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Pointwise Bernstein-Walsh-type inequalities in regions with interior zero angles in the Bergman space

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Abstract. *In this work, we investigate the order of growth of the modulus of an arbitrary algebraic polynomials in the weighted Bergman space, where the contour and the weight functions have some singularities. In particular, we obtain pointwise Berstein-Walsh -type estimation for algebraic polynomials in the unbounded regions with piecewise smooth boundary having interior zero angles.*

Keywords. Algebraic polynomials · Conformal mapping · Smooth curve.

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1 Introduction and Main Results

In many areas of research in mathematics one can be faced following problem: in a given finite Jordan region on the complex plane, in the various spaces, how to undergo a change of norm of the holomorphic function when the given region expands.

We will consider this problem for algebraic polynomial of complex variable in the well known Bergman space, and investigate the increasing of the modulus of polynomials in the exterior of the given region with respect to the norm of the polynomial in the this region.

The first classical results of the this type belong to Bernstein [16], Faber [18] and Walsh [25]. We will give this result in the right to our future arguments.

Let $G \subset \mathbb{C}$ be a finite region, with $0 \in G$, bounded by a Jordan curve $L := \partial G$, $\Omega := extL := \overline{\mathbb{C}} \backslash \overline{G}$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, $\Delta := \{w : |w| > 1\}$ and let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$,

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$\lim_{z\to\infty}\frac{\Phi(z)}{z}>0$, and $\Psi:=\Phi^{-1}$. For $t\geq 1$, $z\in\mathbb{C}$ and $M\subset\mathbb{C}$, we set:

$$L_t:=\{z:\ |\Phi(z)|=t\}\ (L_1\equiv L),\ G_t:=intL_t,\ \Omega_t:=extL_t.$$

$$d(z,M)=dist(z,M):=\inf\{|z-\zeta|:\ \zeta\in M\};$$

Well-known Bernstein -Walsh Lemma [25] says that for any $R>1$

$$|P_n(z)|\leq R^n\,\|P_n\|_{C(\overline{G})},\ \ z\in\Omega, \tag{1.1}$$

where

$$\|P_n\|_{C(\overline{G})}:=\max_{z\in\overline{G}}|P_n(z)|\,.$$

To give an analogue of the estimate (1.1)-type which respect to the other norms, we first define respective weighted spaces. Let $\{z_j\}_{j=1}^m$ be a fixed system of distinct points on curve L which is located in the positive direction. For some fixed R_0, $1<R_0<\infty$, and $z\in G_{R_0}$, consider a so-called generalized Jacobi weight function $h(z)$ being defined as follows:

$$h(z):=h_0(z)\prod_{j=1}^{m}|z-z_j|^{\gamma_j},\ \ z\in G_{R_0}, \tag{1.2}$$

where $\gamma_j>-2$, for all $j=1,2,...,m$, and h_0 is uniformly separated from zero in G_{R_0}, i.e. there exists a constant $c_0:=c_0(G_{R_0})>0$ such that, for all $z\in G_{R_0}$

$$h_0(z)\geq c_0>0.$$

For any $p>0$ and rectifiable curve L, we introduce:

$$\|P_n\|_{\mathcal{L}_p}:=\|P_n\|_{\mathcal{L}_p(h,L)}:=\left(\int_L h(z)\,|P_n(z)|^p\,|dz|\right)^{1/p},\quad 0<p<\infty; \tag{1.3}$$

$$\|P_n\|_{\mathcal{L}_\infty}:=\|P_n\|_{\mathcal{L}_\infty(1,L)}:=\|P_n\|_{C(\overline{G})},\ p=\infty.$$

The Bernstein-Walsh type estimation with respect to norm (1.3) for $p>0$ has been obtained in [19], for $h(z)\equiv 1$ and in [10, Lemma 2.4], for $h(z)$ defined as in (1.2), as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)}\leq R^{n+\frac{1+\gamma^*}{p}}\,\|P_n\|_{\mathcal{L}_p(h,L)},\qquad \gamma^*=\max\{0;\gamma_j:j\leq\ m\}. \tag{1.4}$$

Now, analogously to (1.3), for any $p>0$ and for Jordan region G, lets define:

$$\|P_n\|_{A_p}:=\|P_n\|_{A_p(h,G)}:=\left(\iint_G h(z)\,|P_n(z)|^p\,d\sigma_z\right)^{1/p},\quad 0<p<\infty;$$

$$\|P_n\|_{A_\infty}:=\|P_n\|_{A_\infty(1,G)}:=\|P_n\|_{C(\overline{G})},\ p=\infty,$$

where σ_z is the two-dimensional Lebesgue measure. Clearly, $\|\cdot\|_{A_p}$ and $\|\cdot\|_{\mathcal{L}_p}$ are the quasi-norms (i.e. a norm for $1\leq p\leq\infty$ and a $p-$norm for $0<p<1$).

In [6, Theorem1.1] was studied a similar problem in $A_p(1,G)$ for $p>0$ and for arbitrary Jordan region was obtained the following result: for any $p>0$, $P_n\in\wp_n,\ R_1=1+\frac{1}{n}$ and arbitrary $R,\ R>R_1$, the following is true:

$$\|P_n\|_{A_p(G_R)}\leq c\cdot R^{n+\frac{2}{p}}\,\|P_n\|_{A_p(G_{R_1})},$$

where $c = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right],\ n \to \infty$. Note that the constant c is sharp.

Now, we will give a similar estimation in $A_p(h, G)$ for $h(z) \neq 1$ and with respect to pair (G, G_R). Following [20, p.97], [22], the Jordan curve (or arc) L is called $K-$quasiconformal ($K \geq 1$), if there is a $K-$quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a circle (or line segment). An estimate of the Bernstein-Walsh type for the regions G with quasiconformal boundary and weight function $h(z)$, as defined in (1.2) with $\gamma_j > -2$ for any $p > 0$ was found in [5] as follows:

$$\|P_n\|_{A_p(h,G_R)} \leq c_1\, {R^*}^{n+\frac{1}{p}}\, \|P_n\|_{A_p(h,G)}\,,$$

where $R^* := 1 + c_2(R-1),\ c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent from n and R.

N.Stylianopoulos in [23] replaced the norm $\|P_n\|_{C(\overline{G})}$ with norm $\|P_n\|_{A_2(G)}$ on the right hand side of (1.2) and found a new version of the Bernstein-Walsh Lemma: *For quasiconformal and rectifiable curve L there exists a constant $c = c(L) > 0$ depending only on L such that*

$$|P_n(z)| \leq c_3(L)\frac{\sqrt{n}}{d(z,L)}\, \|P_n\|_{A_2(G)}\, |\Phi(z)|^{n+1}\,,\quad z \in \Omega, \tag{1.5}$$

holds for every $P_n \in \wp_n$, where $c_3(L) > 0$ constant independent from n and z.

Further, analogous results (1.5) for the $\|.\|_{A_p(h,G)}-$quasinorm, for some regions and the weight function $h(z)$ defined as in (1.2) with $\gamma_j > -2$, were obtained: in [8] for $p > 1$ and for regions bounded by piecewise Dini-smooth boundary without cusps; in [9] ($h(z) \equiv 1$) and [13] ($h(z) \neq 1$) for $p > 0$ and for regions bounded by quasiconformal curve; in [7] for $p > 1$ and for regions bounded by piecewise smooth curve without cusps; in [12] for $p > 0$ and for regions bounded by asymptotically conformal curve and in others.

Note that the analogous results to (1.5) for the $\|.\|_{\mathcal{L}_p(h,L)}-$quasinorm, for some curves and the weight function $h(z)$ defined as in (1.2) with $\gamma_j > -1$, were obtained: in [10], [11] and others.

In this work, we investigate similar problems for $z \in \Omega$ in regions bounded by piecewise smooth curve having interior zero angles and for weight function $h(z)$, defined in (1.2), through $\|.\|_{A_p(h,G)}-$quasinorm and $p > 0$.

Let us give some definitions and notations that will be used later in the text.

Let S be rectifiable Jordan curve or arc and let $z = z(s),\ s \in [0, |S|],\ |S| := mes\, S$, denote the natural representation of S.

Definition 1.1 *We say that a Jordan curve or arc $S \in C_\theta$, if S has a continuous tangent $\theta(z) := \theta(z(s))$ at every point $z(s)$. We will write a region $G \in C_\theta$, if $\partial G \in C_\theta$.*

According to [22], we have the following fact:

Corollary 1.1 *If $S \in C_\theta$, then S is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$.*

Now, we shall define a new class of regions with piecewise smooth boundary, where have exterior corners and interior cusps at the boundary points simultaneously.

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general. Also note that, for any $k \geq 0$ and $m > k$, notation $j = \overline{k, m}$ denotes $j = k, k+1, ..., m$.

Definition 1.2 *[7] We say that a Jordan region* $G \in C_\theta(\lambda_1, ..., \lambda_m),\ 0 < \lambda_j \leq 2,\ j = \overline{1, m}$, *if* $L = \partial G$ *consists of the union of finite smooth arcs* $\{L_j\}_{j=1}^m$, *such that they have exterior (with respect to* $\overline{G}$*) angles* $\lambda_j \pi,\ 0 < \lambda_j \leq 2$, *at the corner points* $\{z_j\}_{j=1}^m \in L$, *where two arcs meet.*

Without loss of generality, we assume that these points on the curve $L = \partial G$ are located in the positive direction such that, G has exterior $\lambda_j \pi,\ 0 < \lambda_j < 2,\ j = \overline{0, m_1}$, angle at the points $\{z_j\}_{j=1}^{m_1}$, $m_1 \leq m$, and interior zero angle (i.e. $\lambda_j = 2-$interior cusps) at the points $\{z_j\}_{j=m_1+1}^m$.

It is clear from Definition 1.2, the each region $G \in C_\theta(\lambda_1, ..., \lambda_m),\ 0 < \lambda_j \leq 2,\ j = \overline{1, m}$, may have exterior nonzero $\lambda_j \pi,\ 0 < \lambda_j < 2$, angles at the points $\{z_j\}_{j=1}^{m_1} \in L$, and interior zero angles ($\lambda_j = 2$) at the the points $\{z_j\}_{j=m_1+1}^m \in L$. If $m_1 = m = 0$, then the region G doesn't have such angles, and in this case we will write: $G \in C_\theta$; if $m_1 = m \geq 1$, then G has only $\lambda_i \pi,\ 0 < \lambda_i < 2,\ i = \overline{1, m_1}$, exterior nonzero angles, and in this case we will write: $G \in\ C_\theta(\lambda_i)$; if $m_1 = 0$ and $m \geq 1$, then G has only interior zero angles, and in this case we will write: $G \in C_\theta(2)$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^m \in L$ defined in (1.2) and Definition 1.2 are identical and $w_j := \Phi(z_j)$.

For simplicity of exposition and in order to avoid cumbersome calculations, without loss of generality, we will take $m_1 = 1,\ m = 2$. Then, after this assumption, in the future we will have region $G \in C_\theta(\lambda_1, 2),\ 0 < \lambda_1 < 2$, such that at the point $z_1 \in L$ region G have exterior nonzero $\lambda_1 \pi,\ 0 < \lambda_1 < 2$, and at the point $z_2 \in L$ - interior zero angle and we set $\lambda := \lambda_1$.

According to the "three-point" criterion [14, p.100], every piecewise smooth curve (without any cusps) is quasiconformal.

Now we can state our new results.

Theorem 1.1 *Let* $p > 0;\ G \in C_\theta(\lambda, 2)$, *for some* $0 < \lambda < 2$; $h(z)$ *be defined as in (1.2). Then, for any* $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2$ *and arbitrary small* $\varepsilon > 0$, *we have:*

$$|P_n(z)| \leq c_1 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_{1+1/n})} A_{n;1} \|P_n\|_{A_p},\ z \in \Omega_{1+1/n}, \tag{1.6}$$

where $c_1 = c_1(G, \gamma_1, \gamma_2,\ \lambda, p, \varepsilon) > 0$ *is a constant, independent from* z *and* n,

$$A_{n;1} := \begin{cases} n^{\frac{\widetilde{\gamma} \cdot \widehat{\lambda}}{p}}, & \text{if } \widetilde{\gamma} \cdot \widehat{\lambda} > 1, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} \cdot \widehat{\lambda} = 1, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} \cdot \widehat{\lambda} < 1, \end{cases}; \tag{1.7}$$

$\widehat{\lambda} := \begin{cases} \max\{1; \lambda\} + \varepsilon, & \text{if } 0 < \lambda < 2, \\ 2, & \text{if } \lambda = 2, \end{cases};\ \widetilde{\gamma} := \begin{cases} \widetilde{\gamma}_1, & \text{if } 0 < \lambda < 2, \\ \widetilde{\gamma}_2, & \text{if } \lambda = 2, \end{cases}$ *and* $\widetilde{\gamma}_i := \max\{0; \gamma_i\}$, $i = 1, 2$.

We can take individual cases when the curve L in the both points have the same type of angle: exterior nonzero or interior zero angle. In this case, from Theorem 1.1, we obtain the following:

Corollary 1.2 *Let* $p > 0;\ G \in C_\theta(\lambda_1, \lambda_2)$, *for some* $0 < \lambda_j < 2,\ j = 1, 2$; $h(z)$ *be defined as in (1.2). Then, for any* $P_n \in \wp_n,\ n \in \mathbb{N},\ \gamma_j > -2$ *and arbitrary small* $\varepsilon > 0$, *we have:*

$$|P_n(z)| \leq c_2 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_{1+1/n})} A_{n;2} \|P_n\|_{A_p},\ z \in \Omega_{1+1/n}, \tag{1.8}$$

where $c_2 = c_2(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ *is a constant, independent from* z *and* n, $A_{n;1}$ *defined as in (1.7).*

$$A_{n;2} := \begin{cases} n^{\frac{\widetilde{\gamma}\cdot\widetilde{\lambda}}{p}}, & \text{if } \widetilde{\gamma}\cdot\widetilde{\lambda} > 1, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma}\cdot\widetilde{\lambda} = 1, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma}\cdot\widetilde{\lambda} < 1, \end{cases} \tag{1.9}$$

and $\gamma^* := \max\{0, \gamma_1, \gamma_2\}$, $\widetilde{\lambda} := \max\{1; \lambda_1, \lambda_2\} + \varepsilon$.

Corollary 1.3 *Let* $p > 0$; $G \in C_\theta(2, 2)$; $h(z)$ *be defined as in (1.2). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$ *and arbitrary small* $\varepsilon > 0$, *we have:*

$$|P_n(z)| \le c_3 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L_{1+1/n})} A_{n;3} \|P_n\|_{A_p}, \ z \in \Omega_{1+1/n}, \tag{1.10}$$

where $c_3 = c_3(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ *a the constant, independent from* z *and* n, *and*

$$A_{n,3} := \begin{cases} n^{\frac{2\widetilde{\gamma}}{p}}, & \text{if } \widetilde{\gamma} > 1/2, \\ (n \ln n)^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} = 1/2, \\ n^{\frac{1}{p}}, & \text{if } \widetilde{\gamma} < 1/2. \end{cases} \tag{1.11}$$

Now, we give correspondingly estimate for the $|P_n(z)|$ for the points $z \in G$. In this case we have the following:

Theorem 1.2 *Let* $p > 0$; $G \in C_\theta(\lambda; 2)$, *for some* $0 < \lambda < 2$; $h(z)$ *be defined as in (1.2). Then, for any* $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_j > -2$, $j = 1, 2$, *and arbitrary small* $\varepsilon > 0$, *we have:*

$$|P_n(z)| \le c_4 \frac{A_{n;1}}{d^{2/p}(z, L)} \|P_n\|_{A_p}, \ z \in G,$$

where $c_4 = c_4(G, \gamma_1, \gamma_2, p, \varepsilon) > 0$ *is the constant, independent from* z *and* n, *and* $A_{n;1}$ *defined as in (1.7).*

The sharpness of the estimations (1.6)-(1.11) for some special cases can be discussed by comparing them with the following result:

Remark 1.1 For any $n \in \mathbb{N}$ there exists a polynomial $P_n^* \in \wp_n$, region $G^* \subset \mathbb{C}$, compact $F^* \Subset \Omega \backslash \overline{G}^*$ and constant $c_5 = c_5(G^*, F^*) > 0$ such that

$$|P_n^*(z)| \ge c_5 \frac{\sqrt{n}}{d(z, L_{1+1/n})} \|P_n^*\|_{A_2(G^*)} |\Phi(z)|^{n+1}, \qquad \text{for all } z \in F^*.$$

2 Some auxiliary results

Throughout this work, for the nonnegative functions $a > 0$ and $b > 0$, we shall use the notations "$a \prec b$" (order inequality), if $a \le cb$ and "$a \asymp b$" are equivalent to $c_1 a \le b \le c_2 a$ for some constants c, c_1, c_2 (independent of a and b), respectively.

Lemma 2.1 *[1] Let* L *be a* $K-$*quasiconformal curve*, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \prec d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. *Then*

a) The statements $|z_1 - z_2| \prec |z_1 - z_3|$ *and* $|w_1 - w_2| \prec |w_1 - w_3|$ *are equivalent. So are* $|z_1 - z_2| \asymp |z_1 - z_3|$ *and* $|w_1 - w_2| \asymp |w_1 - w_3|$;
b) If $|z_1 - z_2| \prec |z_1 - z_3|$, *then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^2} \prec \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \prec \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{K^{-2}},$$

where $0 < r_0 < 1,\ R_0 := r_0^{-1}$ *are constants, depending on* G.

Corollary 2.1 *Under the assumptions of Lemma 2.1, for* $z_3 \in L_{r_0}$ $(z_3 \in L_{R_0})$

$$|w_1 - w_2|^{K^2} \prec |z_1 - z_2| \prec |w_1 - w_2|^{K^{-2}}$$

Corollary 2.2 *If* $L \in C_\theta$, *then for all* $\varepsilon > 0$

$$|w_1 - w_2|^{1+\varepsilon} \prec |z_1 - z_2| \prec |w_1 - w_2|^{1-\varepsilon}.$$

Recall that for $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_1 - z_2|\}$, we put $\Omega(z_j,\ \delta_j) := \Omega \cap \{z : |z - z_j| \le \delta_j\}$; $\delta := \min\limits_{1\le j\le m} \delta_j$, $\Omega(\delta) := \bigcup\limits_{j=1}^{2} \Omega(z_j,\ \delta)$, $\widehat{\Omega} := \Omega \setminus \Omega(\delta)$. Additionally, let $\Delta_j := \Phi(\Omega(z_j,\ \delta))$, $\Delta(\delta) := \bigcup\limits_{j=1}^{m} \Phi(\Omega(z_j,\ \delta))$, $\widehat{\Delta}(\delta) := \Delta\backslash\Delta(\delta)$.

The following lemma is a consequence of the results given in [17], [21], [26] and of estimate for the $\left|\Psi'\right|$ (see, for example, [15, Th.2.8]) for $0 < \lambda_j < 2$, $j = \overline{1, m}$:

$$\left|\Psi'(\tau)\right| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{2.1}$$

Lemma 2.2 *[26]. Let* $G \in C_\theta(\lambda_1, ..., \lambda_m),\ 0 < \lambda_j < 2,\ j = \overline{1, m}$. *Then for all* $\varepsilon > 0$:
i) for any $w \in \Delta_j$, $|w - w_j|^{\lambda_j+\varepsilon} \prec |\Psi(w) - \Psi(w_j)| \prec |w - w_j|^{\lambda_j-\varepsilon}$, $|w - w_j|^{\lambda_j-1+\varepsilon} \prec |\Psi'(w)| \prec |w - w_j|^{\lambda_j-1-\varepsilon}$,
ii) for any $w \in \overline{\Delta}\backslash\Delta_j$, $(|w|-1)^{1+\varepsilon} \prec d(\Psi(w), L)| \prec (|w|-1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \prec |\Psi'(w)| \prec (|w| - 1)^{-\varepsilon}$.

Let $\{z_j\}_{j=1}^m$ be a fixed system of the points on L and the weight function $h(z)$ be defined as in (1.2):

Lemma 2.3 *[4] Let L be a* $K-$*quasiconformal curve*; $h(z)$ *is defined in (1.2). Then, for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n = 1, 2, ...,$ *we have*

$$\|P_n\|_{A_p(h,G_R)} \prec \widetilde{R}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)},\ p > 0,$$

where $\widetilde{R} = 1 + c(R - 1)$ *and c is independent from* n *and* R.

Lemma 2.4 *Let* $G \in C_\theta(\lambda_1, ..., \lambda_m),\ 0 < \lambda_j \le 2,\ j = \overline{1, m}$. *Then, for arbitrary* $P_n(z) \in \wp_n$ *and any* $p > 0$, *we have:*

$$\|P_n\|_{A_p(h,G_{1+c/n})} \prec \|P_n\|_{A_p(h,G)},$$

Proof. For $0 < \lambda_j < 2$, $j = \overline{1,m}$, this is follows from Lemma 2.3 and Corollary 1.1 and from the fact, that according to the "three-point" criterion [20, p.100], any piecewise smooth curve without cusps is a quasiconformal. If $\lambda_j = 2$, for all $j = \overline{1,m}$, then the region G have exterior 2π angles (i.e. interior cusps) at the every points z_j, $j = \overline{1,m}$. Then, in the neighborhood of the this points the region G have a boundary with outsite wedge. Therefore, as well known from theory of conformal mappings, the distance from the corner point to the level curve L_R is less than the distance from the other points. Furthermore, the area between boundary L and level curve L_R in the neighborhood of the such corners, having exterior 2π angles, will be smaller less than such in the case of area without angles.

2.1 Proof of Theorem 1.1

Proof. Suppose that $G \in C_\theta(\lambda; 2)$,for some $0 < \lambda < 2$; $h(z)$ be defined as in (1.2).Let $\{\xi_j\}$, $1 \le j \le m \le n$, be the zeros (if any) of $P_n(z)$ lying on Ω. Lets define the function Blaschke with respect to the zeros $\{\xi_j\}$ of the polynomial $P_n(z)$:

$$\widetilde{B}_j(z) := \frac{\Phi(z) - \Phi(\xi_j)}{1 - \overline{\Phi(\xi_j)}\Phi(z)} \ , \ z \in \Omega, \tag{2.2}$$

and let

$$B_m(z) := \prod_{j=1}^{m} \widetilde{B}_j(z), \ z \in \Omega. \tag{2.3}$$

It is easy that the

$$B_m(\xi_j) = 0, \ |B_m(z)| \equiv 1, \ z \in L; \ |B_m(z)| < 1, \ z \in \Omega. \tag{2.4}$$

Then, for each ε_1, $0 < \varepsilon_1 < 1$, there exists circle $\left\{w : |w| = R_1 := 1 + \varepsilon_2, 0 < \varepsilon_2 < \frac{\varepsilon_1}{n}\right\}$ such that for any $j = 1, 2$, the following is holds:

$$\left|\widetilde{B}_j(\zeta)\right| > 1 - \varepsilon_2, \ \zeta \in L_{R_1}.$$

So, from (2.3), we get:

$$|B_m(\zeta)| > (1 - \varepsilon_2)^m \succ 1, \ \zeta \in L_{R_1}. \tag{2.5}$$

For any $p > 0$ and $z \in \Omega$ let us set:

$$Q_{n,p}(z) := \left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2} . \tag{2.6}$$

The function $Q_{n,p}(z)$ is analytic in Ω, continuous on $\overline{\Omega}$, $Q_{n,p}(\infty) = 0$ and does not have zeros in Ω. We take an arbitrary continuous branch of the $Q_{n,p}(z)$ and for this branch, we maintain the same designation. According to Cauchy integral representation for the unbounded region Ω, we have:

$$Q_{n,p}(z) = -\frac{1}{2\pi i} \int\limits_{L_{R_1}} Q_{n,p}(\zeta) \frac{d\zeta}{\zeta - z} \ , \ z \in \Omega_{R_1}.$$

According to (2.2) - (2.6), we have:

$$|P_n(z)|^{p/2} = \frac{\left|B_m(z)\Phi^{n+1}(z)\right|^{\frac{p}{2}}}{2\pi d(z,L_{R_1})} \int\limits_{L_{R_1}} \left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2} |d\zeta| \tag{2.7}$$

$$\prec \frac{\left|\Phi^{n+1}(z)\right|^{\frac{p}{2}}}{d(z,L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)|^{p/2} |d\zeta| .$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we obtain:

$$\left(\int\limits_{L_{R_1}} |P_n(\zeta)|^{\frac{p}{2}} |d\zeta|\right)^2$$

$$\leq \left(\int\limits_{|t|=R_1} h(\Psi(t))\,|P_n(\Psi(t))|^p \left|\Psi'(t)\right|^2 |dt|\right) \cdot \left(\int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))}\right) \tag{2.8}$$

$$\leq \left(\int\limits_{|t|=R_1} h(\Psi(t))\,|P_n(\Psi(t))|^p \left|\Psi'(t)\right|^2 |dt|\right) \cdot \left(\int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))}\right)$$

$$= \left(\int\limits_{|t|=R_1} |f_{n,p}(t)|^p |dt|\right) \cdot \left(\int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))}\right) =: A_n \cdot D_n,$$

where $f_{n,p}(t) := h^{\frac{1}{p}}(\Psi(t))P_n(\Psi(t))(\Psi'(t))^{\frac{2}{p}}$, $|t| = R_1$.

For the estimate integral A_n, we separate the circle $|t| = R_1$ to n equal parts δ_n with $mes\delta_n = \frac{2\pi R_1}{n}$ and by applying the mean value theorem, we get:

$$A_n := \int\limits_{|t|=R_1} |f_{n,p}(t)|^p |dt|$$

$$= \sum_{k=1}^{n} \int\limits_{\delta_k} |f_{n,p}(t)|^p |dt| = \sum_{k=1}^{n} \left|f_{n,p}\left(t_k'\right)\right|^p mes\delta_k, \;\; t_k' \in \delta_k.$$

On the other hand, by applying mean value estimation

$$\left|f_{n,p}\left(t_k'\right)\right|^p \leq \frac{1}{\pi\left(\left|t_k'\right|-1\right)^2} \iint\limits_{\left|\xi-t_k'\right|<\left|t_k'\right|-1} |f_{n,p}(\xi)|^p d\sigma_\xi,$$

we obtain:

$$(A_n)^2 \preceq \sum_{k=1}^{n} \frac{mes\;\delta_k}{\pi\left(\left|t_k'\right|-1\right)^2} \iint\limits_{\left|\xi-t_k'\right|<\left|t_k'\right|-1} |f_{n,p}(\xi)|^p d\sigma_\xi, \; t_k' \in \delta_k.$$

By taking into account, at most two of the discs with center t'_k are intersecting, we have:

$$A_n \preceq \frac{mes\delta_1}{\left(\left|t'_1\right|-1\right)^2} \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi \preceq n \cdot \iint\limits_{1<|\xi|<R} |f_{n,p}(\xi)|^p \, d\sigma_\xi .$$

According to Lemma 2.4, for A_n we get:

$$A_n \preceq n \iint\limits_{G_R \setminus G} h(\zeta) \left|P_n(\zeta)\right|^p d\sigma_\zeta \preceq n \cdot \|P_n\|_p^p . \tag{2.9}$$

To estimate the integral $B_n(w)$, denote by $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$, for any fixed $\rho > 1$,we introduce:

$$\Delta_1(\rho) : = \left\{ t = re^{i\theta} : r > \rho,\ \frac{\varphi_0 + \varphi_1}{2} \le \theta < \frac{\varphi_1 + \varphi_2}{2} \right\},$$
$$\Delta_2(\rho) : = \left\{ t = re^{i\theta} : r > \rho,\ \frac{\varphi_1 + \varphi_2}{2} \le \theta < \frac{\varphi_1 + \varphi_0}{2} \right\};$$

$$\Delta_j : = \Delta_j(1),\ \Omega^j := \Psi(\Delta_j),\ \Omega^j_\rho := \Psi(\Delta_j(\rho));$$
$$L^j : = L \cap \overline{\Omega}^j,\ L = L^1 \cup L^1\ L^j_\rho := L_\rho \cap \overline{\Omega}^j_\rho,\ L_\rho = L^1_\rho \cup L^2_\rho,\ j = 1, 2;$$

and

$$\Phi(L_{R_1}) = \Phi(\bigcup_{j=1}^{2} L^j_{R_1}) = \bigcup_{j=1}^{2} \Phi(L^j_{R_1}) = \bigcup_{j=1}^{2} \bigcup_{i=1}^{3} K^j_i(R_1),$$

where

$$K^j_1(R_1) := \left\{ t \in \Phi(L^j_{R_1}) : \quad |t - w_j| < \frac{c_1}{n} \right\},$$
$$K^j_2(R_1) := \left\{ t \in \Phi(L^j_{R_1}) : \quad \frac{c_1}{n} \le |t - w_j| < c_2 \right\}$$
$$K^j_3(R_1) := \left\{ t \in \Phi(L^j_{R_1}) : \quad c_2 \le |t - w_j| < diam\, \overline{G} \right\},\ j = 1, 2.$$

Then, we get

$$D_n = \int\limits_{|t|=R_1} \frac{|dt|}{h(\Psi(t))} \preceq \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{\prod\limits_{j=1}^{2} |\Psi(t) - \Psi(w_j)|^{\gamma_j}} \tag{2.10}$$

$$\asymp \sum_{j=1}^{2} \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}} := \sum_{j=1}^{2} D_{n,j},$$

since the points $\{z_j\}_{j=1}^m \in L$ are distinct. It remains to estimate the integrals

$$D_{n,j} := \int\limits_{\Phi(L^j_{R_1})} \frac{|dt|}{|\Psi(t) - \Psi(w_j)|^{\gamma_j}} \tag{2.11}$$

for each $j=1,2$. Now, we define

$$D_{n,1}=\sum_{i=1}^{3}\int\limits_{K_i^1(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}}=:D_{n,1}^i, \tag{2.12}$$

for $j=1$,

$$D_{n,2}=\sum_{i=1}^{3}\int\limits_{K_i^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}}=:D_{n,2}^i, \tag{2.13}$$

for $j=2$, and let's estimate they separately.

Case 1.1 Applying Lemma 2.2, we get:

$$\begin{aligned}D_{n,1}^1&=\int\limits_{K_1^1(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}}\prec\int\limits_{K_1^1(R_1)}\frac{|dt|}{|t-w_1|^{\gamma_1(\lambda+\varepsilon)}}\\&\prec\begin{cases}n^{\gamma_1\lambda-1+\varepsilon}, & \text{if } \gamma_1(\lambda+\varepsilon)>1,\\ \ln n, & \text{if } \gamma_1(\lambda+\varepsilon)=1,\\ 1, & \text{if } -2<\gamma_1(\lambda+\varepsilon)<1,\end{cases}\end{aligned} \tag{2.14}$$

if $\gamma_1\geq 0$, and

$$\begin{aligned}D_{n,1}^1&=\int\limits_{K_1^1(R_1)}|\Psi(t)-\Psi(w_1)|^{(-\gamma_1)}|dt|\prec\int\limits_{K_1^1(R_1)}|t-w_1|^{(-\gamma_1)(\lambda-\varepsilon)}|dt|\\&\prec\left(\frac{1}{n}\right)^{(-\gamma_1)(\lambda-\varepsilon)}\cdot mesK_1^1(R_1)\prec 1,\end{aligned} \tag{2.15}$$

if $\gamma_1<0$.

Case 1.2 Analogously, we obtain:

$$\begin{aligned}D_{n,1}^2&=\int\limits_{K_2^1(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}}\prec\int\limits_{K_2^1(R_1)}\frac{|dt|}{|t-w_1|^{\gamma_1(\lambda+\varepsilon)}}\\&\prec\begin{cases}n^{\gamma_1\lambda-1+\varepsilon}, & \text{if } \gamma_1(\lambda+\varepsilon)>1,\\ \ln n, & \text{if } \gamma_1(\lambda+\varepsilon)=1,\\ 1, & \text{if } -2<\gamma_1(\lambda+\varepsilon)<1,\end{cases}\end{aligned} \tag{2.16}$$

if $\gamma_1\geq 0$, and

$$D_{n,1}^2\prec\int\limits_{K_2^1(R_1)}|t-w_1|^{(-\gamma_1)(\lambda-\varepsilon)}|dt|\prec 1, \tag{2.17}$$

if $\gamma_1<0$.

Case 1.3 For all cases of γ_1, we have:

$$D_{n,1}^3=\int\limits_{K_3^1(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_1)|^{\gamma_1}}\prec c_2^{-\gamma_1}\cdot mesK_3^1(R_1)\prec 1. \tag{2.18}$$

Case 1.4 According to the estimation [24, p.181] for arbitrary continuum with simple connected complementary

$$|\Psi(t)-\Psi(w_2)|\succ|t-w_2|^2,$$

we get:

$$D^1_{n,2}+D^2_{n,2}=\sum_{i=1}^{2}\int\limits_{K_i^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}} \tag{2.19}$$

$$\prec \int\limits_{K_1^2(R_1)\cup K_2^2(R_1)}\frac{|dt|}{|t-w_2|^{2\gamma_2}}\prec\begin{cases} n^{2\gamma_2-1}, & \text{if } 2\gamma_2>1,\\ \ln n, & \text{if } 2\gamma_2=1,\\ 1, & \text{if } -2<2\gamma_2<1,\end{cases}$$

if $\gamma_2\geq 0$, and

$$D^1_{n,2}+D^2_{n,2}=\int\limits_{K_1^2(R_1)\cup K_2^2(R_1)}|\Psi(t)-\Psi(w_2)|^{(-\gamma_2)}\,|dt|\prec 1, \tag{2.20}$$

if $\gamma_2<0$.

Case 1.5 For all cases of γ_2, we have:

$$D^3_{n,2}=\int\limits_{K_3^2(R_1)}\frac{|dt|}{|\Psi(t)-\Psi(w_2)|^{\gamma_2}}\prec 1. \tag{2.21}$$

Therefore, comparing (2.10) - (2.21), we obtain:

$$D_n\prec\begin{cases} n^{\widetilde{\gamma}_1(\lambda+\varepsilon)-1}, & \text{if } \widetilde{\gamma}_1>\frac{1}{\lambda}-\varepsilon,\\ \ln n, & \text{if } \widetilde{\gamma}_1=\frac{1}{\lambda}-\varepsilon,\\ 1, & \text{if } -2<\widetilde{\gamma}_1<\frac{1}{\lambda}-\varepsilon,\end{cases}+\begin{cases} n^{2\widetilde{\gamma}_2-1}, & \text{if } \gamma_2>\frac{1}{2},\\ \ln n, & \text{if } \gamma_2=\frac{1}{2},\\ 1, & \text{if } -2<\gamma_2<\frac{1}{2},\end{cases}$$

$$\prec\begin{cases} n^{\widetilde{\gamma}\widehat{\lambda}-1}, & \text{if } \widetilde{\gamma}\widehat{\lambda}>1,\\ \ln n, & \text{if } \widetilde{\gamma}\widehat{\lambda}=1,\\ 1, & \text{if } -2<\widetilde{\gamma}\widehat{\lambda}<1,\end{cases} \tag{2.22}$$

where

$$\widehat{\lambda}:=\begin{cases}\max\{1;\lambda\}+\varepsilon, & \text{if } 0<\lambda<2,\\ 2, & \text{if } \lambda=2.\end{cases};\ \widetilde{\gamma}:=\begin{cases}\widetilde{\gamma}_1, & \text{if } 0<\lambda<2,\\ \widetilde{\gamma}_2, & \text{if } \lambda=2.\end{cases},$$

$$\widetilde{\gamma}_i:=\max\{0;\gamma_i\}, i=1,2.$$

Combining relations (2.7), (2.8), (2.9) and (2.22), we complete the proof.

2.2 Proof of Theorem 1.2

Proof. Suppose that $G\in C_\theta(\lambda;2)$, for some $0<\lambda<2$; $h(z)$ be defined as in (1.2). For each $R>1$, let $w=\varphi_R(z)$ denotes be a univalent conformal mapping G_R onto the B, normalized by $\varphi_R(0)=0,\ \varphi'_R(0)>0$, and let $\{\zeta_j\},\ 1\leq j\leq m\leq n$, be a zeros of $P_n(z)$ lying on G_R. Let

$$b_{m,R}(z):=\prod_{j=1}^{m}\widetilde{b}_{j,R}(z)=\prod_{j=1}^{m}\frac{\varphi_R(z)-\varphi_R(\zeta_j)}{1-\overline{\varphi_R(\zeta_j)}\varphi_R(z)},$$

denotes a Blaschke function with respect to zeros $\{\zeta_j\}, 1\leq j\leq m\leq n$, of $P_n(z)$ ([25]). Clearly,

$$|b_{m,R}(z)|\equiv 1,\ z\in L_R,\ \text{and}\ |b_{m,R}(z)|<1,\ z\in G_R. \tag{2.23}$$

For any $p > 0$ and $z \in G_R$, let us set

$$T_{n.p}(z) := \left[\frac{P_n(z)}{b_{m,R}(z)}\right]^{p/2} .$$

The function $T_{n,p}(z)$ is analytic in G_R, continuous on $\overline{G}_R$ and does not have zeros in G_R. We take an arbitrary continuous branch of the $T_{n,p}(z)$ and for this branch we maintain the same designation. Then, the Cauchy integral representation for the $T_{n,p}(z)$ in G_R gives

$$T_{n,p}(z) = \frac{1}{2\pi i}\int\limits_{L_R} T_{n,p}(\zeta)\frac{d\zeta}{\zeta - z},\ z \in G.$$

Then, according to (2.23), we obtain:

$$\begin{aligned} |P_n(z)|^{p/2} &\leq \frac{|b_{m,R}(z)|^{p/2}}{2\pi}\int\limits_{L_R}\left|\frac{P_n(\zeta)}{b_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta - z|} \\ &\prec \frac{1}{d(z,L)}\int\limits_{L_R}|P_n(\zeta)|^{p/2}|d\zeta| . \end{aligned} \tag{2.24}$$

Multiplying the numerator and the denominator of the last integrand by $h^{1/2}(\zeta)$, replacing the variable $w = \Phi(z)$ and applying the Hölder inequality, we obtain:

$$\begin{aligned} &\left(\int\limits_{L_R}|P_n(\zeta)|^{\frac{p}{2}}|d\zeta|\right)^2 \\ &\leq \left(\int\limits_{|t|=R} h(\Psi(t))\,|P_n(\Psi(t))|^p\left|\Psi'(t)\right|^2|dt|\right)\left(\int\limits_{|t|=R}\frac{|dt|}{h(\Psi(t))}\right) \\ &= \left(\int\limits_{|t|=R}|f_{n,p}(t)|^p|dt|\right)\cdot\left(\int\limits_{|t|=R}\frac{|dt|}{h(\Psi(t))}\right), \end{aligned} \tag{2.25}$$

where $f_{n,p}(t)$ has been defined as in (2.8). Since $R > 1$ is arbitrary, then (2.25) holds also for $R = R_1 := 1 + \frac{\varepsilon_1}{n}$, $0 < \varepsilon_1 < 1$. So, we have:

$$\left(\int\limits_{L_{R_1}}|P_n(\zeta)|^{\frac{p}{2}}|d\zeta|\right)^2 \leq \left(\int\limits_{|t|=R_1}|f_{n,p}(t)|^p|dt|\right)\cdot\left(\int\limits_{|t|=R_1}\frac{|dt|}{h(\Psi(t))}\right) =: A_n\cdot D_n, \tag{2.26}$$

and, A_n and D_n has been defined as in (2.8). Combining relations (2.24), (2.26), with (2.9) and (2.22), we complete the proof.

201 Proof of Remark 1.1.

Proof. Let the region G bounded by smooth curve $L = \partial G$. According to the Corollary 1.1, it is $(1+\varepsilon)-$quasiconformal for arbitrary small $\varepsilon > 0$. Let $\{K_n(z)\}$, $\deg K_n = n$, denote of the system of Bergman polynomials for region G. According to [2], [3] for arbitrary quasidisk, we have

$$K_n(z) = \alpha_n \rho^{n+1} \Phi^n(z) \Phi^{'}(z) A_n(z),\ z \in F \Subset \Omega,$$

where

$$\sqrt{\frac{n+1}{\pi}} \leq \alpha_n \rho^{n+1} \leq c_1 \sqrt{\frac{n+1}{\pi}},$$

for some $c_1 = c_1(G) > 1$ and

$$c_2 \leq |A_n(z)| \leq 1 + \frac{c_3}{\sqrt{|\Phi(z)| - 1}},$$

for some $c_i = c_i(G) > 0,\ i = 2, 3$. Therefore, since $\|K_n\|_{A_2(G)} = 1$, according to (2.1),

we have

$$\begin{aligned} |K_n(z)| &\geq c_2 \sqrt{\frac{n+1}{\pi}}\, |\Phi(z)|^n \frac{|\Phi(z)| - 1}{d(z, L)} \\ &\geq c_3 \frac{\sqrt{n}}{d(z, L)}\, |\Phi(z)|^{n+1} \left(1 - \frac{1}{|\Phi(z)|}\right) \\ &\geq c_4 \frac{\sqrt{n}}{d(z, L)}\, |\Phi(z)|^{n+1}, \end{aligned}$$

and we complete the proof.

References

1. Abdullayev, F.G., Andrievskii, V.V.: *On the orthogonal polynomials in the domains with K -quasiconformal boundary*, Izv. Akad. Nauk Azerb. SSR., Ser. FTM, **1**, 3–7 (in Russian) (1983).
2. Abdullayev, F.G.: Dissertation (Ph.D.), Donetsk (1986).
3. Abdullayev, F.G.: *On the orthogonal polynomials with unique weight*, Izv. AS Azerb. SSR, Ser. FTM, **4**, 7–10 (1986).
4. Abdullayev, F.G.: *On the some properties of the orthogonal polynomials over the region of the complex plane (Part III)*, Ukr.Math.J., **53** (12), 1934–1948 (2001).
5. Abdullayev, F.G.: *On the interference of the weight boundary contour for orthogonal polynomials over the region*, J. of Comp. Anal. and Appl., **6** (1), 31–42 (2004).
6. Abdullayev, F.G., Özkartepe, P.: *An analogue of the Bernstein-Walsh lemma in Jordan regions of the complex plane*, Journal Ineq. and Appl. 570, 1–7 (December, 2013) (2013).
7. Abdullayev, F.G., Gün, C.D.: *On the behavior of the algebraic polynomials in regions with piecewise smooth boundary without cusps*, Ann.Polon.Math., **111**, 39–58 (2014).
8. Abdullayev, F.G., Özkartepe N.P.: *On the Behavior of the Algebraic Polynomial in Unbounded Regions with Piecewise Dini -Smooth Boundary*, Ukr. Math. J., **66** (5), 579–597 (2014).
9. Abdullayev, F.G., Özkartepe N.P.: Uniform and pointwise Bernstein-Walsh-type inequalities on a quasidisk in the complex plane, Bull. Belg. Math. Soc., **23** (2), 285–310 (2016).

10. Abdullayev, F.G. , Özkartepe P.: *On the growth of algebraic polynomials in the whole complex plane*, J. Korean Math. Soc. **52** (4), 699–725 (2015).
11. Abdullayev, F.G., Özkartepe ,P.: *Uniform and pointwise polynomial inequalities in regions with cusps in the weighted Lebesgue space*, Jaen Journal on Approximation, **7** (2), 231–261 (2015).
12. Abdullayev, F.G. , Tunç, T.: *Uniform and pointwise polynomial inequalities in regions with asymptotically conformal curve on weighted Bergman space*, Lobachevcki Journal of Mathematics, **38** (2), 193–205 (2017).
13. Abdullayev, F.G., Tunç, T., Abdullayev, G.A.: *Polynomial inequalities in quasidisks on weighted Bergman space*, Ukranian Mathematical Journal, (2017) (accepted).
14. Ahlfors, L.: Lectures on Quasiconformal Mappings, *Princeton, NJ: Van Nostrand* (1966).
15. Andrievskii, V.V., Belyi, V.I., Dzyadyk, V.K.: Conformal invariants in constructive theory of functions of complex plane, *Atlanta:World Federation Publ.Com.* (1995).
16. Bernstein, S.N.: *Sur l'ordre de la meilleure approximation des fonctions continues par les polynomes de degre donne*, Mem. Cl.Sci. Acad.Roy. Belgique, **4** (2), 1–103 (1912).
17. Gaier, D.: *On the convergence of the Bieberbach polynomials in regions with corners*, Constructive Approximation, **4**, 289–305 (1988).
18. Faber, G.: *Über nach Polynomen fortschreitende Reihen, Sitzungsberichte der Bayrischen Akademie der Wissenschaften*, 157–178 (1922).
19. Hille, E., Szegö, G., Tamarkin, J.D.: *On some generalization of a theorem of A.Markoff*, Duke Math. **3**, 729–739 (1937).
20. Lehto, O., Virtanen, K.I.: *Quasiconformal Mapping in the plane*, Springer Verlag, Berlin (1973).
21. Mergelyan, S.N., *Some questions of Constructive Functions Theory*, Proceedings of the Steklov Institute of Mathematics, **37**, 1–92 (in Russian) (1951).
22. Rickman, S.: *Characterization of quasiconformal arcs*, Ann. Acad. Sci. Fenn., Ser. A, Mathematica., **395** (30) (1966).
23. Stylianopoulos, N.: *Strong asymptotics for Bergman polynomials over domains with corners and applications*, Const. Approx. **38**, 59–100 (2013).
24. Tamrazov, P.M.: Smoothness and Polynomial Approx. *Kiev, Naukova Dumka* (in Russian) (1975).
25. Walsh, J.L.: Interpolation and Approximation by Rational Functions in the Complex Domain, *AMS* (1960).
26. Warschawski, S.E.: *Über das randverhalten der ableitung der ableitung der abbildungsfunktion bei konformer abbildung*, Math.Z., **35**, 321–456 (1932).