such that for an arbitrary point $x \in \partial\Omega$ the inequality $\{B(x,R)\backslash\Omega\} \geq \chi R^n$ holds, where $0 < R \leq R_0$ and $B(x,R)$ is a ball of radius R with center x.

Let the coefficients of (1.1)-(1.3) satisfy the following assumptions. The matrix $\|a_{ij}(x,t)\|$ is a real symmetric one $(x,t) \in Q_T$ and $\xi \in R^n$ the following inequality (condition)holds

$$\gamma\omega(x)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \gamma^{-1}\omega(x)|\xi|^2, \tag{1.4}$$

where the weight $\omega(x)$ will be a non-negative, measurable function and we assumed on the weight $\omega(x)$ in (1.4) have properties typical of the A_p weights and coefficients belong to some suitable weighted L_p spaces, $\gamma \in (0,1]$ $a_{ij}(x,t)$, $c(x,t)$ $b_i(x,t)$, $i,j = \overline{1,n}$ are measurable functions with respect to x,t for every $(x,t) \in Q_T$. Also

$$c(x,t) \leq 0,\ c(x,t) \in L_{n+1}(Q_T), \tag{1.5}$$

$$|b(x,t)| \in L_{n+2}(Q_T), \quad |b(x,t)|^2 + Kc(x,t) \leq 0. \tag{1.6}$$

Assume that the following conditions are true for the weighted functions

$$\Psi(x,t) = \omega(x)\lambda(t)\varphi(T-t),$$

where $\omega(x) \in A_p$ satisfies the Muckenhoupt condition (see [3]), $\lambda(t) \geq 0$

$$\lambda(t) \in C^1([0,T]),\ \varphi(z) \geq 0,\ \varphi'(z) \geq 0,\ \varphi(z) \in C^1([0,T]),$$

$$\varphi(0) = \varphi'(0) = 0,\ \varphi(z) \geq \beta z\varphi'(z) \tag{1.7}$$

where $\beta-$positive constant.

We consider the problem (1.1)-(1.3) with the following data

$$f(x,t) \in L_\infty(Q_T) \cap L_\infty\left(0,T,W_2^1(\Omega)\right) \cap L_1\left(0,T,W_\infty^1(\Omega)\right),$$

$$\frac{\partial f}{\partial t} \in L_1(0,T,L_\infty(\Omega)), \tag{1.8}$$

$$h(x) \in L_\infty(\Omega). \tag{1.9}$$

We introduce some function spaces in Q_T with finite norms

$$\|u\|_{W_{2,\omega}^1(Q_T)} = \left(\int_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^{n} u_{x_i}^2\right)dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_2^2(Q_T)} = \left(\int_{Q_T} \left(u^2 + \sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=1}^{n} u_{x_ix_j}^2\right)dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_2^{2,1}(Q_T)} = \|u\|_{W_2^2(Q_T)} + \|u_t\|_{h_2(Q_T)},$$

$$\|u\|_{W^{2,2}_{2,\,\Psi(Q_T)}} = \left(\left(\int\limits_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_ix_j}^2 + u_t^2\right)\right.\right.$$

$$\left.\left. + \Psi^2(x,t)\, u_{tt}^2 + \Psi(x,t) \sum_{i=1}^n u_{it}^2\right) dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W^{1,2}_{2,\Psi(Q)}} = \left(\left(\int\limits_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + u_t^2\right) + \Psi(x,t)\, u_{tt}^2\right) dxdt\right)^{\frac{1}{2}}.$$

$\overset{0}{W}{}^{1,2}_{2,\Psi}(Q_T)$ the closer of $C^\infty\left(\bar{Q}_T\right)$-functions vanishing on $\Gamma(Q_T)$, with respect to the norm in $W^{1,2}_{2,\Psi}(Q_T)$.

A function $u \in L_2\left(0, T, W^{1,2}_{2,\Psi}(\Omega)\right)$ is called a weak solution of problem (1.1)-(1.3) if the following the integral identity holds

$$\int\limits_0^T \int\limits_\Omega \frac{\partial u}{\partial t} \hat{\varphi} dxdt$$

$$+ \int\limits_\Omega \int\limits_0^T \left[\sum_{i,j=1}^n a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial \hat{\varphi}}{\partial x_i} + \sum_{i=1}^n b_i(x,t)\left(\frac{\partial u}{\partial x_i}\right)\hat{\varphi} + c(x,t)\, u\hat{\varphi}\right] dxdt$$

$$- \int\limits_0^T \int\limits_\Omega \Psi(x,t) \frac{\partial^2 u}{\partial t^2} \hat{\varphi} dxdt = 0 \tag{1.10}$$

for any functions $\hat{\varphi} \in C^\infty\left(\bar{Q}_T\right)$ vanishing near Γ for almost all $(x,t) \in Q_T$. Also for $t \in (0,\tau)$

$$u - f(x,t) \in L_2\left(0, \tau, \overset{0}{W}{}^{1}_{2,\omega}(\Omega)\right).$$

Remark 1.1 Let u be a weak solution of problem (1.1)-(1.3). Since the set of functions from $C^\infty\left(\bar{Q}_T\right)$ vanishing near Γ is dense in $L^2\left(0, T, \overset{0}{W}{}^{1,2}_{2,\Psi}(\Omega)\right)$, the integral identity (1.10) holds for all $\varphi \in L_2\left(0, \tau, \overset{0}{W}{}^{1,2}_{2,\Psi}(\Omega)\right)$ such that

$$\int\limits_{Q_T} \omega(x) \left|\frac{\partial u}{\partial x}\right|^2 dxdt + \int\limits_{Q_T} \Psi(x,t) \left|\frac{\partial u}{\partial t}\right|^2 dxdt + \int\limits_{Q_T} \Psi_t(x,t)|u_t||u| dxdt < \infty.$$

Besides of (1.1), we consider also the regularized equation, where instead of $\omega(x)$ we take $\omega_\varepsilon(x)$, $\Psi(x,t) = \Psi_\varepsilon(x,t)$,

$$\omega_\varepsilon(x) = \max\left\{\omega(x), \omega\left(-\frac{1}{\varepsilon}\right)\right\}$$

for

$$\varepsilon \in (0,1],\ \omega_0(x) = \omega(x) \tag{1.11}$$

$\Psi_\varepsilon(x,t)$ is defined so: for any fixed $\varepsilon \in (0,1)$

$$\Psi_\varepsilon(x,t) = \Psi(x,t) - \frac{\Psi'_x(x,t)\,\varepsilon}{m} + \frac{\Psi'_x(x,t)}{m\varepsilon^{m-1}}\varepsilon^m$$

at $x \in (0,\varepsilon)$, $\Psi_\varepsilon(x,t) = \Psi(x,t)$ at

$$x \in [\varepsilon, 1],\ m = \frac{2}{\beta}. \tag{1.12}$$

Everywhere further we consider the case when $\Psi(x,t) > 0$ at $x > 0$. If $\Psi(x,t) \equiv 0$ then the equation (1.1)-parabolic.

We consider an auxiliary problem, called regularized problem, obtained by (1.1)-(1.3) substituting the weight functions $\omega(x)$ and $\Psi(x,t)$ with $\omega_\varepsilon(x)$ and $\Psi_\varepsilon(x,t)$. Under a solution of this problem we mean a function $u \in L_2\left(0, T, W^{1,2}_{2,\Psi}(\Omega)\right)$ satisfying the condition (1.10) and Remark 1.1 with weights $\omega_\varepsilon(x)$ and $\Psi_\varepsilon(x,t)$.

In what follows we understand as known parameters all numbers from the conditions, norm of functions $f, \varphi(x)$ in respective spaces and numbers that depend only on n, χ, R_0, Ω, $\omega(x)$, $\Psi(x,t)$.

Theorem 1.1 *Let the conditions (1.4)-(1.9) hold with the regularized weights* $\omega_\varepsilon(x), \Psi_\varepsilon(x,t)$. *Then for each* $\varepsilon \in (0,1]$ *there exists a constant* M_1 *depending on known quantities and independent on* ε *such that if* u *is a solution of the regularized problem then*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \left\{\Lambda_1(u(t,x)) + \Lambda_2(u(t,x))\right\} dx + \int \omega_\varepsilon(x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$+\int \Psi^2_\varepsilon(x,t)\left|\frac{\partial^2 u}{\partial t^2}\right|^2 dxdt \le M_1, \tag{1.13}$$

where

$$\Lambda_1(u) = \int_0^u s\omega(s)\,ds,\ \Lambda_2(u) = \int_0^u s\Psi(s,t)\,ds. \tag{1.14}$$

Proof of Theorem 1.1 Let $u(t,x)$ be the solution regularized problem (1.1)-(1.3). We extend function $u(t,x)$ by setting $u(t,x) = \varphi(x)$ for $t < 0$, $x \in \Omega$. Denote

$$\bar{u}(t,x) = u(t,x) - f(t,x).$$

Testing (1.10) with $\hat{\varphi}(x) = \bar{u}(t,x) - \bar{u}(x,t-s)$, we obtain for $s \in (0,T)$, $\tau \in (0, T-s)$

$$\int_0^{\tau+s}\int_\Omega \left\{\frac{\partial \bar{u}}{\partial t}\left[\bar{u}(t,x) - \bar{u}(x,t-s)\right] + \sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial \bar{u}}{\partial x_j}\right.$$

$$\times \frac{\partial}{\partial x_i}\left[\bar{u}(t,x) - \bar{u}(x,t-s)\right] + \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\left[\bar{u}(t,x) - \bar{u}(x,t-s)\right]$$

$$+ c(t,x)\left[\bar u(t,x) - \bar u(x,t-s)\right]\} \,dxdt$$

$$+\int\limits_0^{\tau+s}\int\limits_\Omega \Psi(x,t)\frac{\partial^2 u}{\partial t^2}\left[\bar u(t,x) - \bar u(x,t-s)\right] dxdt = 0.$$

Hence we get by simple calculation

$$\int\limits_0^{\tau+s}\int\limits_\Omega \frac{\partial u}{\partial t}\left[\bar u(t,x) - \bar u(x,t-s)\right] dxdt + \int\limits_0^{\tau+s}\int\limits_\Omega a_{ij}(x,t)\left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$-s\int\limits_\Omega a_{ij}(x,t)\left|\frac{\partial v_0}{\partial x}\right|^2 dxdt + \int\limits_{-s}^{\tau}\int\limits_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\left[\bar u(t,x) - \bar u(x,t-s)\right] dxdt$$

$$+\int\limits_{-s}^{\tau}\int\limits_\Omega c(t,x)\left[\bar u(t,x) - \bar u(x,t-s)\right] dxdt$$

$$+\int\limits_{-s}^{\tau}\int\limits_\Omega \Psi(x,t)\frac{\partial^2 u}{\partial t^2}\left[\bar u(t,x) - \bar u(x,t-s)\right] dxdt = 0.$$

where denote by $v_0(x)$ the solution of problem (1.1)-(1.3) for $t = 0$ with $u(0,x)$ defined by (1.3).

Dividing this equality by s and passing to the limit $s \to 0$, we obtain for almost every $\tau \in (0,T)$ and doing some calculations

$$\int\limits_\Omega a_{ij}(x,t)\left|\frac{\partial u(t,x)}{\partial x}\right|^2 - \int\limits_\Omega a_{ij}(x)\left|\frac{\partial v_0(x)}{\partial x}\right|^2 dxdt$$

$$+\int\limits_0^{\tau}\int\limits_\Omega \sum_{i=1}^n b_i(x,t)\frac{\partial u(t,x)}{\partial x_i} dxdt$$

$$\le \int\limits_0^{\tau}\int\limits_\Omega c(t,x)\left|\frac{\partial u}{\partial x}\right|^2 dxdt. \tag{1.15}$$

Using (1.10) we can write in (1.15)

$$\int\limits_\Omega \omega_\varepsilon(x)\left|\frac{\partial u(\tau,x)}{\partial x}\right|^2 dx + \int\limits_0^{\tau}\frac{\partial u}{\partial t}\bar u(t,x)\,dt + \int \Psi_\varepsilon^2(x,t)\left|\frac{\partial^2 u(\tau,x)}{\partial t^2}\right|^2 dx$$

$$\le C_1\left\{1 + \int\limits_0^{\tau}\int\limits_\Omega \left|\frac{\partial u(t,x)}{\partial x}\right|^2 dxdt\right\}. \tag{1.16}$$

Here and in what follows C_i denote constants depending only on known parameters. The conditions (1.8), (1.9) and Remark 1.1 allow us to substitute $\varphi = \bar u$ in the regularized identity (1.10).

By (1.16) this gives

$$\int\limits_0^{\tau}\frac{\partial u}{\partial t}\left(u(x,t) - f(x,t)\right) dt$$

$$+\int\limits_0^{\tau}\int\limits_\Omega \left\{\sum_{i=1}^n a_{ij}(t,x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i} + c(x,t)\,\bar u + \sum_{i=1}^n b_i(x,t)\frac{\partial u}{\partial x_i}\tilde u\right\} dxdt$$

$$\le \int\limits_0^{\tau}\int\limits_\Omega \left\{\sum_{i,j=1}^n a_{ij}(x,t)\frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_i} - c(x,t)\,u\right\} + c_1\left\{1 + \int\limits_0^{\tau}\int\limits_\Omega \left|\frac{\partial u(t,x)}{\partial x}\right|^2\right\} dxdt. \tag{1.17}$$

We write the first integral from (1.17) in the form

$$\int_0^\tau \frac{\partial u}{\partial t}\left(u-f\left(x,t\right)\right)dx=\int_0^\tau \frac{\partial u}{\partial t}\left(\left|u\right|_{-m}^m-f\left(x,t\right)\right)dt+\int_0^\tau \frac{\partial u}{\partial t}\left(u-\left|u\right|_{-m}^m\right)dt \qquad (1.18)$$

with $m\geq\left\|f\left(x,t\right)\right\|_{L_\infty(Q_T)}, \quad \left|u\right|_{-m}^m=\max\left\{\min\left[u,m\right],-m\right\}.$

Then we can evolute the first and the second integral of the right hand side of (1.18) by using Lemma 1.1 and Lemma 1.2 respectively [3]. So we obtain

$$\int_0^\tau \frac{\partial u}{\partial t}\left(u-f\left(x,t\right)\right)dt=\int_0^\tau\left\{\int_0^{u(x,\tau)} s\omega\left(s\right)ds-\int_0^{h(x)} s\omega\left(s\right)ds\right\}dx$$

$$+\int_0^\tau\left\{\int_0^{u(x,\tau)} s\Psi\left(s,t\right)ds-\int_0^{h(x)} s\Psi\left(s,t\right)ds\right\}dx+\int_0^\tau\int_\Omega\left|u-h\left(x\right)\right|\frac{\partial f}{\partial t}dxdt$$

$$-\int_\Omega\left[u\left(\tau,x\right)-h\left(x\right)\right]f\left(\tau,x\right)dx. \qquad (1.19)$$

Immediately from the definition of (1.14) $\Lambda_1\left(u\right),\ \Lambda_2\left(u\right)$. We deduce

$$u<\varepsilon_1\left(\Lambda_1\left(u\right)+\Lambda_2\left(u\right)\right)+c_{\varepsilon_1} \qquad (1.20)$$

for $u\geq 0$ with arbitrary positive number ε and a constant c_ε depending only on ε_1 and the functions $\omega\left(x\right),\Psi\left(x,t\right)$. Using the condition (1.4)-(1.6), (1.8)-(1.9) and the conditions on $\omega\left(x\right),\Psi\left(x,t\right)$ and the inequality (1.20), we obtain with arbitrary positive number ε_1 and some function $\mu\left(t\right)\in L_1\left(0,T\right)$

$$\left|\int_0^\tau\int_\Omega\omega_\varepsilon\left(x\right)\left|\frac{\partial u}{\partial x}\right|^2\frac{\partial f\left(x,t\right)}{\partial x_j}dxdt\right|+\left|\int_0^\tau\int_\Omega\Psi_\varepsilon^2\left(x,t\right)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\left|\frac{\partial f}{\partial x_j}\right|dxdt\right|$$

$$\leq\varepsilon_1\int_0^\tau\int_\Omega\omega_\varepsilon\left(x\right)\left|\frac{\partial u}{\partial x}\right|^2dxdt+\varepsilon_1\int_0^\tau\int_\Omega\Psi_\varepsilon^2\left(x,t\right)\left|\frac{\partial^2 u}{\partial t^2}\right|^2dxdt$$

$$+\frac{c_2}{\varepsilon_1}\int_0^\tau\int_\Omega\left(\Lambda_1\left(u\right)+\Lambda_2\left(u\right)\right)\mu\left(t\right)dxdt$$

$$+\int_0^\tau\int_\Omega u\frac{\partial f}{\partial t}dxdt\leq c_2\left\{1+\int_0^\tau\int_\Omega\left(\Lambda_1\left(u\right)+\Lambda_2\left(u\right)\right)\mu\left(t\right)dxdt\right\}. \qquad (1.21)$$

We estimate terms in (1.17) involving the function α in standard way by using (1.4)-(1.6), (1.8)-(1.9). Now from (1.17), (1.19), (1.21) and evident estimates for another terms in (1.19), we obtain

$$\int_\Omega\left(\Lambda_1\left(u\left(\tau,x\right)\right)+\Lambda_2\left(u\left(\tau,x\right)\right)\right)dx+\int_0^\tau\int_\Omega\left[\omega_\delta\left(x\right)\left|\frac{\partial u}{\partial x}\right|^2+\Psi_\varepsilon^2\left(x,t\right)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt$$

$$\leq c_3\left\{1+\int_0^\tau\int_\Omega\left[1+\mu\left(t\right)\right]\left[\Lambda_1\left(u\right)+\Lambda_2\left(u\right)\right]dxdt. \qquad (1.22)$$

Now the last inequality and Gronwall's lemma complete the proof of Theorem 1.1.

Theorem 1.2 *Let the assumptions of Theorem 1.1 be satisfied. Then there exists a constant M_2,depending only on known parameters and independent of $\varepsilon \in [0,1]$, such that each solution of regularized problem (1.1)-(1.3) satisfies*

$$\int\limits_{Q_T} \left[\omega_\varepsilon (x) \left| \frac{\partial u}{\partial x} \right|^2 + \Psi_\varepsilon^2 (x,t) \left| \frac{\partial^2 u}{\partial t^2} \right|^2 \right] dxdt \le M_2. \tag{1.23}$$

In order to prove Theorem 1.2 we need auxiliary estimates.

Lemma 1.1 *Assume that the conditions of Theorem 1.1 are satisfied and following inequality*

$$\underset{t\in(0,\tau)}{ess\sup} \int\limits_{\Omega} u^q (t,x)\, dx \le K_1 \tag{1.24}$$

is fulfilled with some numbers $q \in \left(\frac{2n}{n+2}, \frac{n}{2} \right)$, K_1 depending only on known parameters. Then the estimate

$$\underset{t\in(0,\tau)}{ess\sup} \left\{ \int\limits_{\Omega} |u(t,x)|^{\frac{pn}{n-2}} dx + \int\limits_{\Omega} |u(t,x)|^{p-2} \left| \frac{\partial u(t,x)}{\partial x} \right|^2 dx \right\} \le K_2 \tag{1.25}$$

holds with a number $p > 2$ defined by the equality

$$\rho \frac{n}{n-2} = (p-1) \frac{q}{q-1} \tag{1.26}$$

and with a constant K_2 depending only on known parameters.

Proof. Denote

$$m_0 = \|f(x,t)\|_{L_\infty(Q_T)} + \|h(x)\|_{L_\infty(\Omega)} + 1 \tag{1.27}$$

and use following notations for $K \in R^1$ and arbitrary function β defined on Q_T

$$\beta_k(t,x) = [\beta(t,x)]_k = \min\{\beta(t,x), k\},$$

$$\beta_+(t,x) = [\beta(t,x)]_+ = \max\{\beta(t,x), 0\}.$$

We test the integral identity (1.10) with $\varphi(x,t) = signu\, [|u|_k - m_0]^{p-1}$ with $k > m_0$. Using the condition (1.4)-(1.6), (1.8)-(1.9) and Holder inequality we obtain

$$\int\limits_{\Omega} [|u|_k - m_0]_+^{p-2} \left| \frac{\partial u_k}{\partial x} \right|^2 dx \le C_4 \left\{ \int [|u| - m_0]_+^{(p-1)\frac{q}{q-1}} dx \right\}^{\frac{q-1}{q}}. \tag{1.28}$$

From this inequality and the embedding theorem we have

$$\left\{ \int\limits_{\Omega} [|u|_k - m_0]^{\frac{pn}{n-2}} dx \right\}^{\frac{n-2}{n}} \le C_5 \left\{ \int\limits_{\Omega} [|v|_k - m_0]_+^{(p-1)\frac{q}{q-1}} dx \right\}^{\frac{q-1}{q}}. \tag{1.29}$$

Taking into account the restriction on q and the choice of p we deduce (1.25) from (1.28), (1.29), (1.13) and the proof is completed.

Proof of Theorem 1.2 We assume firstly that $\frac{2+\gamma}{1+\gamma}=\frac{n}{2}$. It is simple to check [8] imply

$$|u|\leq C_0 \;\; for\; u<0. \tag{1.30}$$

For proving regularity properties of the function u we need following growth condition

$$\rho_1^{-1}\left(u^{\gamma}+1\right)\leq u\leq\rho_1\left(u^{\gamma}+1\right),\; u>0,\;\; 0\leq\gamma<\frac{2}{n-2} \tag{1.31}$$

with some positive constants ρ_1. From (1.31) implies $u\leq\rho_1\left(\frac{u^{\gamma+1}}{\gamma+1}+u\right)$ for $u>0$ with $\gamma+1<\frac{n}{n-2}$. Remark that such type condition arised in [5] for $n>2$ together with the stronger restriction $\gamma+1<\frac{2}{n-2}$.

From (1.30) and (1.31) we find

$$|u|^{q_0}\leq C_7\left[\Lambda_1(u)+\Lambda_2(u)+1\right]$$

with

$$q_0=\frac{2+\gamma}{1+\gamma}. \tag{1.32}$$

Using (1.32), (1.13) and Lemma 1.1, we obtain (1.25) with ρ_0 defined by the equality

$$\rho_0\frac{n}{n-2}=(\rho_0-1)(2+\gamma).$$

This ρ_0 satisfies the inequality $\rho_0-2>\frac{n}{n-2}>\gamma$.

Consequently, (1.25), (1.31) imply

$$\iint\limits_{\{|u|<2|u|\}}\left[\omega_\varepsilon(x)\left|\frac{\partial u}{\partial x}\right|^2+\Psi_\varepsilon^2(x)\left|\frac{\partial^2 u}{\partial t^2}\right|^2\right]dxdt\leq C_8. \tag{1.33}$$

Here $\{|u|\leq 2|u|\}=\{(t,x)\in Q_T:|u(t,x)|\leq 2|u(t,x)|\}$ and analogous notations we shall use further.

We want to establish a estimate analogous to (1.33) with respect to set $|u|>2u$. Taking into account that $\omega_\varepsilon(u)\leq 1+\omega(0)$ for $u<0$, we can restrict ourselves to the set $\{|u|>2u\}$ .We substitute the test function

$$\varphi=\left\{\left[u-|u_k|_+\right]_k+|u|_k+m_0\right\}\operatorname{sign}u$$

with $k>m_0,\;\bar{\gamma}>0$ in (1.10). After standard calculations we obtain

$$I_1\equiv\iint\limits_{\{|u|<k\}}\left\{\left[u-|u|_+\right]_k+|u|+m_0\right\}^{\bar{\gamma}}\left|\frac{\partial u}{\partial x}\right|^2dxdt\leq C_9\left(I_2+I_3\right), \tag{1.34}$$

where

$$I_2=\iint\limits_{\{|u|_k<u\}}\left(|u|_k+m_0\right)\left\{\left[u-|u|_k\right]_k+|u|_k+m_0\right\}^{\bar{\gamma}-1}\left|\frac{\partial u}{\partial x}\right|\left|\frac{\partial u}{\partial t}\right|dxdt$$

$$I_3=\iint\limits_{Q_T}\left\{(u_++1)^{\gamma-1}\right\}\left(|u|_k+1\right)\left\{|u_+|_{2k}+|u|_k+1\right\}^{\bar{\gamma}}dxdt.$$

The integral I_2 will be estimated is different ways for $\bar{\gamma} \le 1$ and for $\bar{\gamma} > 1$. For $\bar{\gamma} \le 1$ we have

$$I_2 \le \iint\limits_{\{|u|_k<u\}} \left\{ \left\{ (u+m_0)^{\bar{\gamma}} \left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right\} dxdt \right.$$

$$\le 3 \int\int\limits_{\{u>0\}} \left\{ (u+m_0)^{\bar{\gamma}} \right\} \left| \frac{\partial u}{\partial x} \right|^2 + (|u|+m_0)^{\bar{\gamma}} \left| \frac{\partial u}{\partial x} \right|^2 dxdt \le c_{10}. \qquad (1.35)$$

Here we used (1.25) and the inequality

$$\underset{t\in(0,T)}{ess\sup} \int\limits_{\Omega} u_+^{2+\gamma}(t,x)\,dx + \iint\limits_{\{u>0\}} (1+u)^{\gamma} \left[\left| \frac{\partial u}{\partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] dxdt \le c_{11} \qquad (1.36)$$

that follows from (1.13), (1.31).

For $\bar{\gamma} > 1$ we estimate I_2 by using the evident inequality

$$\left[\left[[|u-|u|_k]_+ \right]_k \right] + |u|_k + m_0 \le 2\,|u|_k + m_0$$

on the set $\{|u| \ge k\}$. Then we have

$$I_2 \le \varepsilon_1 I_1 + C_{12} \iint\limits_{\{u>0\}} \left\{ (u+m_0)^{\bar{\gamma}} \left| \frac{\partial u}{\partial x} \right|^2 + \frac{1}{\varepsilon_1^{q-1}} (|u|+m_0)^q \left| \frac{\partial u}{\partial t} \right|^2 \right\} dxdt, \qquad (1.37)$$

where the last integral can be estimated analogously to (1.35).

Using Holders inequality and the embedding theorem we obtain for $\delta \ge 0$

$$\int\limits_{Q_T} |[u_+]_k - f_+(x,t)|^{(2+\gamma)\frac{2}{n}+2+\delta} dxdt \le \int\limits_0^T \left\{ \int\limits_{\Omega} |[u_+]_k - f_+(x,t)|^{2+\gamma} dx \right\}^{\frac{2}{n}}$$

$$\times \left\{ \int\limits_{\Omega} \left(|[u_+]_k - f_+(x,t)|^{1+\frac{\delta}{2}} \right)^{\frac{2n}{n-2}} dx \right\}^{\frac{n-2}{n}} dt$$

$$\le c_{13} \left\{ \underset{t\in(0,T)}{ess} \sup \int\limits_{\Omega} |[u_+]_k - f_+(x,t)|^{2+\gamma} dx \right\}^{\frac{2}{n}}$$

$$\times \int\limits_{Q_T} |[u_+]_k - f_+(x,t)|^{\delta} \left| \frac{\partial}{\partial x} \left([u_+]_k - f_+(x,t) \right) \right|^2 dxdt. \qquad (1.38)$$

Choosing $\delta = 0$, the inequalities (1.13), (1.34) and condition (1.8) imply

$$\int\limits_{Q_T} u_+^{(2+\gamma)\frac{2}{n}+2} dxdt \le C_{14}. \qquad (1.39)$$

We estimate I_3 by Young's inequality and condition (1.8), obtain

$$I_3 \le C_{15} \left\{ 1 + \int\limits_{Q_T} u_+^{\gamma+\bar{\gamma}+2} dxdt + \int\limits_{Q_T} |u|^{\gamma+\bar{\gamma}+2} dxdt \right\}. \qquad (1.40)$$

The integral with u can be estimated by a constant in virtue of the inequality (1.25) in the case that $\bar{\gamma} \in [0, \gamma]$. If γ is such that

$$2\gamma + 2 \leq (2+\gamma)\,\frac{2}{n} + 2,$$

the integral with u_+ and $\bar{\gamma} = \gamma$ in (1.40) can be also estimated by a constant because of the inequality (1.39). In the opposite case we choose $\bar{\gamma}$ satisfying the condition

$$\gamma + \bar{\gamma} + 2 \leq (2+\gamma) \cdot \frac{2}{n} + 2\,.$$

For example we can take $\bar{\gamma} = \bar{\gamma}_1 = \frac{2}{n}$. For such choice of $\bar{\gamma}$ we get from (1.34), (1.36), (1.37), (1.40) $I_1 \leq C_{10}$, which implies

$$\int\limits_{|u|>\{u\}} \left(u - |u|^{\bar{\gamma}}\right) \left|\frac{\partial u}{\partial x}\right|^2 dxdt \leq C_{17}$$

and consequently

$$\int\limits_{|u|>2\{u\}} [u\,(t,x)]^{\bar{\gamma}} \left|\frac{\partial u}{\partial x}\right|^2 dxdt \leq C_{18}. \tag{1.41}$$

From (1.13), (1.33), (1.41) we obtain

$$\int\limits_{Q_T} |u|^{\bar{\gamma}} \left|\frac{\partial u}{\partial x}\right|^2 dxdt \leq C_{19}, \quad \int\limits_{Q_T} |u|^{\bar{\gamma}} \left|\frac{\partial u}{\partial t}\right|^2 dxdt \leq C_{19} \tag{1.42}$$

and this ends the proof of Theorem 1.2 in the case $\frac{2+\gamma}{1+\gamma} < \frac{n}{2}$, $\bar{\gamma} = \gamma$.

If $\bar{\gamma} = \bar{\gamma}_1 < \gamma$, we can iterate our discussions with respect to $\bar{\gamma}$. Using (1.42) we obtain from (1.38)

$$\int\limits_{Q_T} u_+^{(2+\gamma)\frac{n}{2}+2+\bar{\gamma}_1} dxdt \leq C_{20},$$

that allows us to choose $\bar{\gamma}_2 = \min\left\{\gamma, \frac{1}{n}\right\}$. Repeating this argument, if necessary, we can chose $\bar{\gamma}_3 = \gamma$ and we proved the Theorem if $\frac{2+\gamma}{1+\gamma} < \frac{n}{2}$.

If $\frac{2+\gamma}{1+\gamma} = \frac{n}{2}$ we can use Lemma 1.1 with $q^1 < q$ instead of q. We can choose such q' that the corresponding p' satisfies $p' - 2 > \gamma$ and then we keep all discussions of the previous proof. If $\frac{2+\gamma}{1+\gamma} > \frac{n}{2}$, then the boundedness of solutions of the equation (1.1) and the assumption formulated above is will known [3]. Theorem 1.2 is proved.

Lemma 1.2 *Assume that the conditions of Theorems are satisfied and*

$$\underset{t\in(0,T)}{ess\sup}\int\limits_{\Omega} u_+^q\,(t,x)\,dx + \iint\limits_{\{u>1\}} \left[\omega_\varepsilon^2\,(x)\,u^{q-2}\left|\frac{\partial u}{\partial x}\right|^2\right.$$

$$\left. + \Psi_\varepsilon^2\,(x,t)\,u^{q-2}\,(x,t)\left|\frac{\partial u}{\partial t}\right|^2\right] dxdt \leq K_3 \tag{1.43}$$

holds with numbers $q \in \left[\frac{2+\gamma}{1+\gamma}, \frac{n}{2}\right]$, K_3 *depending only on known parameters. Then there exist positive constants*

$$\int\limits_{\{u>1\}} \left[\omega_\varepsilon^2(x)\, u^{q-2+\beta} \left|\frac{\partial u}{\partial x}\right|^2 + \Psi_\varepsilon^2(x,t)\, u^{q-2+\beta} \left|\frac{\partial u}{\partial t}\right|^2\right] dxdt \le K_4. \tag{1.44}$$

Proof. By Theorem 1.2 follows that (1.43) holds for $q = q_0 = \frac{2+\gamma}{1+\gamma}$. We shall prove (1.44) for this value of q. The proof of the lemma for $\frac{2+\gamma}{1+\gamma} < q < \frac{n}{2}$ is the same as for $q = \frac{2+\gamma}{1+\gamma}$. From Lemma 1.1 with $q = \frac{2+\gamma}{1+\gamma}$ we obtain analogously to (1.33)

$$\int\limits_{\{|u|<2|u|\}} \left[\omega_\varepsilon^2(x)\, u^{q_0-2+\beta_1} \left|\frac{\partial u}{\partial x}\right|^2 + \Psi_\varepsilon^2(x,t)\, u^{q_0-2+\beta_1} \left|\frac{\partial u}{\partial t}\right|^2\right] dxdt \le C_{21}, \tag{1.45}$$

$$\beta_1 = \frac{2-(n-2)\gamma}{(1+\gamma)(n-2)}.$$

For the proof of (1.44) it is sufficient to check that the integral I_1 in (1.34) can be estimated by a constant for $\bar\gamma = \gamma + (1+\gamma)\beta_2$ with positive β_2 depending only on γ, u. This estimation of I_1 runs analogously to the corresponding estimation in the proof of Theorem 1.2. Hence we make only some remarks.

We change the inequality (1.35) for $\bar\gamma \le 1$, $\bar\gamma \le \gamma + \frac{1}{2}(\rho_0 - 2 - \gamma)$, $\rho_0 = \frac{q_0(n-2)}{n-2q_0} > 2 + \frac{2}{n-2}$, in the following way

$$I_2 \le 3 \int\limits_{\{u>0\}} \left\{(u+m_0)^\gamma \left|\frac{\partial u}{\partial x}\right|^2 + (|u|+m_0)^{\rho_0-2} \left|\frac{\partial u}{\partial x}\right|^2\right\} dxdt \le C_{22} \tag{1.46}$$

after using Theorem 1.2 and Lemma 1.1. Analogously we change (1.37) for $\bar\gamma > 1$. In order to estimate I_3 we remark that (1.38) and Theorem 1.2 imply

$$\int\limits_{Q_T} u_+^{(2+\gamma)\left(1+\frac{2}{n}\right)} dxdt \le c_{23}. \tag{1.47}$$

From (1.40), (1.47), (1.25) we see that the integral I_3 can be estimated by a constant, provided

$$\gamma + \bar\gamma + 2 \le (2+\gamma)\left(1+\frac{2}{n}\right), \quad \gamma + \bar\gamma + 2 \le \frac{\rho_0 n}{n-2}.$$

But both of these restrictions can be satisfied with $\bar\gamma = \gamma + (1+\gamma)\beta_3$ and some positive β_3 depending only on n, γ. Therefore we can chose positive β_2 such that the integral I_1 with $\gamma = \bar\gamma + 1(1+\gamma)\beta_2$ is estimated by a constant depending only on known parameters. From this estimate and (1.45) we obtain the inequality (1.44).

Lemma 1.3 *Assume that the conditions of Theorem 1.2 are satisfied. Then there exist numbers* $\bar q$, K_3 *depending only on known parameters, such that* $\bar q > \frac{n}{2}$ *and*

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int\limits_{\Omega} u_+^{\bar q}(t,x)\, dx +$$

$$+ \iint\limits_{\{u>1\}} \left[\omega_\varepsilon^2(x)\, u^{\bar q-2} \left|\frac{\partial u}{\partial x}\right|^2 + \Psi_\varepsilon^2(x,t)\, u^{\bar q-2} \left|\frac{\partial^2 u}{\partial t^2}\right|^2\right] dxdt \le K_3. \tag{1.48}$$

Proof. We substitute the function

$$\varphi + [u_k - m_0]_+^2 \left\{1 + [u_k - m_0]^3\right\}^r, \quad r \in \left(-\frac{2}{3}, \infty\right) \tag{1.49}$$

in the integral identity (1.10). Then using Lemma 1.1 from [3], we can evaluate the first summand of (1.49) to obtain

$$\int_0^\tau \frac{\partial u}{\partial t} \varphi dt = \int_\Omega \Lambda^{(\tau)} (u(\tau, x))\, dx, \tag{1.50}$$

where

$$\Lambda^{(\tau)}(u) = \int_0^u (\omega(s) + \Psi(s,t)) \left([S_k - m_0]_+^2\right) \left\{\frac{1}{2} + [S_k - m_0]^3 r\right\} ds$$

$$\geq \frac{1}{3(r+1)} \left\{\frac{1}{2} + [u_k - m_0]^3\right\}^{r+1}, \tag{1.51}$$

for $u > m_0$. Here $S_k = \min[s, k]$ and the value of u is analogous. We write the derivative of $\hat{\varphi}$ in the form

$$\frac{\partial \bar{\varphi}}{\partial x_i} = \left[\tilde{\Phi}^{(r)}(u_k) \frac{\partial u}{\partial x_i}\right] \chi(m_0 < u < k), \tag{1.52}$$

where $\chi(m_0 < u < k)$ is the characteristic function of the set $\{m_0 < u < k\}$ and the function $\tilde{\Phi}^r(u)$ satisfies for $r > -\frac{2}{3}$ the estimate

$$c_{24} k(r) \Phi^{(r)}(u)\, \omega(x) \leq \tilde{\Phi}^r(u) \leq c_{25}(r+1) \Phi^{(r)}(u)\, \omega(x), \tag{1.53}$$

with $k(r) = \min(1, 2 + 3r)$,

$$\Phi^{(r)}(u) = [u - m_0]_+ \left\{\frac{1}{2} + [u - m_0]^3\right\}^r. \tag{1.54}$$

Using (1.50)-(1.53) and conditions (1.4)-(1.6), (1.8) we obtain from (1.10) with the function φ defined by (1.49)

$$\int_\Omega \left\{\frac{1}{2} + [u_k(\tau, x) - m_0]_+^3\right\}^{r+1} dx + \int_0^\tau \int_\Omega \omega_\varepsilon^2(x) \Phi^{(r)}(u_k) \chi(m_0 < u < k) \left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$+ \int_0^\tau \int_\Omega \Psi_\varepsilon^2(x,t) \Phi^{(r)}(u_k) \chi(m_0 < u < k) \left|\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$+ \frac{r+1}{k(r)} \int_0^\tau \int_\Omega (1 + |u|) [u_k - m_0] \Phi^{(r)}(u_k)\, dxdt. \tag{1.55}$$

Let us assume now that for some $q \in \left[\frac{2+\gamma}{1+\gamma}, \frac{n}{2}\right]$ the inequality (1.43) is fulfilled. Then we obtain from Lemma 1.2 that the first integral of the right hand side of (1.55) can be estimated by a constant independent on k for $r = \frac{1}{2}[q - 3 + \beta]$. We shall check now that the second integral of the right hand site of (1.55) for $r = \frac{1}{3}\left[q - 3 + \beta'\right]$ and some positive

$\beta^{'}$ depending only on γ can be also estimated by a constant independent on k. Analogously to inequalities (1.38), (1.39) we obtain from (1.43)

$$\int\limits_{Q_T} u_+^{q(1+\gamma)\left(1+\frac{2}{n}\right)} dxdt \leq C_{27}. \quad (1.56)$$

From (43) and Lemma 1.1 we have

$$ess \sup_{t\in(0,\tau)} \int\limits_{\Omega} |u\,(t,x)|^{\frac{qn}{n-2q}}\, dx \leq C_{28}. \quad (1.57)$$

(1.56), (1.57) imply the needed estimate for the last integral in (1.55) provided

$$\beta^{'} \leq \frac{1}{1+\gamma}\left\{q\,(1+\gamma)\left(1+\frac{2}{n}\right)+\gamma\right\}-q,\ \ \beta^{'} \leq \frac{1}{1+\gamma}\left\{\frac{qn}{n-2q}+\gamma\right\}-q.$$

For that purpose it is sufficient to chose $\beta^{'} = \frac{\gamma}{1+\gamma}$. We proved that for $\bar{\beta} = \min\left(\beta, \beta^{'}\right)$ the left hand side of (1.55) is estimated by constant depending only on known parameters if $r = \frac{1}{3}\left(q-3+\bar{\beta}\right)$. This estimate implies that the inequality (1.43) is fulfilled with $q+\bar{\beta}$ instead of q. We can guarantee also by small change of $\bar{\beta}$ that the number $\frac{1}{\beta}\left[\frac{n}{2}-\frac{2+\gamma}{1+\gamma}\right]$ is not integer, and denote by N its integer part. Recalling that the estimate (1.43) is fulfilled with $q = q_0 = \frac{2+\gamma}{1+\gamma}$ and choosing the sequence $q_i = q_0 + i\bar{\beta}$. We obtain after $N+1$ iterations our previous discussing that the inequality (1.43) is fulfilled with $q = q_{N+1} > \frac{n}{2}$. onsequently the inequality (1.48) is satisfied with $q_{N+1}$and this ends the proof of Lemma 1.3.

Theorem 1.3 *Let the assumptions of Theorem 1.2 be satisfied. Then the estimates*

$$\|u(x,t)\|_{L_\infty(Q_T)} \leq M_3, \quad \left|u\left(t,x^{'}\right)-u\left(t,x^{''}\right)\right| \leq M_4\left|x^{'}-x^{''}\right|^{\eta} \quad (1.58)$$

hold for arbitrary $t \in [0,\tau]$, $x^{'}$, $x^{"} \in \Omega$ *with* $\eta \in (0,1)$ *and constants* M_3, M_4, φ *depending only on known parameters and independent of* ε.

Proof. The result of Theorems follows immediately from the estimates (1.30), (1.48), the conditions (1.4)-(1.6), (1.8) and the assumption on the set Ω. It is necessary to apply only well known results on regularity of solutions of elliptic equations to equation (1.1) (see, for example, [3]).

$$\omega^{'}(z) \leq \rho_2\omega(z),\ \rho_2 > 0 - \text{const}. \quad (1.59)$$

Theorem 1.4 *Let the conditions (1.4)-(1.6), (1.7), (1.8)-(1.9), (1.31), (1.59) be satisfied. Then there exists a constant* M_5, *depending only on known parameters and independent of* $\varepsilon \in \left[0, \frac{1}{M_5}\right]$, *such that each solution of problem (1.1)-(1.3) satisfies*

$$ess \sup\{|u\,(t,x)| : (t,x) \in Q_T\} \leq M_5. \quad (1.60)$$

Theorem 1.5 *Let the conditions (1.4)-(1.6), (1.7), (1.8)-(1.9), (1.31) (1.59) be satisfied. Then the initial-boundary value problem (1.1)-(1.3) has at least one solution in the sense of (1.10).*

Theorem 1.6 *Let the conditions (1.4)-(1.6), (1.7), (1.8)-(1.9), (1.31), (1.59) be satisfied and assume additionally that the functions* $a_{ij}(x,t)$, $b(x,t)$, $c(x,t)$ *are locally Lipschitzian with respect to* x. *Then the initial-boundary value problem (1.1)-(1.3) has a unique solution.*

For proof we use, proof of existance of solutions.

We consider for $\delta = \left[\frac{1}{M_5}\right]$ the initial boundary value problem (1.10). By Theorem 1.4 arbitrary solutions u of modify problem (1.10) satisfy the a priori estimate (1.60). We see that a solution of modify problem with $\delta = \frac{1}{M_5}$ is automatically a solution of problem (1.1)-(1.3).

Proof of uniqueness.

For proving the uniqueness of the solution for problem (1.1)-(1.3) we assume that there exists two solutions u_1, u_2. By Theorem 1.2, 1.3, we have for $j = 1, 2$

$$\|u_j\|_{L_\infty(Q_T)} + \left\|\frac{\partial u_j}{\partial x}\right\|^2_{L_{2,\omega}(Q_T)} + \left\|\frac{\partial u_j}{\partial t}\right\|^2_{L_{2,\Psi}(Q_T)} \leq M, \tag{1.61}$$

with some constant M.

The proof of Theorem 1.6 will be given in four steps corresponding to four different choices of test functions in the integral identities (1.10).

First step. We test (1.10) for $u = u_1$

$$\varphi_1 = \frac{1}{\omega(x)\Psi(x,t)}[u_1 - u_2]$$

and for $u = u_2$ with $\varphi_2 = u_1 - u_2$.

The result we obtain

$$\int\limits_\Omega |u_1(\tau,x) - u_2(\tau,x)|^2 dx + \int\limits_{Q_T} \left[\left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 + \left|\frac{\partial(u_1-u_2)}{\partial t}\right|^2\right] dxdt$$

$$\leq c_{29} \int\limits_{Q_T} \left[\left(1 + \left|\frac{\partial u_1}{\partial x}\right|\right) + 1 + \alpha(t,x)\right] |u_1 - u_2|^2 dxdt. \tag{1.62}$$

Second step. We test the integral identity (1.10) for $u = u_i,\ i = 1, 2$ with $\varphi_2 = u_1 - u_2$.Taking the difference of the obtained equalities, applying condition (1.4)-(1.6) and the inequalities of Cauchy and Poincare, we get

$$\int\limits_\Omega \left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 \leq C_{30} \int (u_1 - u_2)^2 dx. \tag{1.63}$$

Third step. We test the integral identity (1.10) for $u = u_1$ with

$$\varphi_3 = \frac{1}{\omega(x)}\Psi(x,t)[\exp(Nu_1) - \exp(Nu_2)], \tag{1.64}$$

and for $u = u_2$ with

$$\varphi_4 = N[u_1 - u_2]_+ \exp(Nu_2), \tag{1.65}$$

where N is a positive number depending only on known parameters and satisfying

$$N\omega^2(s) + 2\omega'(s) + N\Psi^2(s,t) + 2\Psi(s,t) \geq 1,$$

for

$$|s| \leq M,$$

with the constant M from (1.61).

Finally we obtain

$$\int\limits_{\Omega} |u_1(\tau,x) - u_2(\tau,x)|^2 dx + \int\limits_{Q_T} |u_1 - u_2|^2 \left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 \right) dxdt$$

$$\leq C_{30} \int\limits_{Q_T} \left\{ \left[\left|\frac{\partial (u_1 - u_2)}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u_2}{\partial x}\right|^2 \right] \right.$$

$$\left. + |u_1 - u_2|^2 + (1 + \gamma(t,x)) |u_1 - u_2|^2 dxdt \right\}. \tag{1.66}$$

Fourth step. Let $\varphi_j(x), j = 1, ...J$ be a partition satisfying the conditions

$$\sum_{j=1}^{J} \varphi_j^2(x) = 1, \quad \left|\frac{\partial \varphi_j}{\partial x}\right| < \frac{K_0}{R}$$

for $x \in \Omega$

$$\varphi_j(x) \in C^\infty(R^n), \quad \sup \varphi_j < B(x_j, R), \quad J \leq \frac{K_0}{R^n}, \quad R < 1, \tag{1.67}$$

where $B(x_j, R)$ is a ball of radius R with to be fixed chosen later on. We the integral identity (1.10) for $u = u_1$ with

$$\varphi = \sum_{i=1}^{J} \varphi_j^2 |u_1 - u_2|^2. \tag{1.68}$$

After some calculations imply immediately

$$\int\limits_{Q_T} |u_1 - u_2|^2 \left(\left|\frac{\partial u_1}{\partial x}\right|^2 + \left|\frac{\partial u_1}{\partial t}\right|^2 \right) dxdt \leq C_{31} \int\limits_{Q_T} \left\{ R^\gamma \left(\left|\frac{\partial u_1 - u_2}{\partial x}\right|^2 \right.\right.$$

$$\left.\left. + \left|\frac{\partial (u_1 - u_2)}{\partial t}\right|^2 \right) + \left(\frac{1}{R^2}\right) |u_1 - u_2|^2 \right\} dxdt. \tag{1.69}$$

Proof of Theorem 1.6. Applying Cauchy's inequality to the term in (1.62) involving the derivative of u_1 and choosing a suitable value of R, we obtain from (1.69), (1.66), (1.62), (1.63)

$$\int\limits_{\Omega} |u_1(\tau,x) - u_2(\tau,x)|^2 dx + \int\limits_{Q_+} \left(\left|\frac{\partial (u_1 - u_2)}{\partial x}\right|^2 + \left|\frac{\partial (u_1 - u_2)}{\partial t}\right|^2 \right) dxdt$$

$$\leq c_{32} \int (1 + |\alpha|) |u_1 - u_2|^2 dxdt. \tag{1.70}$$

We estimate the integral on the right hand site of (1.70) by Holders inequality and use condition on α, to get

$$ess \sup_{\tau \in (0,\theta)} \int\limits_{\Omega} |u_1(\tau,x) - u_2(\tau,x)|^2 dx + \int\limits_{Q_\theta} \left(\left|\frac{\partial (u_1 - u_2)}{\partial x}\right|^2 + \left|\frac{\partial (u_1 - u_2)}{\partial t}\right|^2 \right) dxdt$$

$$\leq C_{33} \left\{ \int\limits_{Q_\theta} |u_1 - u_2|^{2p_1'} dxdt \right\}^{\frac{1}{p_1'}} + C_{34} \int_0^\theta \left\{ \int\limits_{\Omega} |u_1 - u_2|^{2p_2'} dx \right\}^{\frac{1}{p_2'}} dt, \tag{1.71}$$

for an arbitrary $\theta \in (0,T)$. Estimating the first integral on the right hand site of (1.71) by Holders inequality, using the embedding $V^2(Q_T) < L_{\frac{2(n+2)}{n}}(Q_T)$ (see.[3]) and setting $q_1 = n + 2 - p_1' n$, we find for arbitrary $\varepsilon \in (0,1)$ and a constant C_{33} depending only on n

$$\left\{ \int\limits_{Q_\theta} |u_1 - u_2|^{2p_1'} dxdt \right\}^{1/p_1'}$$

$$\leq \left(\int\limits_{Q_\theta} |u_1 - u_2|^2 dxdt \right)^{\frac{q_1}{2p_1'}} \left(\int\limits_{Q_\theta} |u_1 - u_2|^{\frac{2(n+2)}{n}} dxdt \right)^{\frac{1}{p_1'} - \frac{q_1}{2p_1'}}$$

$$\leq \varepsilon^{-\frac{2p_1'}{q\uparrow}} \int\limits_{Q_\theta} |u_1 - u_2|^2 dxdt + C_{33} \varepsilon^{\frac{2p_1'}{2p_1' - q_1}} \left\{ ess \sup_{\tau \in (0,\theta)} \int\limits_{\Omega} |u_1(\tau,x) - u_2(\tau,x)|^2 dx \right.$$

$$\left. + \int\limits_{Q_\theta} \left| \frac{\partial (u_1 - u_2)}{\partial x} \right|^2 dxdt \right\}. \qquad (1.72)$$

In analogous way we estimate the last integral in (1.71). We define γ to be solution of the equation

$$\frac{2-\gamma}{2p_1' - \gamma} = \frac{1}{2} \left(\frac{1}{p_1'} + \frac{n-2}{n} \right).$$

We find

$$\int\limits_0^\theta \left\{ \int\limits_{\Omega} |u_1 - u_2|^{2p_2'} dx \right\}^{\frac{1}{p_1'}} dt \leq C_{35} \left\{ \varepsilon^{-\frac{2p_1'}{\gamma}} \int\limits_{Q_\theta} |u_1 - u_2|^2 dxdt \right.$$

$$\left. + \varepsilon^{\frac{2p_1'}{2p_1' - \gamma}} \left[ess \sup_{\tau \in (0,\theta)} \int\limits_{\Omega} |u_1 - u_2|^2 dx + \int\limits_{Q_\theta} \left| \frac{\partial |u_1 - u_2|}{\partial x} \right|^2 dxdt \right] \right\}. \qquad (1.73)$$

The inequalities (1.71)-(1.73) imply with suitable ε

$$\int\limits_{\Omega} |u_1(\theta,x) - u_2(\theta,x)|^2 dx \leq C_{36} \int\limits_{Q_\theta} |u_1 - u_2|^2 dxdt, \qquad (1.74)$$

for arbitrary $\theta \in (0,\tau)$. Finally, Gronwall's lemma yields $u_1 = u_2$.

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