Transactions of NAS of Azerbaijan, Issue Mathematics, **37** (4), 5–11 (2017).
Series of Physical-Technical and Mathematical Sciences

Boundedness of the parabolic maximal operator in Orlicz spaces

Gulnara A. Abasova

Received: 10.05.2017 / Revised: 12.07.2017 / Accepted: 26.09.2017

Abstract. *This paper deals with the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$.*

Keywords. Parabolic maximal operator; Orlicz space.

Mathematics Subject Classification (2010): 42B20 · 42B25 · 42B35

1 Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators etc, from one Lebesgue space to another one is well studied by now. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of whose eigenvalues have positive real part. Let $A_t = t^P \ \ (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$\begin{aligned}
&(a) \ \ \rho(A_t x) = t\rho(x), \ \ t > 0, \ \ \text{for every} \ \ x \in \mathbb{R}^n; \\
&(b) \ \ \rho(0) = 0, \ \ \rho(x-y) = \rho(y-x) \geq 0 \\
&\qquad \text{and} \ \ \rho(x-y) \leq k(\rho(x-z) + \rho(y-z)); \\
&(c) \ \ dx = \rho^{\gamma-1} d\sigma(w) d\rho, \ \ \text{where} \ \ \rho = \rho(x), w = A_{\rho^{-1}} x \\
&\qquad \text{and } d\sigma(w) \text{is a } C^\infty \text{measure on the ellipsoid } \{w : \rho(w) = 1\}.
\end{aligned}$$

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Moreover, we always assume the following properties on ρ:

G.A. Abasova
Azerbaijan State University of Economics, Baku, Azerbaijan
E-mail: abasovag@yahoo.com

(d) For every x,

$$c_1|x|^{\alpha_1} \le \rho(x) \le c_2|x|^{\alpha_2} \text{ if } \rho(x) \ge 1$$
$$c_3|x|^{\alpha_3} \le \rho(x) \le c_4|x|^{\alpha_4} \text{ if } \rho(x) \le 1$$

and

$$\rho(\theta x) \le \rho(x) \text{ for } 0 < \theta < 1.$$

Here α_i and c_i $(i = 1, \dots, 4)$ are some positive constants. Similar properties hold for ρ^* which is associated with the matrix P^*.

There are some important examples for the above spaces:

1. Let $(Px, x) \ge (x, x)$ $(x \in \mathbb{R}^n)$. In this case, $\rho(x)$ is defined by the unique solution of $|A_{t^{-1}}x| = 1$, and $k = 1$. This space is just the one studied by Calderon and Torchinsky in [2].

2. Let P be a diagonal matrix with positive diagonal entries, and let $\rho(x)$ be the unique solution of $|A_{t^{-1}}x| = 1$.

2_a) If all diagonal entries are greater than or equal to 1, this space was studied by E.B. Fabes and N.M. Riviere [3]. More precisely they studied the weak $(1,1)$ and L^p estimates of the singular integral operators on this space in 1966.

2_b) If there are diagonal entries smaller than 1, then ρ satisfies the above $(a) - (d)$ with $k \ge 1$.

Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([1], [3]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x, r) = \{y \in \mathbb{R}^n : \rho(x - y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x, r)| = v_n r^\gamma$, where v_n is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x, r) = \mathbb{R}^n \setminus \mathcal{E}(x, r)$ be the complement of $\mathcal{E}(x, r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x, r) = B(x, r)$. Note that in the standard parabolic case $P = (1, \dots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \quad x = (x', x_n).$$

Let $f \in L_1^{\text{loc}}(\mathbb{R}^n)$. The parabolic maximal function $M^P f$ is defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x, t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy.$$

If $P = I$, then $M \equiv M_0^I$ is the classical Hardy-Littlewood maximal operator.

It is well known that maximal operator play an important role in harmonic analysis (see [9]).

In this work, we prove the boundedness of the parabolic maximal operator M^P in Orlicz spaces.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2 On Young functions and Orlicz spaces

Orlicz space was first introduced by Orlicz in [6,7] as a generalizations of Lebesgue spaces L^p. Since then this space has been one of important functional frames in the mathematical analysis, and especially in real and harmonic analysis. Orlicz space is also an appropriate substitute for L^1 space when L^1 space does not work.

First, we recall the definition of Young functions.

Definition 2.1 *A function* $\Phi : [0,\infty) \to [0,\infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim\limits_{r\to\infty} \Phi(r) = \infty$.

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{2.1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad r \geq 0$$

for some $C > 1$.

Lemma 2.1 *[5] Let* Φ *be a Young function with canonical representation*

$$\Phi(t) = \int_0^t \varphi(s)ds, t > 0.$$

(1) *Assume that* $\Phi \in \Delta_2$. *More precisely* $\Phi(2t) \leq A\Phi(t)$ *for some* $A \geq 2$. *Set* $\beta = \log_2 A$. *If* $p > \beta + 1$, *then the following inequality is valid:*

$$\int_t^\infty \frac{\varphi(s)}{s^p}ds \lesssim \frac{\Phi(t)}{t^p},\ t > 0.$$

(2) *Assume that* $\Phi \in \nabla_2$. *Then the following inequality is valid:*

$$\int_0^t \frac{\varphi(s)}{s}ds \lesssim \frac{\Phi(t)}{t},\ t > 0.$$

Definition 2.2 *(Orlicz Space). For a Young function* Φ, *the set*

$$L^\Phi(\mathbb{R}^n) = \left\{ f \in L^1_{\rm loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$, *then* $L^\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. *If* $\Phi(r) = 0$, $(0 \leq r \leq 1)$ *and* $\Phi(r) = \infty$, $(r > 1)$, *then* $L^\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. *The space* $L^\Phi_{\rm loc}(\mathbb{R}^n)$ *is defined as the set of all functions* f *such that* $f\chi_{\mathcal{E}} \in L^\Phi(\mathbb{R}^n)$ *for all parabolic balls* $\mathcal{E} \subset \mathbb{R}^n$.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t > 0$, let $m(\Omega,\ f,\ t) = |\{x \in \Omega : |f(x)| > t\}|$. In the case $\Omega = \mathbb{R}^n$, we shortly denote it by $m(f,\ t)$.

Definition 2.3 *The weak Orlicz space*

$$WL^\Phi(\mathbb{R}^n) = \{f \in L^1_{\rm loc}(\mathbb{R}^n) : \|f\|_{WL^\Phi} < \infty\}$$

is defined by the norm

$$\|f\|_{WL^\Phi} = \inf\left\{\lambda > 0\ :\ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda},\ t\Big)\ \le 1\right\}.$$

We note that $\|f\|_{WL^\Phi} \le \|f\|_{L^\Phi}$,

$$\sup_{t>0} \Phi(t) m(\Omega,\ f,\ t) = \sup_{t>0} t\, m(\Omega,\ f,\ \Phi^{-1}(t)) = \sup_{t>0} t\, m(\Omega,\ \Phi(|f|),\ t)$$

and

$$\int_\Omega \Phi\Big(\frac{|f(x)|}{\|f\|_{L^\Phi(\Omega)}}\Big)dx \le 1, \qquad \sup_{t>0} \Phi(t) m\Big(\Omega,\ \frac{f}{\|f\|_{WL^\Phi(\Omega)}},\ t\Big) \le 1, \tag{2.2}$$

where $\|f\|_{L^\Phi(\Omega)} = \|f\chi_\Omega\|_{L^\Phi}$ and $\|f\|_{WL^\Phi(\Omega)} = \|f\chi_\Omega\|_{WL^\Phi}$.

The following analogue of the Hölder's inequality is well known (see, for example, [8]).

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)|dx \le 2\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\tilde\Phi}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 2.2 *Let Φ be a Young function and $\mathcal{E}$ be a parabolic balls in $\mathbb{R}^n$. Then*

$$\|\chi_{\mathcal{E}}\|_{L^\Phi} = \|\chi_{\mathcal{E}}\|_{WL^\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 2.1, Lemma 2.2 and (2.1) we get the following estimate.

Lemma 2.3 *For a Young function Φ and for the parabolic balls $\mathcal{E} = \mathcal{E}(x, r)$ the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)|dy \le 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)\|f\|_{L^\Phi(\mathcal{E})}.$$

3 Parabolic maximal operator in the Orlicz spaces $L_\Phi(\mathbb{R}^n)$

In this section the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ have been obtained.

Theorem 3.1 *Let Φ any Young function. Then the parabolic maximal operator M^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$ bounded in $L_\Phi(\mathbb{R}^n)$.*

Proof. At first proved that the parabolic maximal operator M^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Phi(\mathbb{R}^n)$.

We take $f \in L_\Phi$ satisfying $\|f\|_{L_\Phi} = 1$ so that the modular

$$\rho_\Phi(f) := \int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq 1.$$

We know that by Jensen inequality

$$\Phi\left(\frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} |f(y)|dy\right) \leq \frac{1}{|\mathcal{E}|}\int_{\mathcal{E}} \Phi(|f(y)|)dy \tag{3.1}$$

for all parabolic balls $\mathcal{E}$. Using (3.1) and definition of parabolic maximal operator we have

$$\Phi(M^P f(x)) \leq M^P[(\Phi \circ f)(x)]. \tag{3.2}$$

Using (3.2) and weak (1,1) boundedness of parabolic maximal operator we get

$$\begin{aligned}
|\{x : M^P f(x) > t\}| &= |\{x : \Phi(M^P f(x)) > \Phi(t)\}| \\
&\leq |\{x : M^P(\Phi \circ f)(x) > \Phi(t)\}| \\
&\leq \frac{C}{\Phi(t)}\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \\
&\leq \frac{C}{\Phi(t)} \leq \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})},
\end{aligned}$$

since $\|f\|_{L_\Phi} = 1$ and $\frac{1}{C}\Phi(t) \geq \Phi\left(\frac{t}{C}\right)$, if $C \geq 1$.

Since $\|\cdot\|_{L_\Phi}$ norm is homogeneous the inequality

$$|\{x : M^P f(x) > t\}| \leq \frac{1}{\Phi(\frac{t}{C\|f\|_{L_\Phi}})}$$

is true for every $f \in L_\Phi$.

Now proved that for $\Phi \in \nabla_2$ the parabolic maximal operator M^P is bounded in $L_\Phi(\mathbb{R}^n)$.

Let $\Lambda > 0$ and $f \in L_\Phi \backslash \{0\}$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \Phi\left(\frac{M^P f(x)}{\Lambda}\right) dx &= \int_{\mathbb{R}^n}\int_0^{\frac{M^P f(x)}{\Lambda}} \varphi(s)dsdx \\
&= \int_{\mathbb{R}^n}\int_0^\infty \chi_{\{s\in[0,\infty):\frac{M^P f(x)}{\Lambda}>s\}}\varphi(s)dsdx \\
&= \int_0^\infty \varphi(s)\int_{\mathbb{R}^n} \chi_{\{x\in\mathbb{R}^n:M^P f(x)>\Lambda s\}}dxds \\
&= \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{\lambda}{\Lambda}\right)\left|\{x \in \mathbb{R}^n : M^P f(x) > \lambda\}\right| d\lambda \\
&= \frac{2}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left|\{x \in \mathbb{R}^n : M^P f(x) > \lambda\}\right| d\lambda.
\end{aligned}$$

From the maximal inequality

$$\left|\{x \in \mathbb{R}^n : M^P f(x) > 2\lambda\}\right| \lesssim \frac{1}{\lambda} \int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f(x)|dx$$

and change the order of integration

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi\left(\frac{M^P f(x)}{\Lambda}\right) dx &\lesssim \frac{1}{\Lambda}\int_0^\infty \varphi\left(\frac{2\lambda}{\Lambda}\right)\left(\int_{\{x\in\mathbb{R}^n:|f(x)|>\lambda\}} |f|\right)\frac{d\lambda}{\lambda} \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n} |f(x)| \left(\int_0^{|f(x)|} \varphi\left(\frac{2\lambda}{\Lambda}\right)\frac{d\lambda}{\lambda}\right) dx \\ &\lesssim \frac{1}{\Lambda}\int_{\mathbb{R}^n} |f(x)| \left(\int_0^{2\Lambda^{-1}|f(x)|} \varphi(\lambda)\frac{d\lambda}{\lambda}\right) dx.\end{aligned}$$

Now we use Lemma 2.1 which yields

$$\left(\int_0^{2\Lambda^{-1}|f(x)|} \varphi(\lambda)\frac{d\lambda}{\lambda}\right) \lesssim |f(x)|^{-1}\Lambda\Phi\left(\frac{2|f(x)|}{\Lambda}\right),$$

if $f(x) \neq 0$. Recall that $k\Phi(t) \le \Phi(kt)$ for $k \ge 1$ and $t > 0$, assuming Φ convex. Therefore, it follows that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{M^P f(x)}{\Lambda}\right) dx \le c_0 \int_{\mathbb{R}^n} \Phi\left(\frac{2|f(x)|}{\Lambda}\right) dx \le \int_{\mathbb{R}^n} \Phi\left(\frac{c_0|f(x)|}{\Lambda}\right) dx.$$

Here c_0 is a constant we would like to shed light on. Choosing $\Lambda = c_0\|f\|_{L_\Phi}$, we obtain

$$\int_{\mathbb{R}^n} \Phi\left(\frac{M^P f(x)}{\Lambda}\right) dx \le 1.$$

This means

$$\|M^P f\|_{L_\Phi} \le \Lambda = c_0\|f\|_{L_\Phi}$$

from the definition of the norm.

Remark 3.1 Note that Theorem 3.1 in the isotropic case $P = I$ were proved in [4].

References

1. Besov, O.V., Il'in, V.P., Lizorkin,P.I.: *The L_p-estimates of a certain class of non-isotropically singular integrals,* Dokl. Akad. Nauk SSSR, **169**, 1250-1253 (1966) (in Russian).
2. Caldereon, A.P., Torchinsky A.: *Parabolic maximal function associated with a distribution,I.* Advances in math., **16**, 1-64 (1975).
3. Fabes, E.B., Rivère, N.: *Singular integrals with mixed homogeneity,* Studia Math., **27**, 19-38 (1966).
4. Kita, H.: *On maximal functions in Orlicz spaces*, Proc. Amer. Math. Soc. **124**, 3019-3025 (1996).
5. Kokilashvili, V., Krbec, M.M.: *Weighted Inequalities in Lorentz and Orlicz Spaces*. World Scientific, Singapore, (1991).

6. Orlicz, W.: Über eine gewisse Klasse von Räumen vom Typus B, Bull. Acad. Polon. A, 207-220 (1932).; reprinted in: Collected Papers, PWN, Warszawa, 217-230 (1988).
7. Orlicz, W., Über Räume (L^M), Bull. Acad. Polon. A, 93-107 (1936).; reprinted in: Collected Papers, PWN, Warszawa, 345-359 (1988).
8. Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces, *M. Dekker, Inc., New York* (1991).
9. Stein, E.M.: *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ (1993).